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# LEM

# WORKING PAPER SERIES

# Rethinking volatility scaling in firm growth

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# Rethinking volatility scaling in firm growth

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**Abstract**

We revisit the size-volatility relationship in firm growth using administrative data on French manufacturing firms. Departing from the log-log linear decay commonly reported by other studies, we find a two-regime pattern: volatility declines steeply with size for small firms, but flattens for larger ones. We relate this new fact to the presence of resources misallocation as captured by imperfect correlation between size and productivity at the firm level. To explain the nexus between these two facts, we develop a stochastic model where firms face a number of risky business opportunities for which they compete. Two key features characterize this competition process. First, larger firms are more intensively exposed to competition dynamics. Second, firms with higher productivity are more likely to see business opportunities turning into positive, rather than negative, growth episodes. We analytically show that only when the correlation between firm size and productivity is lower than 1 the model is able to reproduce the volatility scaling we observed in the data. Simulations suggest that finite sample approximations of our asymptotic result are satisfactory in a reasonable portion of the parameter space. We conclude showing that in France industries populated by firms with higher correlation between size and productivity are associated with steeper average size-volatility decays consistent with the model's main prediction. Our findings suggest that the existence of resources misallocation, shaping the size-volatility relation, affects the relevance of the granularity channel in explaining aggregate fluctuations (Gabaix, 2011).

**Keywords**: volatility scaling, granularity, resource allocation
**JEL codes**: D40, L20

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# 1 Introduction

When the distribution of firm sizes is fat-tailed, idiosyncratic fluctuations of the largest firms fail to average out at the aggregate level, contributing to macroeconomic fluctuations. This transmission mechanism, known as the *granularity channel*, has been found to explain approximately one-third of the variation in output growth in the United States (Gabaix, 2011).[1] One key premise underlying this result is that the standard deviation of a firm's growth rate is independent of its size. However, the growth volatility of business firms has been found to log-linearly decay with size with a slope approximately equal to $-0.2$. In presence of this negative size-volatility relationship the relevance of the granular argument may be mitigated, a possibility discussed in Calvino et al. (2018) and recently shown to reduce the explanatory power of the granular channel from about one-third to at most 15 percent of aggregate fluctuations (Yeh, 2023).[2]

This paper revisits the size-volatility relationship using sales data encompassing the entire population of active employers in the French manufacturing sector.[3] Contrary to previous findings, our analysis uncovers a new feature in the size-volatility relationship: the rate at which volatility declines with size in a log-log space is not constant, but systematically attenuates across the size distribution, from being steep for small firms to moderate for larger ones.[4] We show that the exact shape of this non-linear decay varies over time and more importantly across industrial sectors, pointing to the presence of underlying structural heterogeneity. This new feature of the size-volatility relationship has direct implications for the granularity channel. Being an asymptotic result, the quantitative relevance of the granularity argument depends on the regime that governs volatility among the largest firms. We show that, in the case of French manufacturing, this regime displays a very mild decay approximately flattening, thereby mitigating concerns like those raised in Yeh (2023). We relate the existence of this non-linearity in the size-volatility relation to another well-established empirical finding in the industrial dynamics literature: the positive but imperfect correlation between firm size and productivity. This fact has been interpreted as evidence that resource allocation tends to favor more productive firms, but does so imperfectly (Bartelsman et al., 2013). In our sample of French firms, this correlation is also positive and lower than one – around 0.2 on average – and it exhibits substantial variation over time and across industries.

We characterize the size-volatility-productivity nexus by proposing a model in which the industry is conceptualized as an "islands economy": a setting composed of a fixed pool of firms operating – without direct competition – in a number of independent sub-industries of equal size. The term "islands economy" captures the idea that firms do not compete directly within their own sub-industries (Sutton, 2001; Klepper and Thompson, 2006). Firms are heterogeneous in both productivity and size, where size is defined in terms of the number of sub-industries in which a firm is active. Size evolves over time according to a simple stochastic process that captures competitive dynamics. At the beginning of each period, firms face a fixed number of

[1] See also Carvalho and Grassi (2019). Similar evidence has been found studying firms sales in international markets (di Giovanni and Levchenko, 2012; di Giovanni et al., 2014, 2017, 2018, 2024), mergers and acquisitions (Chan and Qi, 2025), local geographical markets (Jannati et al., 2020) and and production networks (Acemoglu et al., 2012; Devereux et al., 2023).

[2] The negative relationship between size and volatility of growth rates is not the only factor possibly affecting the granular channel, see for example Pastén et al. (2024).

[3] Garicano et al. (2016) use the same dataset to examine how size-contingent labor regulations affect equilibrium outcomes and welfare. Relevant to this work, they find that the firm size distribution in France is significantly fat-tailed, closely approximating a power law, consistently with evidence from other developed economies (Axtell, 2001).

[4] Simultaneously to the development of this paper the same feature has been highlighted also in Moran (2020).

new business opportunities for which they compete.[5] Each opportunity is risky: it may result in the acquisition of a new sub-industry or in the loss of one currently held by the firm. This mechanism, which governs the birth and death of sub-industries and their allocation across firms, drives the dynamics of firm size. Two key features characterize this competition process. First, larger firms are more likely to grab business opportunities. Second, firms with higher productivity are more likely to see these opportunities turning into an increase of the number of sub-industries they are active in, thereby increasing their size.[6]

We analytically show that when firm size and productivity are perfectly correlated – i.e. resource allocation systematically favors more productive firms – the model necessarily yields a log-log linear decay in growth volatility with respect to a firm's size, at odds with the empirical evidence we document for France. By contrast, when the correlation is imperfect,[7] the model predicts a marked deviation from this behavior: volatility continues to decline with size at a constant rate only among small firms, but this decline progressively attenuates and eventually flattens for the largest firms. Moreover, the weaker the correlation, the earlier the flattening occurs, and the broader the range over which volatility becomes almost size-invariant as predicted by the Gibrat's law (Gibrat, 1931). Furthermore, under favorable market conditions – represented by an increase in the number of business opportunities – the volatility gap between small and large firms diminishes, as large firms become relatively more volatile. Numerical analyses indicate that, while finite-size industries exhibit deviations from the asymptotic results in our theorems, these deviations are too small to affect our qualitative conclusions and vanish quickly as industry size grows. Our model suggests that imperfect correlation between size and productivity is associated with more volatile large firms and hence, ceteris paribus, with a higher relevance of the granularity channel in explaining aggregate fluctuations.

We conclude our investigations by taking the model back to the data and running reduced-form regressions. We show that industries populated by firms with higher correlation between size and productivity are associated with steeper average size-volatility decays consistent with the model's main prediction. Moreover we observe a compression in the volatility gap between small and large firms in industries where reallocation mechanisms are more intense.

This paper makes three main contributions to the literature. First, it extends the long-standing body of research on the statistical properties of industrial dynamics initiated by Gibrat (1931), focusing on the relationship between firm size and growth volatility. Empirical evidence has consistently shown that larger firms tend to exhibit lower growth volatility compared to smaller firms, with this relationship approximated by a log-linear decay, typically with a slope close to -0.2. This relationship has been established using sales data for U.S. publicly traded firms (Stanley et al., 1996; Sutton, 1997; Amaral et al., 1997; Lee et al., 1998; Koren and Tenreyro, 2013), global data on pharmaceutical firms (Bottazzi and Secchi, 2006b; Riccaboni et al., 2008), and employment data for manufacturing firms across 20 developed economies (Calvino et al., 2018). Yeh (2023) provides similar evidence for the U.S. using population-level employment data.[8] For the first time, this paper provides evidence that the size-volatility decay in France is not uniform, but instead features two distinct

[5]In our framework, a business opportunity denotes the chance to compete for industry shares. As detailed in Section 3, these opportunities may lead to either positive or negative size changes, depending on the competition's outcome

[6]In the remaining of the paper we consider a single period version of this process describing the dynamics of a firm's size from $t$ to $t+1$ while holding productivity constant.

[7]In the remaining of this paper we consider only positive or zero correlations.

[8]An inverse size-volatility relationship has also been documented at the sectoral (Castaldi and Sapio, 2008) and national (Lee et al., 1998; Canning et al., 1998; Castaldi and Dosi, 2009) levels.

regimes. While small firms benefit from diversification effects as they grow, resulting in reduced volatility, these effects vanish above a certain size threshold.[9]

Second, this paper contributes to theoretical models of firm growth that explain the existence of systematic deviations from Gibrat's law. Existing models typically represent firms as ensembles of heterogeneous sub-units. Firms' size (i.e. the sum of sub-units) evolves via a multiplicative process driven by the ability to grab business opportunities and transform them into positive shocks. These opportunities are assumed to be independent and conceptualised as segments of an industry where firms can operate without competitors. Different types of mechanisms have been proposed to explain the size-volatility relation. Stanley et al. (1996) and Amaral et al. (1997) model firms as having an internal hierarchical structure, Sutton (2002) as simple "partitions of integers", while Bottazzi and Secchi (2006b) propose a branching process encompassing a diversification dynamics driven by economies of scope. Fu et al. (2005), Buldyrev et al. (2007) and Schwarzkopf et al. (2010) represent three other examples of compositional models where the variance-size relation is explained assuming that sub-units a firm is composed by are not necessarily of equal size.[10] The two works closest to this paper are Wyart and Bouchaud (2003) and Gabaix (2011). While they differ in the way they model the statistics of the number of sub-units composing each firm, they can accommodate both the slow decay of growth volatilities with size and the non-Gaussian distribution of growth rates (Moran et al., 2024).[11] However, they cannot account for the two-regime size-volatility relationship observed in this paper. By introducing productivity as a second dimension of firm heterogeneity, our model provides a novel explanation for the shape of this relationship based on the existence and the strength of the size-productivity correlation. Our explanation emphasizing the role of productivity heterogeneity also links with a vast body of research in the industrial dynamics and international trade literatures (see e.g. Hopenhayn, 1992; Melitz and Redding, 2014; Dosi et al., 2017; Carvalho and Grassi, 2019, among many others).

Finally, linking with the modern misallocation literature (see, e.g. Hsieh and Klenow, 2009; Bartelsman et al., 2013; Garicano et al., 2016; Foster et al., 2001), our results suggest that the strength of the granularity channel in explaining aggregate fluctuations depends to the degree of allocative efficiency in an economy. Our contribution is to show that this mechanism operates not only impacting directly firms size, as in Ifergane (2025), but also distorting the shape of the size-volatility relation.

## 2 Motivating evidence

In this section, we use administrative data on French firms to produce empirical evidences that motivate our theoretical explorations in Section 3. First, we reassess the relationship between size and growth volatility and show that in France it displays significant deviations from the log-log linear decay identified so far in the related literature. Second, we show that the correlation between size and productivity among French firms – found positive but far from 1 – varies within a year across sectors and displays also noticeable heterogeneity in the way it evolves over time again across different sectors.

