An econometric analysis of the Blanchard-Watson bubble model

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Introduction

The paper by Blanchard and Watson (1982) investigates the nature and the presence of bubbles in financial markets and whether the presence of bubbles in a particular market can be detected statistically.

The purpose of the present paper is to analyse a simple bubble model suggested by Blanchard and Watson as a statistical model.

We want

- to find conditions for stationarity and infinite variance of the process
- discuss the behavior of the least squares autoregressive estimator

The question we want to discuss is whether a bubble model can create the long swings, or persistence, which are observed in many macro variables.

A simple autoregressive bubble model

Let $\rho > 1$

$$y_t = s_t \rho y_{t-1} + \varepsilon_t, \;\; t = 1, \ldots, N,$$

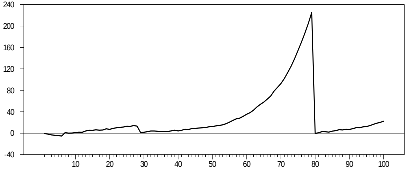
s_t is i.i.d. binary $(0,1)$ variables with $p = P(s_t = 1) = 1 - q$
ε_t is i.i.d. $(0, \sigma^2)$ and $\{s_t\}$ independent of $\{\varepsilon_t\}$.
y_t is explosive when $s_t = 1$ and the bubble bursts when $s_t = 0$
Persistence of y_t is defined to mean that

$$\hat{\rho}_T = \frac{\sum_{t=1}^T y_t y_{t-1}}{\sum_{t=1}^T y_{t-1}^2}$$

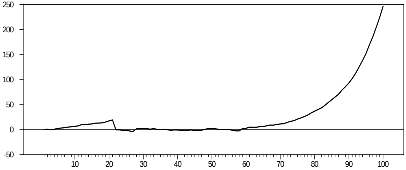
is close to one
The question raised by Frydman and Goldberg (2007) is if this model can create long swings, or persistence, when the variance of y_t is infinite.
Comments on the model

Plot of bubble process, rho = 1.10, p = 0.95

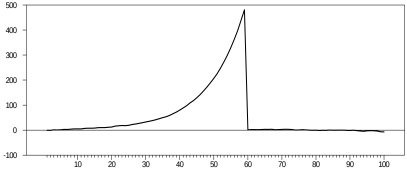
number of periods is 5 with start 0,7,9,29,80



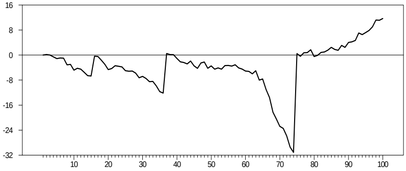
number of periods is 6 with start 0,22,28,37,48,59



number of periods is 4 with start 0,5,60,79



number of periods is 5 with start 0,16,37,75,80



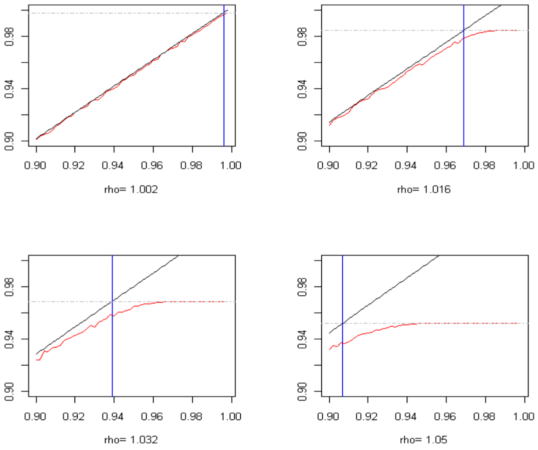


Figure: Median of 200 simulations of $\hat{\rho}_n$ for $N = 10.000$ as function of p.
For values of $\rho = (1.002, 1.016, 1.032, 1.050)$.
For $p < \rho^{-2} = (0.996, 0.968, 0.938, 0.907)$ $\hat{\rho}_n \approx p\rho$.
For $p > \rho^2$ the limit is equal to ρ^{-1}, so that $\hat{\rho}_n \xrightarrow{P} \min(p\rho, \rho^{-1})$.

Properties of the process y_t

The model

$$y_t = s_t \rho y_{t-1} + \varepsilon_t$$

Lemma

Let $\rho > 1$, then y_t is stationary if $p < 1$.