[9] See Moran et al. (2024) for a similar reasoning. Interestingly they note that also with Compustat data on US publicly traded firms the volatility decay for the largest firms seems to become extremely slow.

[10] See also Moran and Riccaboni (2024), Scharfenaker (2022) and Fontanelli (2024) for recent reviews on these models.

[11] The non-Gaussian nature of the distribution of growth rates of business firm is another notable departure form the Gibrat's law. See Stanley et al. (1996) and Bottazzi and Secchi (2006a). The growth rates distribution is scale-invariant as well, as it is tent shaped (i.e. Laplace) for countries (Lee et al., 1998; Fagiolo et al., 2008) and sectors (Castaldi and Sapio, 2008).

| | | limited liability | simplified | other | micro | total |
|---|---|---|---|---|---|---|
| Legal type | | | | | | |
| | 2000 | $55,930$ | $637$ | $30,645$ | $37,166$ | $124,378$ |
| | 2019 | $47,739$ | $33,477$ | $3,809$ | $10,820$ | $95,845$ |
| | | | average | std.dev | median | n.obs |
| Total sales ($1,000$ euro) | | | | | | |
| | 2000 | | $8,132.5$ | $207,481.0$ | $792.6$ | $87,212$ |
| | 2019 | | $10,737.9$ | $342,786.6$ | $676.6$ | $85,025$ |
| Multi-factor productivity | | | | | | |
| | 2000 | | $26.6$ | $30.8$ | $22.8$ | $87,212$ |
| | 2019 | | $43.21$ | $83.7$ | $34.2$ | $85,025$ |

**Table 1:** Sample composition in terms of legal type (top-panel).

**Data and variables definition**. Our analysis relies on the FICUS/FARE databases produced by the French National Institute of Statistics and Economic Studies (INSEE). These datasets provide annual snapshots of the population of French enterprises (legal units) within the productive system, integrating administrative data with ad-hoc surveys. We focus on firms in the manufacturing sector (SIC codes 10-33) over the period 2000-2019,[12] excluding those engaged in tobacco manufacturing (SIC code 12) and the production of coke and petroleum products (SIC code 19). We also exclude individual companies and companies benefiting from a simplified accounting and tax regime if they remain below certain size thresholds ("Micro-entreprise regime"), since they potentially introduce mechanical distortions in firm size dynamics.[13] Beyond standard limited liability companies, our sample includes simplified joint-stock companies not imposing any minimum capital requirement, corporations and various forms of partnerships we classify as "other". Table 1 (top-panel) reports the composition of our sample in the first and last year in terms of the legal type of the entity. The shift in the composition of legal types mirrors institutional reforms aimed at easing the organization of business activities through more flexible and less burdensome legal vehicles. In particular, the 2008 "Law for the Modernisation of the Economy" was explicitly designed to simplify business creation, remove barriers to corporate forms, and separate simplified individual businesses from formal corporate entities.

We extract data on Total Sales (variable CATOTAL) and deflate it using a sector-specific price index at the industry level to proxy $S_{ijt}$,[14] the size of firm $i$ with main activity in sector $j$ in year $t$. We define the corresponding logarithmic growth rate as $g_{it} = \log S_{it} - \log S_{it-1}$. We then compute a multi-factor productivity index $a_{ijt}$ following the methodology described by Levinsohn and Petrin (2003) with the correction proposed by Ackerberg et al. (2015). We use number of employees (variable EFFSALM), fixed assets (variable IMMOCOR), material costs (variables ACHAMAR, AUTACHA and ACHAMPR), investments (variable INVCORP) and value added (variable VAHT) to proxy for labor, capital and intermediate inputs, investments and output.[15] Finally $\rho_{jt}$ represents the Spearman rank correlation index between size and productivity of

[12]FICUS/FARE data cover the French economy since 1994. However information about the type of legal unit becomes available starting in 2000 and we stop before the Covid-19 pandemic.

[13]These benefits fall under the "micro-entreprise" regime, which has undergone several revisions to its turnover thresholds since its introduction in 2003. Major adjustments occurred in 2009, 2018, 2020, and 2023, raising the threshold for commercial activities from approximately 80K euro to 190K euro. Before 2003 most of these entities were typically organized as individual companies.

[14]All nominal variables are deflated using the SNA A38 industry specific price deflators provided by INSEE, the French National Statistical Office.

[15]Capital is constructed using the perpetual inventory method, where we apply the SNA A38 industry-specific depreciation rates

firms active in sector $j$ in year $t$. Our final sample consists of approximately 1.7 million firm-year observations.

**Size-volatility relation**. To identify the size-volatility relation, we follow a standard approach which we refer to as the "bins methodology". It is a two-step procedure. Within each sector or year, observations are first sorted by size and grouped into bins, each containing an approximately equal number of observations. Next, for each bin, we compute the average size and the standard deviation of growth rates for firms in the bin. The number of bins is chosen ad-hoc to ensure that each one of them includes a sufficient number of observations to achieve sufficient precision in computing average size and volatility and, at the same time, to avoid an over-smoothing of the size-volatility relation. In practice, when we consider data for firms active in the whole manufacturing industry and pooled across years, we use 1000 bins. When we sub-set data by year or sector (2 digit), we use 500 bins.

An alternative approach, known as the "blocks methodology" and used for example in Koren and Tenreyro (2013) and Yeh (2023), identifies the same relation using average sizes and growth volatilities computed for each firm over rolling time windows (blocks). The key difference between the two approaches is that the latter allows for within-firm variation in growth volatilities. This within-firm variation may potentially be relevant for larger and older firms, but less so for smaller and younger ones. However, using blocks instead of bins, mechanically results in the loss of short-lived firms. These are typically small, more volatile, and hence important in shaping non-linearities in the size-volatility relationship. This is the main reason we favor bins over blocks in our investigation. In any case, Yeh (2023) provides indirect evidence that the two procedures do not give highly different results in the identification of the size-volatility relation and that results are not altered when using blocks changing the length of the time-window from 3, to 5 and 10 years. This further reassures in choosing bins over blocks.

Figure 1 reports the size-volatility relation for different years pooling observations across sectors (top panel) and for different sectors pooling observations across years (bottom panel). Both panels provide evidence supporting the idea that larger firms exhibit less volatile growth trajectories than smaller ones. Further this result does not appear to be merely an artifact of aggregation, but is likely to reflect a characteristic of firm growth dynamics. This evidence is perfectly in line with a well-established finding in the literature (see e.g. Stanley et al., 1996; Bottazzi and Secchi, 2006b; Calvino et al., 2018; Yeh, 2023, among many others).[16]

However, further inspection of Figure 1 suggests a new noticeable finding. The decline in volatility does not follow the usual log-log linear pattern across the entire size distribution observed in the literature. Instead, the relationship appears to follow two distinct regimes. For firms below a certain size threshold, that for illustrative purposes in all figures we set to 350K euro (light-blue line), volatility steadily declines with size implying that growth trajectories become significantly more stable as size increases. Above the 350K euro threshold the size-volatility relation flattens and the stabilizing effect of growing larger seems to diminish significantly. This is confirmed with simple log-log linear regressions fitted on firms below and above the threshold. For the former the average slope (dark-violet dashed line) is found to be approximately $-0.53$ and $-0.49$ across years (top panel) and across sectors (bottom panel) respectively. For the latter the two averages

provided by INSEE, the French National Statistical Office.

[16]Appendix A shows that our result remains qualitatively the same if we replace Total Sales with Value Added as proxy of size (Appendix A.1) and the standard deviation with the mean absolute deviation as proxy of volatility (Appendix A.2). Further it provides evidences in support of our decision of removing companies under the simplified "Micro-entreprise" regime (Appendix A.3).

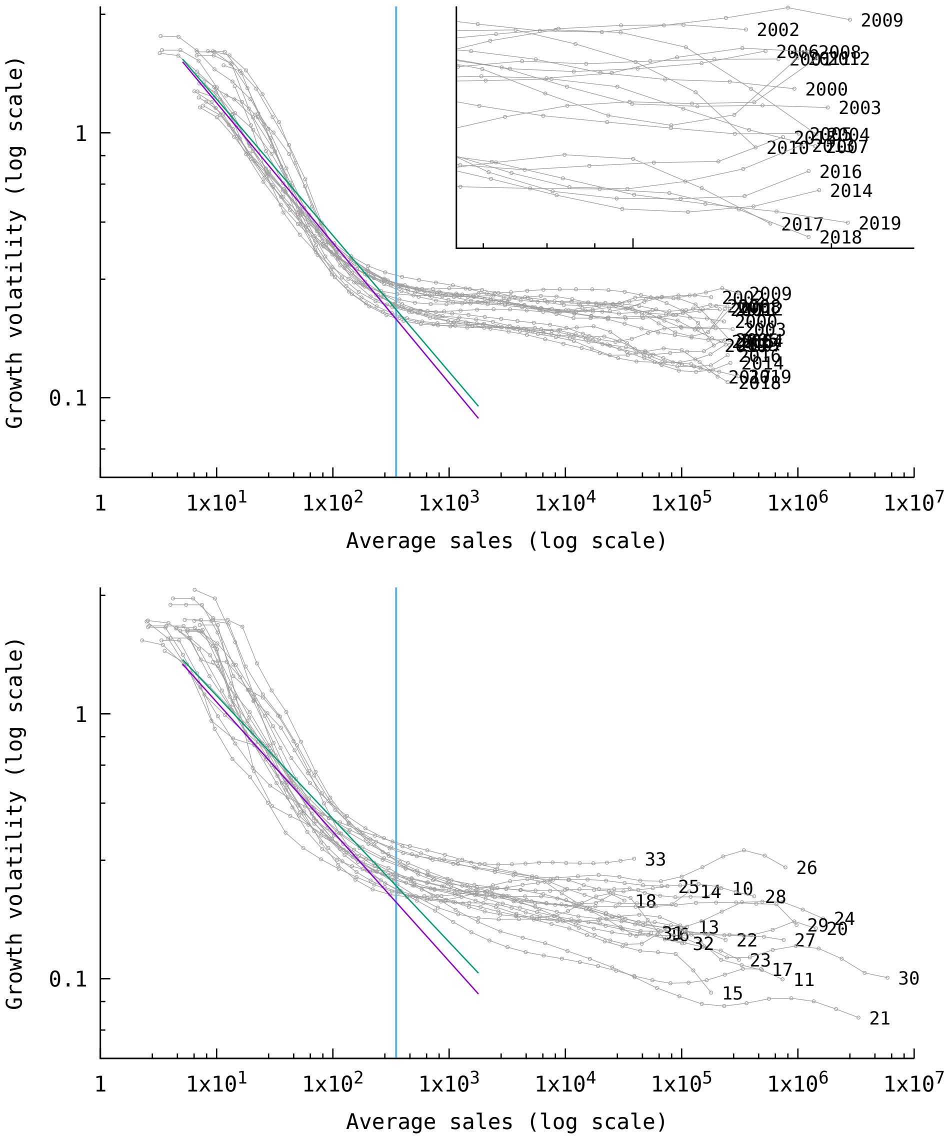


**Figure 1:** Kernel regression estimate of the size-volatility relationship across years (top panel) and industrial sectors (bottom panel). The inset in the top panel zooms in the scaling of variance for largest firms. Kernel regressions are computed on a 32 points regular grid using an Epanenchnikov kernel and a rule-of-thumb bandwidth selection procedure. Binned relations are built including in the sample non individual firms active in the manufacturing industry in the time window 2000-2019. We report also a linear fit estimated on bins recording sales lower or equal than $350K$ (light-blue solid line) euro. The average of the estimated slopes (dark-violet solid line) is $-0.53$ and $-0.49$ for years and sectors respectively while the estimated slopes (dark-green solid line) pooling observations are $-0.51$ and $-0.52$ respectively.