- *If $p\rho < 1$, the mean is finite and equal zero*
- *If $p\rho^2 < 1$ the variance is finite and equal to $\sigma^2/(1 - p\rho^2)$.*
- *If $p\rho^2 > 1$ the variance is infinite.*

The proof applies the drift criterion.
Although the process is explosive in the intervals where $s_t = 1$, it collapses to ε_t if $s_t = 0$, and the bubble bursts. It is this repeated collapse that creates a stationary process, which starts each period in a new ε.

But, what happens to the autoregressive estimator?

Description of y_t, its product moments, and the estimator

Burst times are increasing $0 = T_0^* < T_1^* < T_2^* < \cdots < T_n^* \leq N$
Burst lengths $T_i = T_i^* - T_{i-1}^*$ have geometric distribution
$P(T_i = m) = p^{m-1}q, m = 1, 2, \ldots$
For the first period $t = 0, 1, \ldots, T_1 - 1$ we have

$$
\begin{aligned}
y_t &= \sum_{v=0}^{t} \rho^{t-v}\varepsilon_v = \rho^t\varepsilon_0 + \rho^{t-1}\varepsilon_1 + \cdots + \varepsilon_t = \rho^t \sum_{v=0}^{t} \rho^{-v}\varepsilon_v \\
y_t &= \rho^t \sum_{v=0}^{T_1-1} \rho^{-v}\varepsilon_v - \rho^{-1} \sum_{v=0}^{T_1-t-2} \rho^{-v}\varepsilon_{v+1+t} = \rho^t Z_1 + \ldots
\end{aligned}
$$

$$
\begin{aligned}
\sum_{t=1}^{T_1} y_{t-1}^2 &= \frac{\rho^{2T_1} - 1}{\rho^2 - 1} Z_1^2 + \ldots \\
\sum_{t=1}^{T_1} y_t y_{t-1} &= \sum_{t=1}^{T_1-1} y_t y_{t-1} + y_{T_1} y_{T_1-1} \approx \sum_{t=1}^{T_1-1} \rho^{2t-1} Z_1^2 = \rho \frac{\rho^{2(T_1-1)} - 1}{\rho^2 - 1} Z_1^2
\end{aligned}
$$

A sort of proof

Similar expressions can be found for the i'th period leading to an analysis of the estimator

$$\begin{aligned}
\hat{\rho}_n &= \frac{\sum_{t=1}^{T_n^*} y_t y_{t-1}}{\sum_{t=1}^{T_n^*} y_{t-1}^2} \approx \frac{\sum_{i=1}^n \rho \frac{\rho^{2(T_i-1)}-1}{\rho^2-1} Z_i^2}{\sum_{i=1}^n \frac{\rho^{2T_i}-1}{\rho^2-1} Z_i^2} \\
&= \rho \frac{\sum_{i=1}^n (\rho^{2(T_i-1)}-1) Z_i^2}{\sum_{i=1}^n (\rho^{2T_i}-1) Z_i^2} = \rho \frac{\sum_{i=1}^n \rho^{2(T_i-1)} Z_i^2}{\sum_{i=1}^n \rho^{2T_i} Z_i^2} = \rho^{-1}
\end{aligned}$$

Theorem

For $p < 1$ (stationarity), $p\rho^2 < 1$ (variance finite) and $E(\varepsilon_t^4) < \infty$ we have

$$\hat{\rho}_n \xrightarrow{P} p\rho,$$

$$n^{1/2}(\hat{\rho}_n - p\rho) \xrightarrow{d} N(0, \Sigma),$$

where $\Sigma = q([pq\rho^2 E(y_{t-1}^4) + \sigma^2 E(y_{t-1}^2)]/E(y_{t-1}^2)^2$.

Calculation

$$\begin{aligned} E(y_t y_{t-1}) &= E((s_t \rho y_{t-1} + \varepsilon_t) y_{t-1}) = p\rho E(y_{t-1}^2) \\ \frac{E(y_t y_{t-1})}{E(y_{t-1}^2)} &= p\rho \end{aligned}$$

Theorem

For $p < 1$ (stationarity), $p\rho^2 > 1$ (infinite variance) and $E(\varepsilon_t^4) < \infty$ we have

$$\hat{\rho}_n \xrightarrow{P} \rho^{-1}.$$

Discussion of one difficulty and two tricks

$$\hat{\rho}_n = \frac{\sum_{t=1}^{T_n^*} y_t y_{t-1}}{\sum_{t=1}^{T_n^*} y_{t-1}^2} = \frac{\sum_{i=1}^{n} \rho \frac{\rho^{2(T_i-1)}-1}{\rho^2-1} Z_i^2 + \dots}{\sum_{i=1}^{n} \frac{\rho^{2T_i}-1}{\rho^2-1} Z_i^2 + \dots}$$

Difficulty: We want the order of magnitude of the main term $\sum_{i=1}^{n} \rho^{T_i} Z_i^2$.