drops to $-0.05$ and $-0.08$.[17]

While these estimates are provided for illustrative purposes only and should be interpreted with caution,[18] they are nonetheless informative about three aspects. First, the growth dynamics of both small and large firms violate Gibrat's law postulating that firm growth is independent of size. Second, this violation is more flagrant for small firms yielding slope estimates clustering around $-0.5$,[19] a value clearly outside the range $[-0.24, -0.16]$ covering values found in the literature. This indicates that small firms benefit from near-perfect statistical diversification, behaving like aggregations of small sub-units subject to idiosyncratic shocks (Moran et al., 2024). Large firms, on the contrary, show limits in their capabilities of exploiting the benefits of diversification when they grow larger. This new feature in the size-volatility relation may affect the quantitative relevance of the granularity channel. Since the granular hypothesis is an asymptotic result, it is valid in the limit of large firms. For this reason, what matters is not the size-volatility decay per se, but the regime in action for the largest firms only. This regime has just been shown to be almost flat for large sizes in the French manufacturing sectors, limiting the relevance of concerns like those raised in Yeh (2023). Interestingly, Figure 1 shows that the way this new pattern in the size-volatility relation unfolds differs across year (cfr. the inset in the top-panel) and even more across industrial sectors, a fact which is at the basis of our third contribution.

**Size-productivity correlation**. Next we compute the Spearman rank correlation between size and productivity, denote it with $\rho_{j,t}$, and look at its empirical distributions across industrial sectors for the usual time span 2000-2019.[20] Figure 2 (top-panel) reports candlestick summaries of these distributions with boxes and whiskers representing inter-quartile and min-max ranges respectively, and the black line-points the corresponding medians. The same figure (bottom-panel) displays the time evolution of $\rho_{j,t}$ for each individual sector. Visual inspection of these figures suggests three main facts.

First, size-productivity correlations are generally positive, but much smaller than 1. The grand average (across sectors and years) is about 0.17, with an average (across years) standard deviation (across sectors) of about 0.09. This correlation between firm size and productivity has recently been emphasized as an indicator of the degree of resource misallocation in the market (see Bartelsman et al., 2013). More specifically, if resources were allocated entirely at random, the correlation between size and productivity would be zero. At the other extreme, a perfectly efficient allocation would yield a perfect positive correlation between the two variables. A positive correlation coefficient below one – such as the one observed in our data – indicates that resource allocation tends to favor more productive firms, but the process remains imperfect and affected by various distortions (see e.g. Hsieh and Klenow, 2009; Bartelsman et al., 2013; Garicano et al., 2016). The same literature has shown that the size–productivity correlation is generally lower, and thus misallocation more severe, in European countries (including France) compared to the United States.[21]

[17] The same figure reports, for comparison, fitted lines estimated pooling observations across years and across sectors (dark-green solid line). They look very similar.

[18] It is clear for example that they mechanically depend on the exact value of the size threshold.

[19] Coefficients of variation for these estimates suggest the existence of more than moderate variation around this mean.

[20] We use the Spearman rank correlation to overcome the difficulty of defining the Pearson correlation index in case size has an infinite variance. Nevertheless, all the results we present in this section are unaltered when we employ the Pearson correlation index instead.

[21] Bartelsman et al. (2013) employ an Olley-Pakes decomposition of productivity and find that the size-productivity covariance term of the decomposition explains a smaller portion of productivity in European countries than in United States. The result that the size-productivity covariance explains a small share of overall productivity has also been confirmed by other studies focusing on

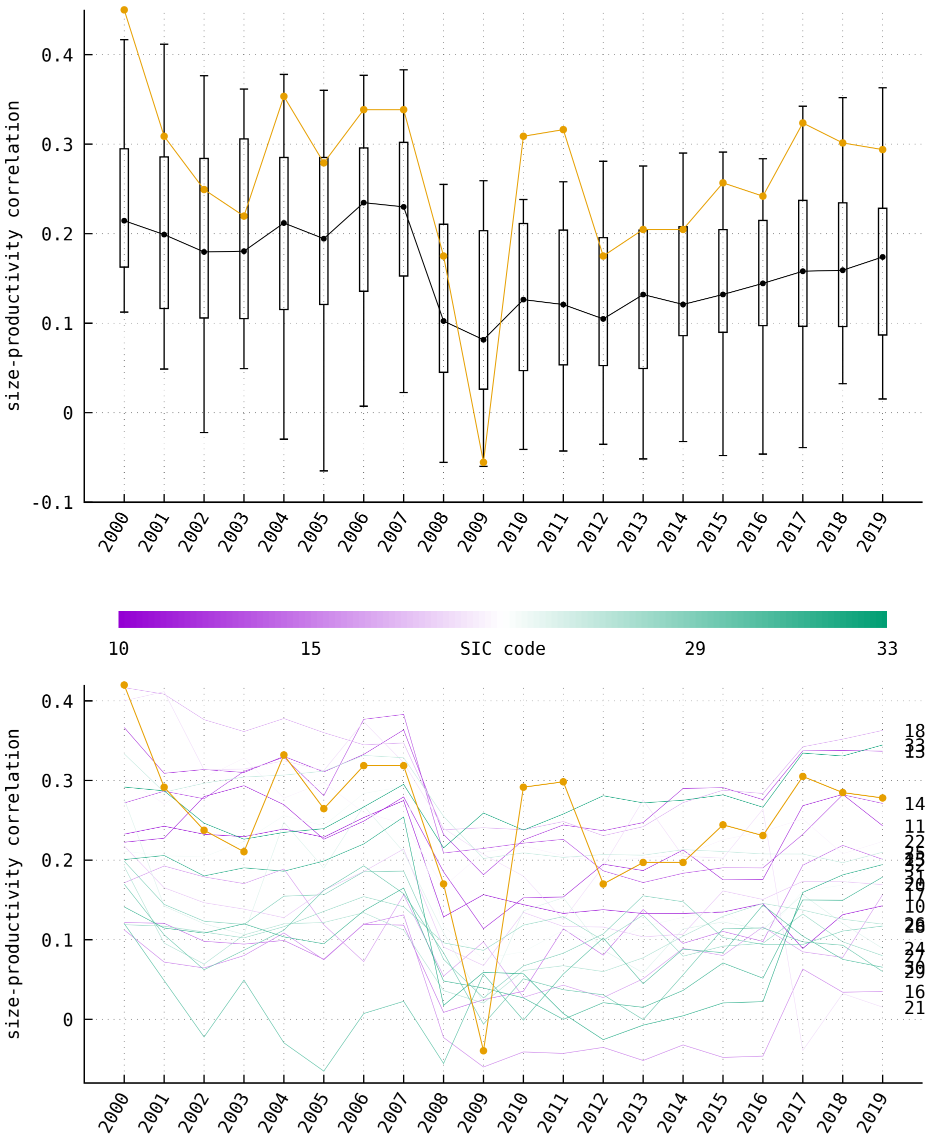


**Figure 2:** Top-panel shows candlesticks of the distribution of size-productivity correlations across industrial sectors in different years where boxes and whiskers represent inter-quartile and min-max range respectively. Black line-points present the time evolution of the median. Bottom-panel shows heterogeneous time variation across sectors of $\rho_{j,t}$ for each industrial sector. The color scheme assigns 2-digit sectors, ranging from 10 to 33, to a palette that transitions from dark-violet to dark-green. In both panels gold line-points represent the time evolution of the French GDP growth on a different y-scale.

Second, again in the top-panel, we observe an apparent time variation in the (median) degree of misallocation, interestingly with the lowest median correlations observed in in 2008 and 2009, the years of the Great Recession. To dig deeper in this direction, the same figure reports the time evolution of the French GDP growth rate (gold line-points) displaying a comovement with $\rho_{j,t}$. The correlation between the two is found to be about $0.72$. This evidence is consistent with a growing body of research showing that allocative efficiency varies over the course of the business cycle. Foster et al. (2016) find that allocative efficiency in the U.S. improved during recessions prior to 2008, consistent with the cleansing effect hypothesis,[22] but also that this pattern was reversed during the Great Recession. Bartelsman et al. (2019) report similar findings for a sample of nine European countries (including France), where the Great Recession proved far less productivity-enhancing than previous downturns. A comparable decline in allocative efficiency during the Great Recession is also documented for France and Germany by Libert (2017), Domini and Moschella (2022) and Grebel et al. (2023).

Third, the degree of misallocation and the way it comoves with GDP growth vary across sectors. Within each year, we observe a non-negligible variation in size-productivity correlations across sectors with an average min-max range of $0.34$ associated with a coefficient of variation larger than $10$. Looking at the time evolution of $\rho_{j,t}$ for each individual sector uncovers significant heterogeneity in their trajectories. Correlations with the French GDP growth are always positive with one exception (Rubber and plastic products) but range from $-0.04$ to $0.89$ with an average across sectors of $0.51$. This significant heterogeneity in how market reallocation responds to macroeconomic fluctuations is again consistent with the existing literature. In particular, Bartelsman et al. (2019) show that country-industry pairs more exposed to trade experienced a marked decline in allocative efficiency during the Great Recession. Likewise, Grebel et al. (2023) find that small firms – typically more vulnerable to credit constraints – suffered a larger post-recession drop in allocative efficiency compared to medium and large firms, especially in France. Firms with higher export intensity were also disproportionately affected, losing market share to firms with intermediate export intensity.

Summing up, our investigations suggest three main results that will guide our theoretical explorations in the next section. First, in French manufacturing sectors we observe a size-volatility decay that is not log-log linear over the entire size distribution, it is steeper for small firms and flattens for large ones. Second, the correlation between size and productivity is on average positive, but far from being perfect. Third, these two statistical properties display a noticeable degree of heterogeneity across sectors and over time.

## 3 A model for the size-volatility relation

In this section we introduce and analyze a simple model that accounts for deviations from the log-log linear size-volatility relation by linking them to the degree of correlation between firms size and productivity. We follow the same approach used in Sutton (2001), Klepper and Thompson (2006) and Bottazzi and Secchi (2006a), developing an "islands model" of industry competition that highlights structural drivers of the growth rate

France and employing data similar to ours (see e.g. Grebel et al., 2023).

[22]The "cleansing effect" hypothesis of recession (Caballero and Hammour, 1994, 1996) posits that downturns can enhance productivity by reallocating resources away from less productive firms toward more efficient ones.

of firms and of the size-volatility relation.[23] Together with the key role of the degree of correlation between size and productivity, our model puts emphasis on the number of sub-industries a firm is active in and on the role of market shares reallocation.

**Setup**

Let us consider an industry populated by $N$ firms indexed by $i$. Firms are heterogeneous in both size $S_{i,t}$ and productivity $a_i$. This notation reflects the fact that our model describes the dynamics of a firm's size from $t$ to $t+1$ while holding productivity constant. At the beginning of the period, each firm's size and productivity are drawn from two generic probability distributions, whose cumulative distribution functions (CDFs) are denoted by $F_s$ and $F_a$, respectively.[24] Following the logic of island models, we assume that at the beginning of the period the industry has a total size $M_t$, divided into a number of independent sub-industries of size $z > 0$.[25] Each sub-industry belongs to one and only one firm such that a firm's size is proportional to the number of sub-industries in which it is active. Larger firms can thus be rationalized as those operating in more sub-industries in line with existing empirical evidence (Hymer and Pashigian, 1962; Boeri, 1989; Sutton, 2001; Bottazzi and Secchi, 2006b; Crouzet and Mehrotra, 2020).

While firms face no competition within their own sub-industry, they compete for the allocation of sub-industries, a process that drives changes in firm size. We model this process through "business opportunities", defined as chances to compete for and capture shares of the overall industry. At time $t$, each firm is assigned a number of business opportunities. These opportunities are all risky in that they may materialize as either positive or negative growth shocks. As a result, firms may experience no change, an increase, or a decrease in the number of sub-industries in which they are active. Accordingly, the size of the entire industry may expand, stay the same or contract depending on the sign of the correlation between size and productivity. This competitive process captures three distinct dynamics: the assignment of new sub-industries to incumbent firms, the loss of existing sub-industries by incumbents, and the reallocation of existing sub-industries across existing firms.[26] The exact characterization of a firm's growth rate $g_{i,t}$ requires to specify (i) how risky business opportunities are allocated to firms and (ii) how these market opportunities are transformed into positive *vs.* negative growth shocks. We explain how we model these two mechanisms here below.

**Market opportunity assignment (hp1)** - A given number of business opportunities $H$ is allocated to the $N$ firms in the assignment phase.[27] The positive parameter $k$ captures the intensity of the allocation and reallocation processes and it measures the degree of industry turbulence. When $k$ is 0, there is no turbulence at all,[28] when $k = 1$ the equivalent of the whole industry size $M_t$ is converted in new business opportunities. More in general the larger $k$ the more intense are the allocation and reallocation processes. The number of market opportunities assigned to firm $i$ depends on its market share $s_{i,t} = \frac{S_{i,t}}{M_t}$ and it is denoted as $h_i(H, s_{i,t})$.

[23] Islands models belong to a long tradition of works (among a non- exhaustive list let us cite Ijiri and Simon, 1967; Stanley et al., 1996; Bottazzi and Secchi, 2006b,a; Fontanelli et al., 2023) that have developed stochastic models of industry evolution.

[24] Along the paper we consider the minimal requirement that both posses a finite first moment.

[25] The assumption that sub-industries are independent is in line with empirical evidence showing weak correlation among sub-industries (see e.g. Bottazzi et al., 2001; Sutton, 2002).

[26] Unlike in Sutton (2001) and Klepper and Thompson (2006), in our model we consider only incumbent firms, the model does not allow for entry.

[27] To simplify the notation, we omit the subscript $t$ from $H$ and $N$ since they are constant in our framework.

[28] In this case there is no firms size dynamics.

We assume that the assignment of each market opportunity follows a Bernoulli trial with probability equal to $s_{i,t}$ and, accordingly, the number of new opportunities assigned to a firm, $h_i(H, s_{i,t})$, follows a Binomial distribution with parameters $H$, and $s_{i,t}$ respectively. This assignment mechanism models the idea that larger firms, operating in a higher number of sub-industries compared to small firms, are more exposed to competition

Market opportunities assigned to a firm can result in either positive or negative shocks. While we do not explicitly distinguish between entry into a new sub-industry and the reallocation of an existing one, positive shocks model scenarios in which an incumbent firm enters a new sub-industry or acquires a sub-industry previously occupied by another firm. Negative shocks, by contrast, capture the loss of a sub-industry by an incumbent. We model this transformation as follows.

**Market opportunity realization (hp2)** - Once market opportunities have been assigned to firms, each of them becomes a micro-shock $\epsilon_i$ in the realization phase on the basis of firms' relative productivities, as determined by $F_a$. The higher (respectively, lower) it is the position of a firm in the productivity ladder, the higher it is the probability for a market opportunity to become a positive (respectively, negative) micro-shock. Formally, the micro-shock $\epsilon_h = \epsilon(F_a)$ is described as a Bernoulli trial taking positive value $z$ with probability equal to $F_a$ and negative value $-z$ with the residual probability equal to $1 - F_a$. This realization mechanism simply assumes that the probability of a positive or negative growth shock from a market opportunity is proportional to firms' relative productivities. Accordingly, more productive firms will expand on average, while less productive firms will shrink, in line with a large strand of previous models where productivity determines firm competiteveness and the growth chances of a firm (see e.g. Jovanovic, 1982; Hopenhayn, 1992; Melitz, 2003; Dosi et al., 2017; Fontanelli et al., 2023). In this set-up a firm's growth rate and its conditional variance can be characterized as follows.[29]

**The firm growth rate function** - The two mechanisms of market opportunity assignment and shock realization return a firm growth rate that is equal to the sum of a random number $h_i(H, s_{i,t})$ of micro-shocks. The firm growth rate $g_i$ in the model is thus determined as follows:

$$g_i := \frac{S_{i,t+1} - S_{i,t}}{S_{i,t}} = \frac{\sum_{h=0}^{h_i(H,s_{i,t})} \epsilon_h(F_a(a_i))}{S_{i,t}} \quad . \tag{1}$$

This growth rate function implies that a firm growth rate $g_i$ in our model is a function of firm market share $s_{i,t}$ and of firm productivity $a_i$.

**Conditional growth rate variance** - The variance of a firm growth rate conditional on firm size can be written as

$$V(g_i|S_{i,t}) = \frac{1}{S_{i,t}^2} V\left(\sum_{h=0}^{h(H,s_{i,t})} \epsilon(F_a(a_i))\Big| S_{i,t}\right) \quad , \tag{2}$$

which indicates that, conditional on size, the volatility of the firm growth rate depends the assignment of the $H$ market opportunities via the function $h(H, s)$, and on the productivity distribution $F_a$, which accounts for

[29]The market reallocation induced by the assignment and shock realization mechanisms in our model produces positive (negative) growth on average when the size-productivity correlation is positive (negative). When the correlation is null instead, aggregate industry growth is zero. Moreover, the results of the model are preserved when the sign of the micro-shock is swapped (i.e. the micro-shock is equal to -$\epsilon(F_a)$). In this case, the model generates negative growth in presence of a positive size-productivity correlation as larger and more productive firms are the ones accounting for largest declines in size.

heterogeneity in productivity levels $a_i$ and determines the type of firm growth shocks via the function $\epsilon(F_a)$.

In our model, the mechanism driving a firm's growth rate depends on both its size and productivity. Larger (smaller) firms, on average, capture a greater (smaller) number of market opportunities during the assignment phase. These opportunities translate into positive or negative growth rates depending on the firm's relative productivity. As a result, the relationship between size and productivity shapes the properties of growth rate volatility in Equation (2), including its dependence on size. In the remainder of this section, we examine this dependence under two alternative scenarios.

**Scenario I (perfect correlation)**

**Analytical result**. We begin with a preliminary result that considers a scenario in which firms' productivity $a_i$ is perfectly correlated with size $S_{i,t}$, that is that is $\rho = 1$ with $\rho = \text{corr}(a_i, S_{i,t})$ and we state the following proposition.[30]

**Proposition 1.** *Let $H$ denote the number of market opportunities available in an industry of size $M_t$. Suppose that business opportunities are assigned and realized according to hypotheses (hp1) and (hp2), and assume that firm productivity $a_i$ is perfectly correlated with size $S_{i,t}$, i.e. $\rho = corr(a_i, S_{i,t}) = 1$. Then, in the limit as $M_t \to \infty$ and $H \to \infty$ such that $\frac{H}{M_t} = k < \infty$, the standard deviation of firm growth rates, $\sqrt{V(g_i \mid S_{i,t})}$, decays as a log-linear function of $S_{i,t}$ with a constant slope equal to $-0.5$.*

*Proof.* *See Appendix B.* □

Proposition 1 establishes two results. First, when size and productivity are perfectly correlated, the size-volatility relationship is necessarily log-log linear. Second, the slope of this log-log linear decay is equal to $-0.5$. The proof applies the law of total variance involving multiple random variables (Bowsher and Swain, 2012), showing that $V(g_i \mid S_{i,t})$ can be decomposed into three components: (i) a term capturing the contribution of the assignment of market opportunities to firms $h(H, s_{i,t})$, (ii) a component accounting for heterogeneity in firm productivity $F_a$ and proportional to $V[F_a|S_{i,t}]$ conditional on size, and (iii) a residual term. Under perfect correlation, the second component vanishes, as productivity is entirely determined by size. The remaining terms are shown to scale as $S_{i,t}^{-1}$.[31]

The intuition behind Proposition 1 is straightforward. Small firms face a lower probability of being assigned business opportunities and thus of competing for sub-industries. Further, when size and productivity are perfectly correlated, small firms face also a higher likelihood of encountering business opportunities that translate into negative growth shocks. For large firms it is the opposite. They are more likely assigned business opportunities that translate into positive growth shocks with higher probability. In any case in this scenario, differences in growth rate volatility across firms are driven solely by firm size and more specifically by the number of sub-industries in which a firm operates. When the number of business opportunities becomes large, firm growth benefits from near-perfect statistical diversification: large firms effectively behave

[30] Throughout this section, we consider the case of a positive correlation between size and productivity. However, in the proofs of Propositions 1 and 2, we clarify that the results also hold under a negative correlation.

[31] Note that this proposition could be modified to generate a log-log linear size-volatility relationship with a slower decay, more in line with empirical evidence, by introducing highly heterogeneous sub-industry sizes, as in Moran et al. (2024). We set aside this extension for the sake of clarity, given the main focus of the present paper.

as aggregations of many small sub-units exposed to idiosyncratic growth shocks. In line with the Central Limit Theorem, this yields a decay slope of $-0.5$.

**Numerical analysis**. Although Proposition 1 formally proves that the volatility of growth rates decays log linearly with size at a rate -0.5 under perfect correlation between productivity and size, we complement this analytical result with numerical simulations.[32] The goal is to assess how closely finite-sample industries reproduce the theoretical limit. This exercise allows us to verify that the stylized result remains robust under realistic levels of heterogeneity in firm size and to quantify the speed at which convergence to the limit occurs.