Introduction of exponential waiting times

Trick 1: Let U_i be an exponential variable with parameter $\lambda = -\log p$ so that $P(U \geq x) = e^{-\lambda x} = p^x$
Define the waiting time $T_i = [U_i] + 1$ then $P(T_i = m) = qp^{m-1}$

$$\sum_{i=1}^n \rho^{2U_i} Z_i^2 \leq \sum_{i=1}^n \rho^{2T_i} Z_i^2 \leq \rho^2 \sum_{i=1}^n \rho^{2U_i} Z_i^2.$$

Let $\alpha = -\log p / \log \rho < 2$. The tail index, $\alpha/2$, of ρ^{2U} is defined by

$$P(\rho^{2U} > x) = e^{-\frac{\lambda \log x}{2 \log \rho}} = x^{\frac{\log p}{2 \log \rho}} = x^{-\alpha/2},$$

For independent $W > 0$, $E(W^\alpha) < \infty$, the tail index of $\rho^{2U} W^2$ is also $\alpha/2$:

$$P(\rho^{2U} W^2 > x) = E[P(\rho^{2U} W^2 > x)|W)] = E(x/W^2)^{-\alpha/2} = x^{-\alpha/2} E(W^\alpha)$$

and then, Feller (1971, Chapter IX, Section 8),

$$\sum_{i=1}^n \rho^{2U_i} W_i^2 = O_P(n^{2/\alpha})$$

Making Z_1 independent of T_1

Trick 2

Because Z_i is not independent of T_i or U_i we use another idea

$$
\begin{aligned}
Z_1 &= \sum_{v=0}^{T_1-1} \rho^{-v}\varepsilon_v = \sum_{v=0}^{\infty} \rho^{-v}\varepsilon_v - \sum_{v=T_1}^{\infty} \rho^{-v}\varepsilon_v \\
&= \sum_{v=0}^{\infty} \rho^{-v}\varepsilon_v - \rho^{-T_1}\sum_{v=0}^{\infty} \rho^{-v}\varepsilon_{v+T_1} = Z_1^* + Z_1^{**}
\end{aligned}
$$

because $\sum_{v=0}^{\infty} \rho^{-v}\varepsilon_v$ is independent of T_1 and the remainder $O_P(\rho^{-T_1})$. Then the order of magnitude of $\sum_{i=1}^{n} \rho^{2T_i} Z_i$ is the order of magnitude of

$$
\sum_{i=1}^{n} \rho^{2U_i} Z_i^*
$$

which is given by Feller (1971, Chapter IX, Section 8) as $n^{2/\alpha}$.

Conclusion

The statistical analysis of the simple bubble model

$$y_t = s_t \rho y_{t-1} + \varepsilon_t$$

shows two results which appear a bit surprising

- The process y_t is stationary of $p < 1$
- The autoregressive estimator converges to $\min(p\rho, \rho^{-1})$

References

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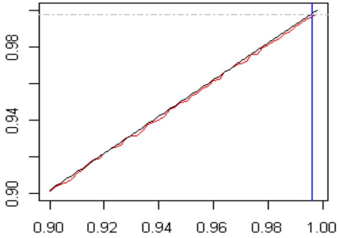
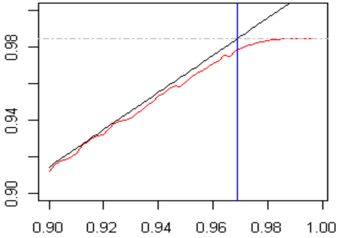
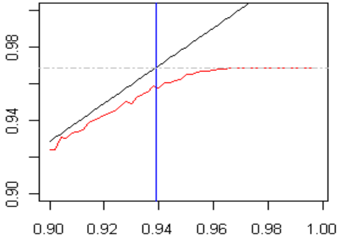


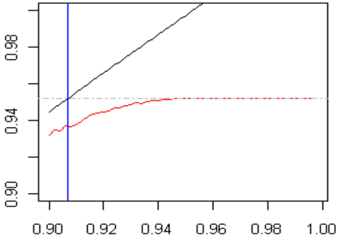
rho= 1.002



rho= 1.016



rho= 1.032



rho= 1.05