To implement the simulation under the assumptions of Proposition 1, we construct artificial industries populated by $N$ firms whose sizes are independently drawn from a Pareto distribution with shape parameter equal to $\alpha$. In this scenario a firm's productivity is set equal to its size, imposing perfect correlation of the corresponding ranks. This guarantees that larger firms have both a higher probability of getting business opportunities, and that these turn in positive, rather than negative, growth shocks. The total number of available business opportunities $H$ is set proportional to industry size $M = \sum_i S_i$ with $k = H/M$ representing the industry turbulence parameter.

In each of the Monte Carlo iterations we simulate the growth process as follows. First, each firm draws a number of business opportunities $h_i$ from a Binomial distribution with success probability equal to its relative size $S_i/M$. Second, each opportunity results in a positive or negative growth shock of size $z$ based on the firm's relative productivity rank as captured by the corresponding cumulative distribution function. The resulting growth rate is computed as the sum of the shocks divided by firm size as in Equation (1). For each firm we compute $R$ different growth rates and their standard deviation is the proxy of growth volatility $\sqrt{V[g_i|S_i]}$. We then estimate the logarithmic size-volatility relation using both linear (OLS estimator) and non-parametric regressions (local-linear kernel estimator). This procedure is detailed in Algorithm 1 in Appendix C.

The simulation results reported in Figure 3 confirm that the theoretical benchmark derived in Proposition 1 holds robustly in finite samples. Across all simulated $N \in (50, 100, 250, 500)$, we recover a clear log-linear relationship between firm size and growth rate volatility, with estimated slopes very close to the theoretical benchmark of $-0.5$. While smaller industries exhibit slightly more dispersion around the fitted line, the average decay rate remains remarkably stable, indicating that even relatively sparse economies approximate well the limiting behavior predicted by the theory.

Figure 4 displays also the convergence of the estimated slope in the size-volatility relation to the theoretical benchmark of $-0.5$ as the number of firms active in the industry increases, under three different values of the Pareto tail exponent $\alpha$ $(1.75, 1.5, 1.25)$. As expected, convergence improves with larger $N$, but the speed and precision of convergence depend on the heaviness of the tail. For $\alpha = 1.75$, slope estimates are already very close to the theoretical value for moderate values of $N$,[33] and the confidence intervals shrink quickly. As $\alpha$ decreases, indicating a fatter tail in the size distribution, the slope converges more slowly and with greater variability across Monte Carlo simulations. This effect arises from the mechanics of our stochastic model in the presence of an abnormally large firm. Indeed, the presence of a very large firm implies, ceteris paribus, a large and highly concentrated industry with a large number of available business opportunities. The vast majority of these opportunities are allocated to the largest firm, while smaller firms receive either

[32] To simplify the notation, we omit in this sub-section the index $t$.

[33] Recall that, in our set-up, industry size $M$ is proportional to $N$. So the higher the latter, the larger the industry size.

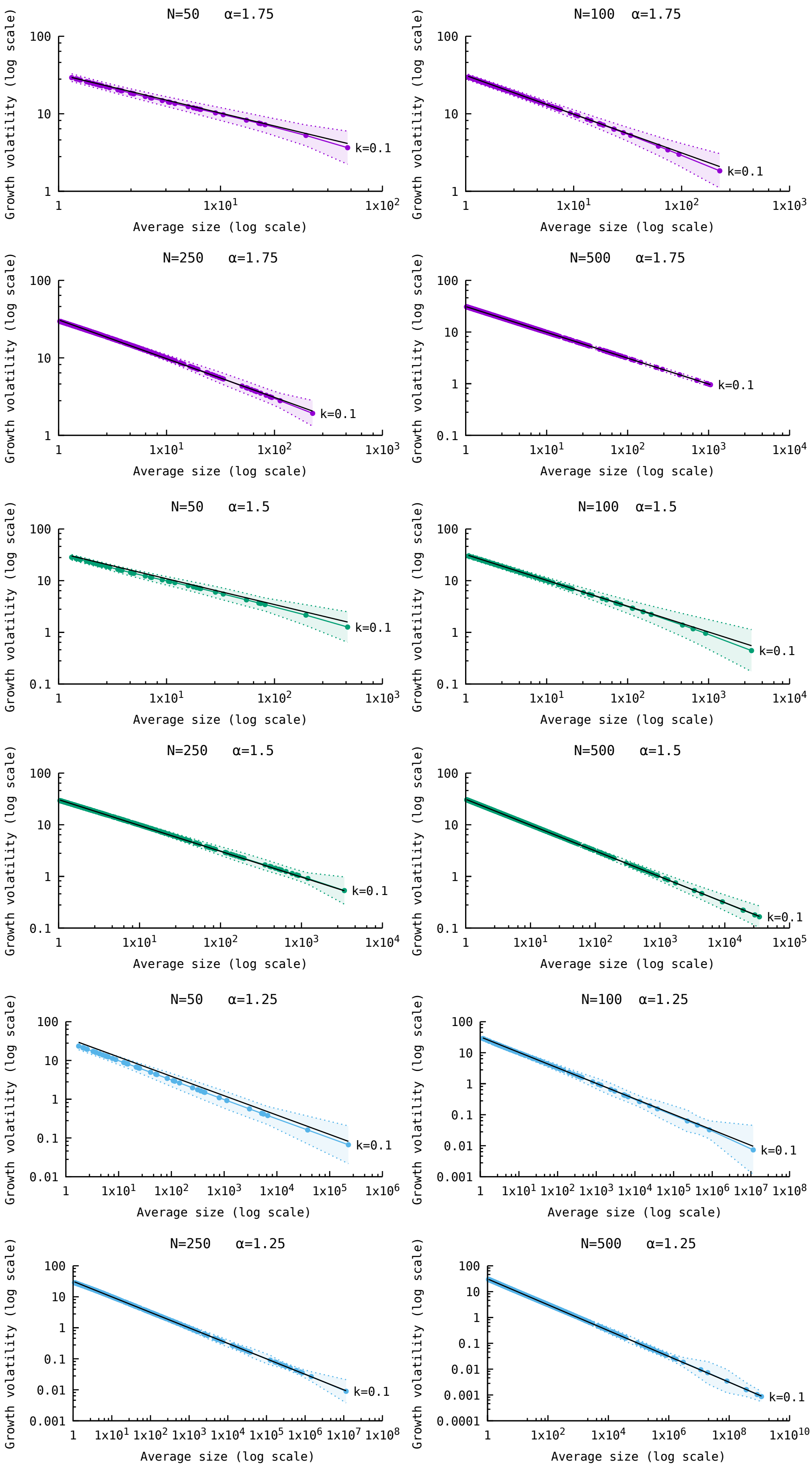


**Figure 3:** Local-linear kernel estimates of the size-volatility relation for industries populated with a different number of firms ($N$) whose size is drawn from Pareto distributions with diverse shape parameter together with a 99% asymptotic confidence band ($\alpha = 1.75$ dark-violet points, $\alpha = 1.5$ dark-green points and $\alpha = 1.25$ light-blue points) and the corresponding $-0.5$ benchmark (solid black line). Estimates are obtained using the whole sample of artificial data generated in one single Monte Carlo iteration and is evaluated for the plot trimming the bottom and top 1%. Bandwidth is least-squares cross-validated. Industry turbulence $k$ is set to 0.1 in all panels.

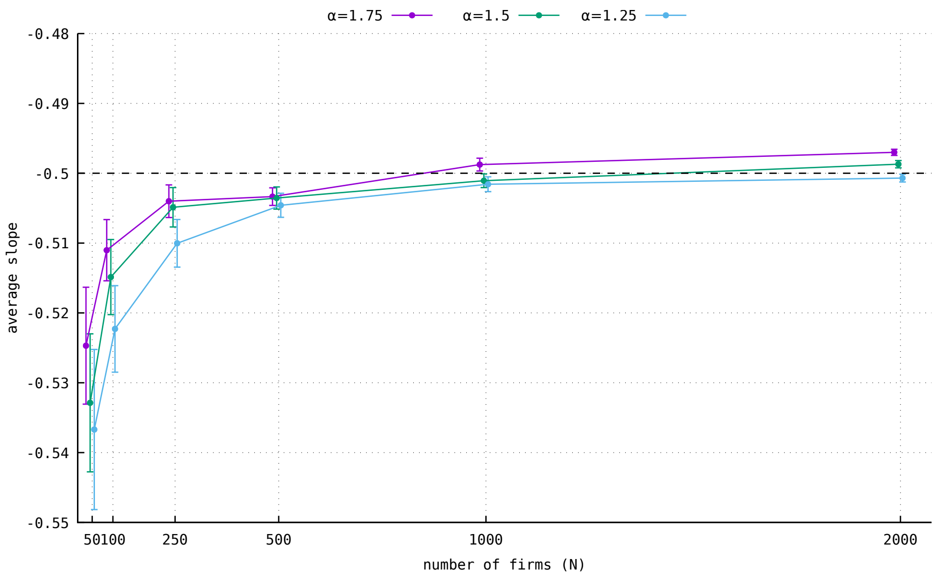


**Figure 4:** Average slope of the size-volatility relation across 100 Monte Carlo iterations for industries populated with a different number of firms ($N$) whose size is drawn from Pareto distributions with diverse shape parameter together with a 99% confidence band. Estimates are obtained using the whole sample of artificial data generated in one single Monte Carlo iteration and is evaluated for the plot trimming the bottom and top 1%. Bandwidth is least-squares cross-validated. Industry turbulence $k$ is set to 0.1.

none or only a few over different Monte Carlo iterations. When productivity is perfectly correlated with size, this leads to a polarized structure: small firms tend to have similar sizes and, on average, alternate between zero and a low number of business opportunities, resulting in relatively high growth volatility; by contrast, the largest firm always absorbs a disproportionate share of opportunities, thereby exhibiting extremely low volatility. This imbalance disrupts the smooth size-volatility relation, occasionally producing abnormally low slope coefficients. In one such case, the Hirschamn-Herfindal index reaches 0.98, with the largest firm accounting for 99% of the entire industry. This firm receives on average more than 98% of the business opportunities, leading to a growth volatility roughly three orders of magnitude smaller than that of the smaller firms.[34] In all cases, however, convergence looks fairly monotonic and stabilizes near $-0.5$ for $N \geq 500$.

### Scenario II (imperfect correlation)

**Analytical result**. We now consider the case where firm size and productivity are positively but imperfectly correlated, that is, $0 \leq \rho \leq 1$, with $\rho = \text{corr}(a_i, S_{i,t})$. In this context, we state the following proposition.

**Proposition 2.** *Let $H$ denote the number of market opportunities and $M_t$ the total market size, with $\lim_{M_t, H \to \infty} H/M_t = k < \infty$. Suppose that market opportunities are assigned and realized according to assumptions (hp1) and (hp2), and that firm productivity $a_i$ is imperfectly correlated with firm size $S_{i,t}$. Then, in the limit as $M_t, H \to \infty$, the standard deviation of firm growth rates, $\sqrt{V(g_i \mid S_{i,t})}$, exhibits a non-linear decay with firm size: for small firms it declines approximately as a power law with exponent $-0.5$, whereas for large firms it flattens and converges to a horizontal asymptote. An increase in industry turbulence $k$ shifts upwards the size-volatility schedule and reduces the volatility gap between small and large firms.*

[34] This case occurs with $N = 50$, $\alpha = 1.75$, and $k = 0.1$ when the random seed is set to 46. Out of $28,094$ business opportunities, this firm receives on average about $28,064$. The growth volatility of the largest firm is approximately $0.001$, while the average for the 47 smallest firms is $20.42$, with a coefficient of variation as low as $2.7$.

*Proof. See Appendix B.* □

Proposition 2 delivers two results. First, when size and productivity are not perfectly correlated, the size-volatility relationship no longer follows a purely log-linear decay. Instead, it exhibits two distinct regimes: for small firms, volatility declines approximately log-linearly with size, with a slope of $-0.5$; for larger firms, the decline attenuates and eventually flattens, converging to a horizontal asymptote. Second, holding all else constant, an increase in the intensity of industry turbulence ($k$) shifts the entire size-volatility relation upward and simultaneously reduces the volatility gap between small and large firms. The proof follows the same strategy used in Proposition 1, relying on the law of total variance. However, in this scenario, the second term in the variance decomposition of $V(g_i \mid S_{i,t})$ no longer vanishes, thereby introducing a strictly positive lower bound to firm-level volatility that prevents further decay for sufficiently large firms.

The intuition behind Proposition 2 departs from that of Proposition 1 in an important aspect. In both scenarios, the assignment mechanism implies that small firms have a lower probability of receiving market opportunities than large firms, which tend to capture the bulk of available opportunities. This alone explains the higher volatility of smaller firms, due to limited diversification and infrequent participation in market reallocation. However, once the assumption of perfect correlation between size and productivity is relaxed, a key mechanism is altered: large size no longer guarantees high productivity. Large firms may still face probabilities of experiencing positive or negative shocks that are not substantially different from those of smaller firms, thereby limiting their ability to stabilize growth through scale alone. This residual volatility undermines the convergence toward vanishing variance and results in a flattening of the size-volatility relation at the upper end of the size distribution.

An increase in the intensity of turbulence $k$ shifts the entire size-volatility relation upward by increasing the number of reallocative shocks per unit of market size. As more market opportunities are distributed across firms, all firms become more exposed to idiosyncratic fluctuations in their market positions. This effect is particularly relevant for large firms: under imperfect correlation between size and productivity, their increased exposure is not matched by a corresponding rise in their ability to convert opportunities into positive growth. As a consequence, volatility remains elevated even at high sizes, and the flattening of the size-volatility relation occurs more rapidly. The net result is a narrower volatility gap between small and large firms and a more turbulent growth environment across the entire size spectrum.

**Numerical analysis**. We now turn to a numerical analysis of the model under imperfect correlation between size and productivity to assess how the theoretical predictions of Proposition 2 manifest in finite samples. We adopt the same simulation framework detailed in Algorithm 1, maintaining the distributional assumptions for firm size and the assignment-realization mechanisms for market opportunities. The only additional element we specify is the procedure used to generate firm-level productivity with a target correlation $0 < \rho < 1$ with firm size, which governs the degree of alignment between the size and productivity rankings in the simulation. To introduce partial randomness in the allocation of ranks while preserving the overall structure, we implement a partial rank perturbation procedure. Starting from the baseline rank order of firm sizes, a fixed fraction of firms (denoted as "perturbation fraction", $pf$) is randomly selected, and the ranks within this subset are randomly reshuffled. The remaining firms retain their original ranks. This method preserves the marginal distribution of ranks, but disrupts their precise alignment with firm size, allowing for controlled deviations from perfect rank-size correlation. This procedure is detailed in Algorithm 2 in Appendix C.

The simulation results reported in Figure 5 confirm that the theoretical benchmark derived in Proposition 2 holds robustly in finite samples. For all simulated $N \in (50, 100, 250, 500)$, and with a perturbation ratio $pf$ of $0.75$,[35] we observe clear deviations from the log-log linear relationship between firm size and growth rate volatility. To help in visualizing these deviations, we report, for $k = 0.1$ only, the log-log linear benchmark with slope $-0.5$ (black solid line). Figure 5 show that these deviations depend critically on the tail parameter $\alpha$ of the firm size distribution. In particular, fatter tails (i.e., lower values of $\alpha$) magnify the departure from linearity, making the two-regime structure more apparent: a sharp decline in volatility for small firms, followed by a clear flattening for larger ones.

To provide a quantitative assessment of these differences, Figure 6 (top panel) reports, for this scenario as well, the convergence of the average estimated slope as the number of firms in the industry increases, under three different values of the Pareto tail exponent. We begin by computing averages using the entire samples generated in each Monte Carlo simulation. Although fitting a linear model to data generated by an explicitly non-linear process may appear unconventional, we justify this approach on the grounds that the linear approximation offers a succinct and interpretable summary of deviations from the log-log linear benchmark.[36] As the number of firms $N$ increases, the average slope converges toward asymptotic values of approximately $-0.3$, $-0.2$, and $-0.1$ for $\alpha$ equal $1.75$, $1.5$, and $1.25$, respectively (dashed lines). This behavior confirms that sharper differences between volatility regimes for small and large firms emerge when the firm size distribution becomes more fat-tailed and market concentration is higher.[37] We then recompute average slopes using the same Monte Carlo samples this time trimmed to include only firm sizes between the $5th$ and $30th$ percentiles, with the aim of characterizing the size-volatility decay among smaller firms.[38] In line with Proposition 2, we find that these restricted-sample slopes converge toward $-0.5$ as $N$ increases, regardless of the value of $\alpha$.

Next, we explore the effect of breaking the perfect correlation between size and productivity on the size-volatility relation. In the bottom panel of Figure 6 we manipulate the perturbation fraction $pf$ to reduce the correlation from 1 to $0.9$, $0.75$ and $0.5$ for the industries displayed in the two top-panels of the same figure. We show that even a relatively minor decrease in correlation induces an apparent deviation of the size-volatility from the log-log linear benchmark identified in Proposition 1 with a lower bound for the volatility that scales-up with the perturbation factor. This evidence is once again in line with our theoretical predictions and provides a justification to use our strategy to measure the extent to which the size-volatility relation deviates from the log-log linear benchmark also when we manipulate the size-productivity correlation.

Finally, comparing the curves in the top panels of Figure 5, we observe that increasing the intensity of industry turbulence $k$ from $0.1$ to $0.2$ and $0.5$ systematically shifts the size-volatility relation upward and compresses the volatility gap between small and large firms, in line with Proposition 2. A higher intensity of turbulence implies a higher horizontal asymptote for the growth volatility of large firms.[39]

[35]The perturbation ratio is approximately equal to one minus the correlation between size and productivity. Setting $pf = 0.75$ implies a rank correlation of about $0.25$, a value not far to what found for EU countries.

[36]When we take the model to data in Section 4, we exploit this idea.

[37]In these simulations, the average Herfindahl-Hirschman index across Monte Carlo iterations is $0.43$, $0.61$, and $0.83$ for $\alpha$ equal $1.75$, $1.5$, and $1.25$, respectively.

[38]The choice of the 5th and 30th percentiles, while ad hoc, is consistent with the illustrative purpose of the exercise.

[39]In the proof of Proposition 2 we show that the asymptote is proportional to $k^2$.

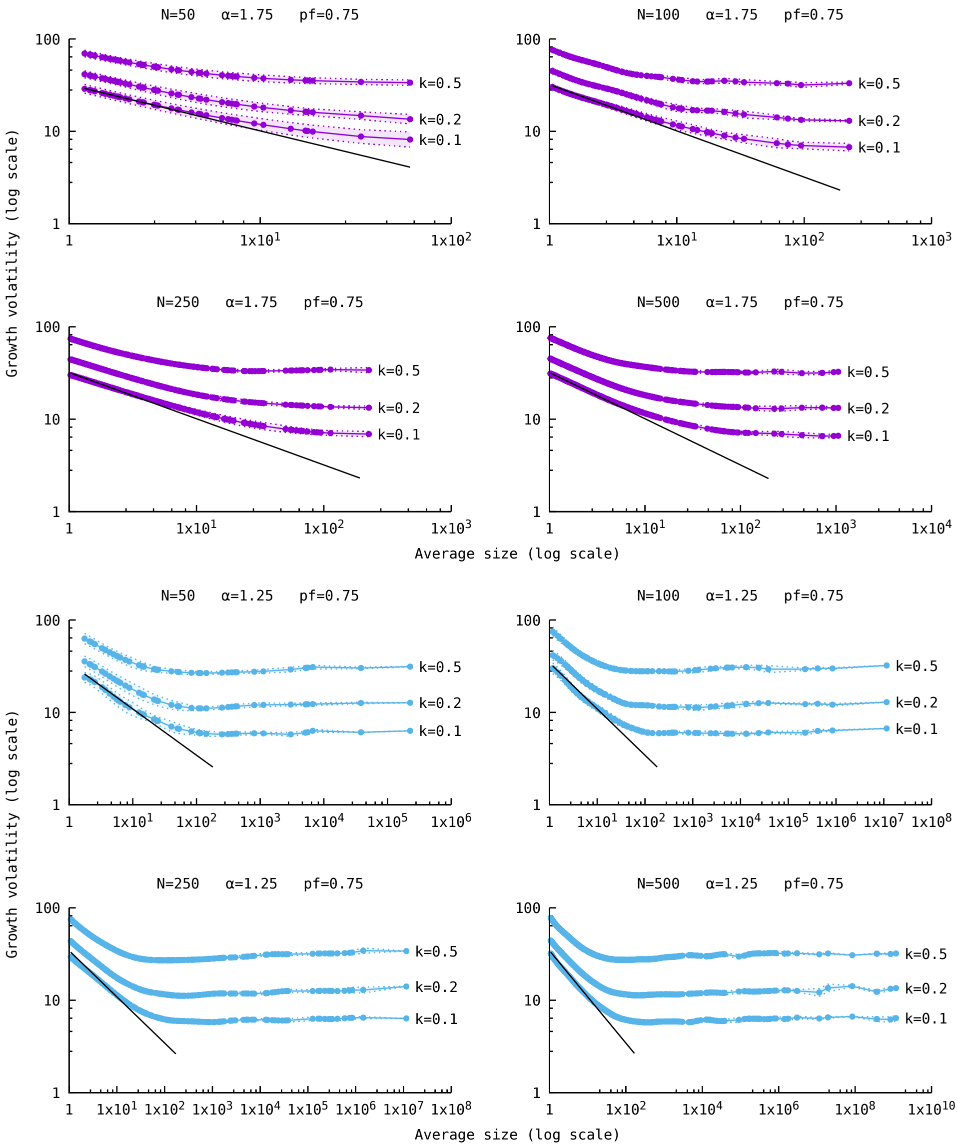


**Figure 5:** Local-linear kernel estimates of the size-volatility relation for industries populated with a different number of firms ($N$) whose size is drawn from Pareto distributions with diverse shape parameter. Perturbation fraction $pf$ is set to 0.75 and the intensity of industry turbulence $k$ to 0.1, 0.2 and 0.5 in all panels ($\alpha = 1.75$ dark-violet points and $\alpha = 1.25$ light-blue points). We report a 99% asymptotic confidence band and, for $k = 0.1$, the corresponding $-0.5$ benchmark (solid black line). Estimates are obtained using the whole sample of artificial data generated in one single Monte Carlo iteration and is evaluated for the plot trimming the bottom and top 1%. Bandwidth is least-squares cross-validated.

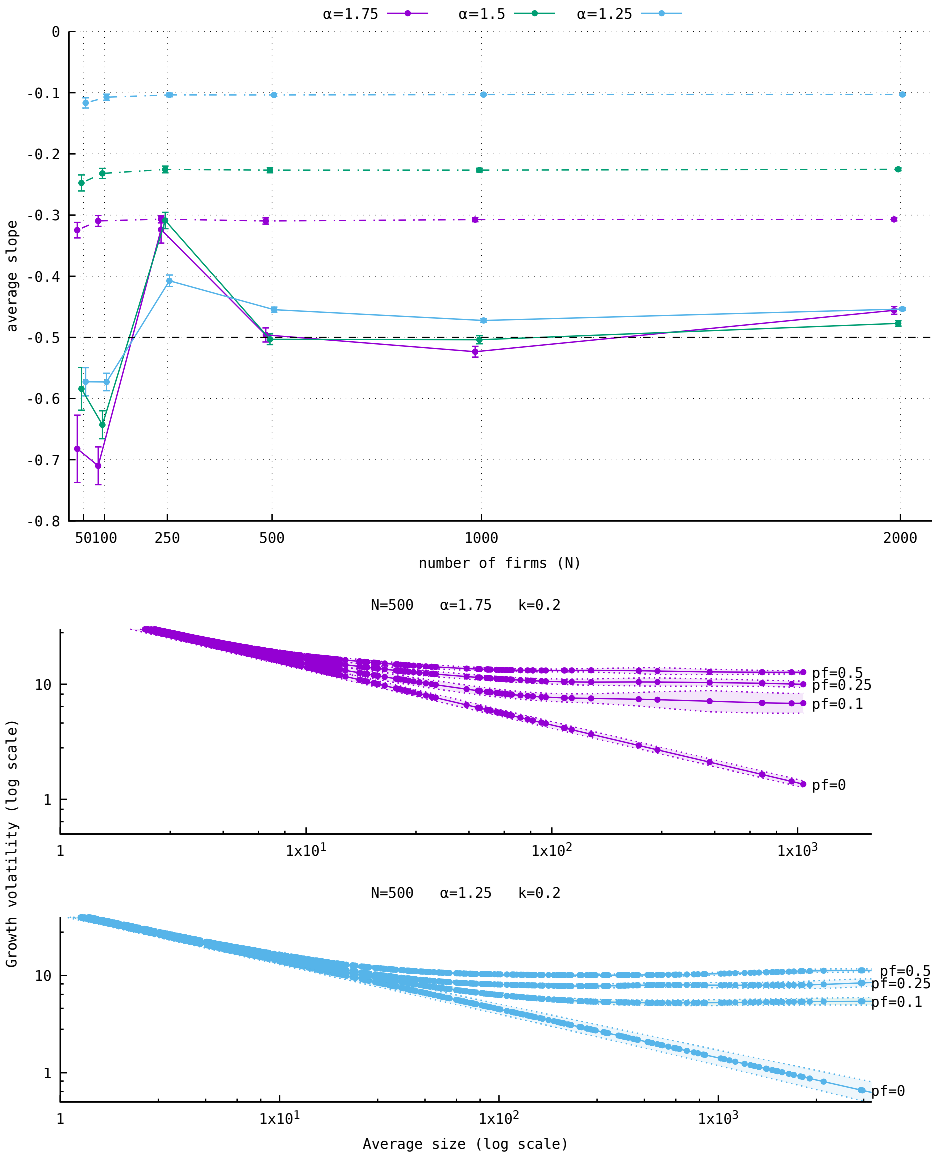


**Figure 6:** Top panel reports the average slope of the size-volatility relation across 100 Monte Carlo iterations together with a 99% confidence band estimated on entire samples in each Monte Carlo iteration (dashed line) and on trimmed samples including sizes between the $5^{th}$ and $30^{th}$ percentile only (solid lines). Bottom panel reports local-linear kernel estimates of the size-volatility relation for different values of the size-productivity correlation as measured by the perturbation-factor $pf$. Industries are populated with 500 firms with $k = 0.2$ and $\alpha = 1.75$ (dark-violet points) and $\alpha = 1.25$ (light-blue points).

| | | average | std.dev | min | max | n.obs |
|---|---|---|---|---|---|---|
| $r_{j,t}$ | | | | | | |
| | 30 bins | $-0.133$ | 0.058 | $-0.334$ | 0.029 | 440 |
| | 50 bins | $-0.135$ | 0.053 | $-0.304$ | 0.065 | 440 |
| $\rho_{j,t}$ | | | | | | |
| | mfp | 0.698 | 0.115 | 0.325 | 0.876 | 440 |
| | lp | 0.287 | 0.107 | 0.050 | 0.596 | 440 |
| $k_{j,t}$ | | | | | | |
| | proxy.1 | 0.127 | 0.047 | 0.049 | 0.560 | 440 |
| | proxy.2 | 0.120 | 0.056 | 0.032 | 0.918 | 440 |
| | proxy.3 | 0.040 | 0.046 | 0.000 | 0.501 | 440 |
| number of firms | | 3721.3 | 4583.7 | 258.0 | 25626.0 | 440 |

**Table 2:** Descriptive statistics of different proxies for the three main variables used in estimations and for the number of firms active in each industry-year pair. $r_{j,t}$ is the estimated slope of the size-volatility relation using 30 or 50 bins, $\rho_{j,t}$ the Spearman rank correlation index between size and productivity, where the latter is either a multi-factor productivity or a labor productivity index and, $k_{j,t}$ the turbulence index that we measure with the sum of the absolute values of the firm-level difference in size divided by the industry size (proxy.1), with the sum of the firm-level differences in market shares (proxy.2) and with the industry aggregate growth rate (proxy.3).

# 4 Model meets data: reduced-form regression analysis

Our theoretical framework predicts a within-industry size-volatility relationship that departs from log-log linearity. In particular, it shows that the emergence of two distinct regimes: an approximately log-log linear decline in volatility among smaller firms, and a flattening pattern among larger ones depending on the degree of correlation between firm size and productivity. Conditional on this correlation, the extent of market turbulence within the industry, captured in the model by the parameter $k$, further shapes the pattern. In this section, we examine the extent to which these theoretical predictions are supported by the data using simple reduced-form regression models.

We reconsider the French administrative data used in Section 2 to motivate our work. We proxy $S_{i,t}$, the size of a firm $i$ in year $t$, with its annual Total Sales (expressed in $1,000$ euro), estimate its productivity $a_i$ following the methodology described by Levinsohn and Petrin (2003) with the correction proposed by Ackerberg et al. (2015) and compute for each year $t$ and 2-digit industry $j$ in the Manufacturing sector the Spearman rank correlation index between the two, denoted as $\rho_{j,t}$. Next, we build for each industry-year pair three different proxies of the industry turbulence parameter $k_{j,t}$. First we consider the sum of the absolute values of the firm-level difference in size between $t+1$ and in $t$ divided by the industry size $\frac{\sum_{\forall i \in j} |S_{i,t+1} - S_{i,t}|}{\sum_{\forall i \in j} S_{i,t-1}}$ (proxy.1). Second, we compute the sum of the firm-level differences in market shares between $t+1$ and in $t$ as in $\sum_{\forall i \in j} |s_{i,t+1} - s_{i,t}|$, where $s_{i,t}$ are firms' market shares (proxy.2). Third, we simply compute the industry aggregate growth rate as $|\log S_{j,t+1} - \log S_{j,t}|$, where $S_{j,t}$ represents the size of industry $j$ in year $t$ (proxy.3).[40] Finally, we estimate for each industry-year the slope of a log-log linear relation between size and growth volatility, denoted $r_{j,t}$.[41] In light of our motivating evidence (cfr. in particular the right-panel in Figure 1) these estimated slopes represent in our context an approximate but informative and interpretable index summarizing the extent to which we observe deviations from the log-log linear benchmark. In Table 2

[40] We employ the absolute value in all these proxies because we are interested in the amount of reallocation due to changes in sectoral production, rather than in the sign of its growth.

[41] In the baseline specification we set the number of bins to 30 in each year-industry. We check that our results remain stable and significant when changing the number of bins.

| Dependent variable $r_{j,t}$ (30 bins) | | (1) | (2) | (3) | (4) |
|---|---|---|---|---|---|
| $\rho_{j,t}$ | | | | | |
| | mfp | -0.109*** | | -0.097** | -0.105** |
| | | (0.038) | | (0.038) | (0.042) |
| | lp | | -0.083* | | |
| | | | (0.047) | | |
| $k_{j,t}$ | | | | | |
| | proxy.1 | 0.288** | 0.281** | | |
| | | (0.135) | (0.133) | | |
| | proxy.2 | | | 0.176** | |
| | | | | (0.081) | |
| | proxy.3 | | | | 0.098 |
| | | | | | (0.079) |
| $\mathbf{X}_{j,t}$ controls | | Yes | Yes | Yes | Yes |
| Industry FE | | Yes | Yes | Yes | Yes |
| Year FE | | Yes | Yes | Yes | Yes |
| Observations | | 440 | 440 | 440 | 440 |
| Adjusted R-squared | | 0.473 | 0.470 | 0.462 | 0.443 |

**Table 3:** Estimates of the regression (3) using French industry-year data. The slope of the size-volatility relation $r_{j,t}$ is estimated using the bin methodology with 30 bins for each 2-digit industry and year pair. $\rho_{j,t}$ represents the Spearman rank correlation between size and productivity, while the turbulence index $k_{j,t}$. Productivity is proxied with a multi-factor index (mfp) and with a single-factor index (lp) while industry turbulence is proxied with the sum of the absolute values of the firm-level difference in size divided by the industry size (proxy.1), with the sum of the firm-level differences in market shares (proxy.2) and with the industry aggregate growth rate (proxy.3). Errors are clustered at the industry level.

we report few statistics describing the main variables involved in the estimation comparing different proxies and methods to compute them. We show that $r_{j,t}$ does not seem to be significantly affected by the choice of the number of bins and that it is typically estimated over a relatively large number of firms. Moreover, as expected, multi factor and single factor productivity indexes may well differ across sector-year pairs, but both register only positive correlations spanning a similar and relatively large portion of the interval $[0, 1]$. Finally, the three proxies for industry turbulence look in a first approximation similar among each other, but still present different averages and ranges of variation. Overall, these statistics corroborate the idea that our dependent variable is robustly estimated and not too sensitive to the arbitrary choice of the number of bins and that exploring various proxies for $\rho_{j,t}$ and $k_{j,t}$ in our estimations is an informative exercise to assess the robustness of our findings.

Next we move to our reduced-form econometric exercise and estimate with OLS the following specification

$$r_{j,t} = \alpha + \beta_1 \rho_{j,t} + \beta_2 k_{j,t} + \beta_{\mathbf{X}} \mathbf{X}_{j,t} + \mu_j + \mu_t + \epsilon_{j,t} \tag{3}$$

where all the variables are standardized and where $\mu_j$ and $\mu_t$ represent industry and year fixed effects and $\mathbf{X}_{j,t}$ is a vector of industry-year varying controls including the (log) number of firms and the (log) industry size. Standard errors are clustered at the level of industries.

Table 3 reports the main estimates of the reduced-form regression (3) using French industry-year data. Column (1) presents our preferred specification, which computes the size-productivity correlation using multi-factor productivity and proxies industry turbulence as the sum of absolute firm-level changes in size relative to total industry size (proxy.1). The estimated coefficient on $\rho_{j,t}$, $\beta_1$, is negative and statistically significant: industries where size and productivity are more strongly correlated exhibit steeper declines in volatility with firm size, consistent with a closer adherence to the log-log linear benchmark. The coefficient on the turbulence proxy, $\beta_2$, is positive and significant, indicating that greater market turbulence is associated with a flatter size-volatility relation, suggesting more uniform growth volatility across the firm size distribution. Taken together, these results align with the model's prediction in Proposition 2, namely that stronger size-productivity correlation and lower turbulence widen the volatility gap between small and large firms, while weaker correlation and greater turbulence compress it. Column (2) tests the robustness of our findings by replacing the multi-factor productivity measure with labor productivity (value added per employee) for computing the size-productivity correlation. Results confirm the baseline pattern both for $\beta_1$ and $\beta_2$: estimates remain significantly negative for the former and significantly positive for the latter. Finally, columns (3) and (4) explore our regression model using alternative proxies for industry turbulence, that is the reallocation of market shares (proxy.2) and the industry growth rate (proxy.3), respectively. Also these results corroborate our main findings: $\beta_1$ estimates remain negative and significant while estimates for $\beta_2$ display the correct sign, but lose statistical significance in one case, that is when the industry growth rate is used suggesting that this proxy may be a noisier measure of industry turbulence not considering the firm-to-firm reallocations.

The results reported in Table 3 survive a battery of robustness checks where we change to $50$ the number of bins used to estimate the slope of the size-volatility relation, when we replace $r_{j,t}$ with a simple correlation between the (log) size and the (log) growth volatility, when we replace the proxy of multi-factor productivity with the variant proposed in Wooldridge (2009), when we estimate the model weighting observations by industry shares, and when we replace Spearman rank with a simple linear correlation. Results of these robustness checks are reported in Appendix E.

## 5 Concluding remarks

We revisited the empirical and theoretical underpinnings of the size-volatility relationship in firm growth dynamics, with a focus on the French manufacturing sector. Using comprehensive administrative data of French manufacturing firms between 2000 and 2019, we document a robust non-linearity in the size-volatility relation: while small firms exhibit a steep decline in volatility as they grow, this pattern progressively flattens for larger firms. This two-regime structure challenges the standard log-log linear decay widely reported in the literature and suggests a more nuanced mechanism linking size to growth volatility.

To interpret this empirical regularity, we propose a simple stochastic model of industry dynamics grounded in an "islands" framework (e.g. Sutton, 2001; Klepper and Thompson, 2006; Bottazzi and Secchi, 2006a), in which firms compete for market opportunities and differ in productivity. A key insight of the model is that the shape of the size-volatility relation depends on the efficiency of resource allocation in the economy, measured by the correlation between firm size and firm productivity. When the correlation is perfect, growth volatility decays linearly in log-log space with a slope of -0.5. When the correlation is imperfect – a condition supported by our empirical findings – the decay attenuates and flattens at higher firm sizes. Importantly, the model predicts and explains the observed cross-sector heterogeneity in the strength and shape of this relationship.

Our results carry implications for the quantitative relevance of the granularity channel in explaining aggregate fluctuations (see Gabaix, 2011; Carvalho and Grassi, 2019). The flattening of the size-volatility curve among large firms implies that the macroeconomic influence of granular shocks may remain substantial even in the presence of size-dependent volatility. Moreover, the strength of this influence is linked to the degree of resource misallocation: economies with weaker size-productivity correlations – i.e., greater misallocation – tend to exhibit flatter size-volatility relationships and stronger granular effects. A second implication of our work is also that structural policies that boost market reallocation may have unexpected consequences on overall volatility. In this regard, our model suggests that an increase in market reallocation not only increases firm growth volatility regardless of firm size. It also changes the shape of the relation between firm growth volatility and firm size, by flattening it and inducing a smaller volatility gap between small and large firms.

By bridging empirical evidence with a structurally interpretable model, our work contributes to the literature on industrial dynamics, firm growth, and macroeconomic volatility. It emphasizes the importance of resource misallocations in shaping the aggregate impact of idiosyncratic firm-level shocks and calls for further research into how policy and institutional factors may influence the link between market reallocation processes and macroeconomic fluctuations.

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# A Size-volatility relation - robustness checks

## A.1 Using value added to proxy for size

In this appendix we reproduce Figure 1 replacing total sales with value added as a proxy of size. We keep the same procedure used in Section 2 with the only difference that we reduce the threshold below which we estimate the log-log linear fit from $350K$ to $100K$ euro. The main message remains unaffected.

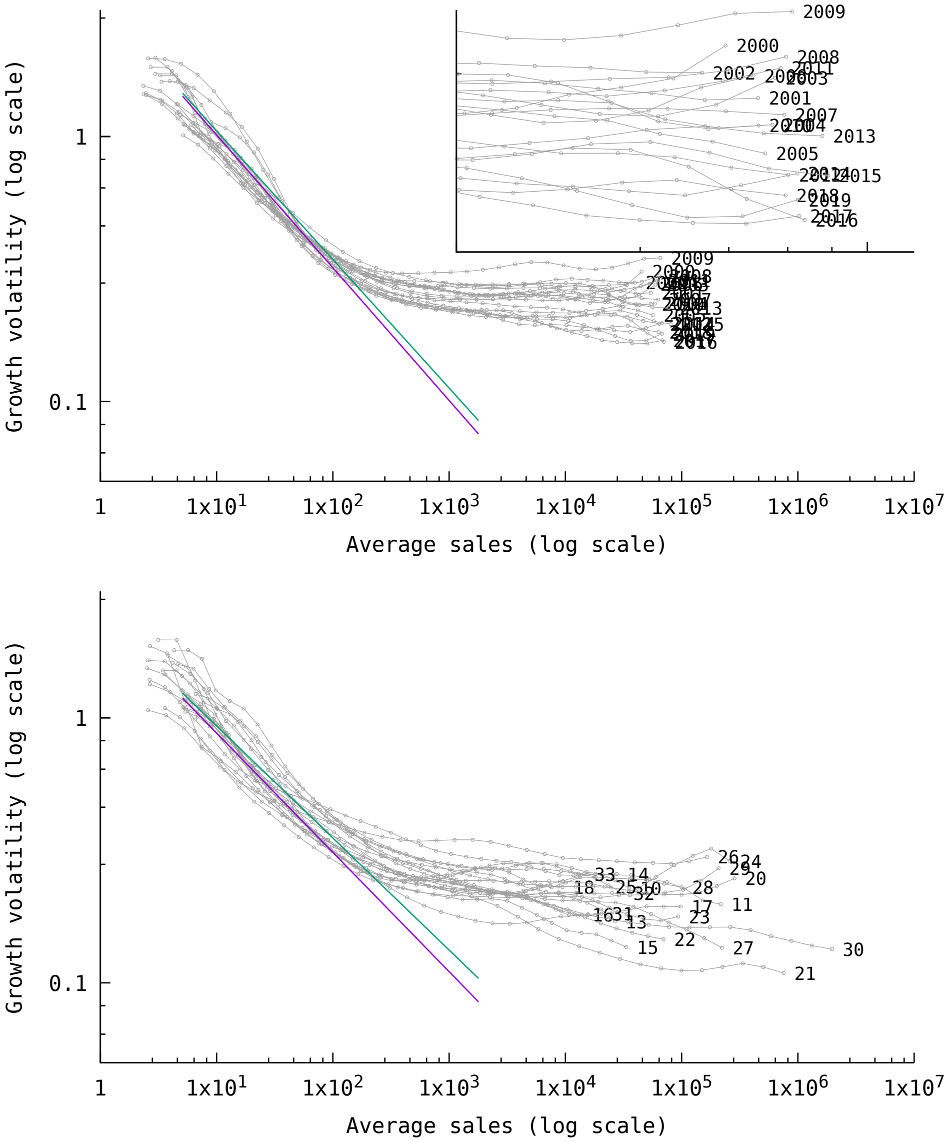


**Figure A.1:** Kernel regression estimate of the size-volatility relationship across years (top panel) and industrial sectors (bottom panel). The inset in the top panel zooms in the scaling of variance for largest firms. Kernel regressions are computed on a 32 points regular grid using an Epanenchnikov kernel and a rule-of-thumb bandwidth selection procedure. Binned relations are built including in the sample non individual firms active in the manufacturing industry in the time window 2000-2019. We report also a linear fit estimated on bins recording value added lower or equal than $100K$ (light-blue solid line) euro. The average of the estimated slopes (dark-violet solid line) is $-0.50$ and $-0.45$ for years and sectors respectively while the estimated slopes (dark-green solid line) pooling observations are $-0.49(0.01)$ and $-0.42(0.01)$ respectively with asymptotic standard error in parenthesis.

## A.2 Using mean absolute deviation to proxy for volatility

In this appendix we reproduce Figure 1 replacing the Standard Deviation with the Mean Absolute Deviation to estimate growth volatilities. We keep the same procedure used in Section 2. While the main message remains unaffected we observe an anomalous behavior of the industrial sector "Beverage industry" (SIC 11) in the bottom panel highlighted with a dashed red line. A deeper look at the data reveals that in the first two bins (out of 500), those associated with the smallest firms, there are duplicated values that concentrate around to the mean. In this situation the mean absolute deviation of the growth rates for these two bins mechanically shrinks to zero.[42] Figure A.2 reports the size-volatility scaling profile for this sector obtained by removing the two bins with a solid red line, showing an almost perfect overlap for most of the size support.

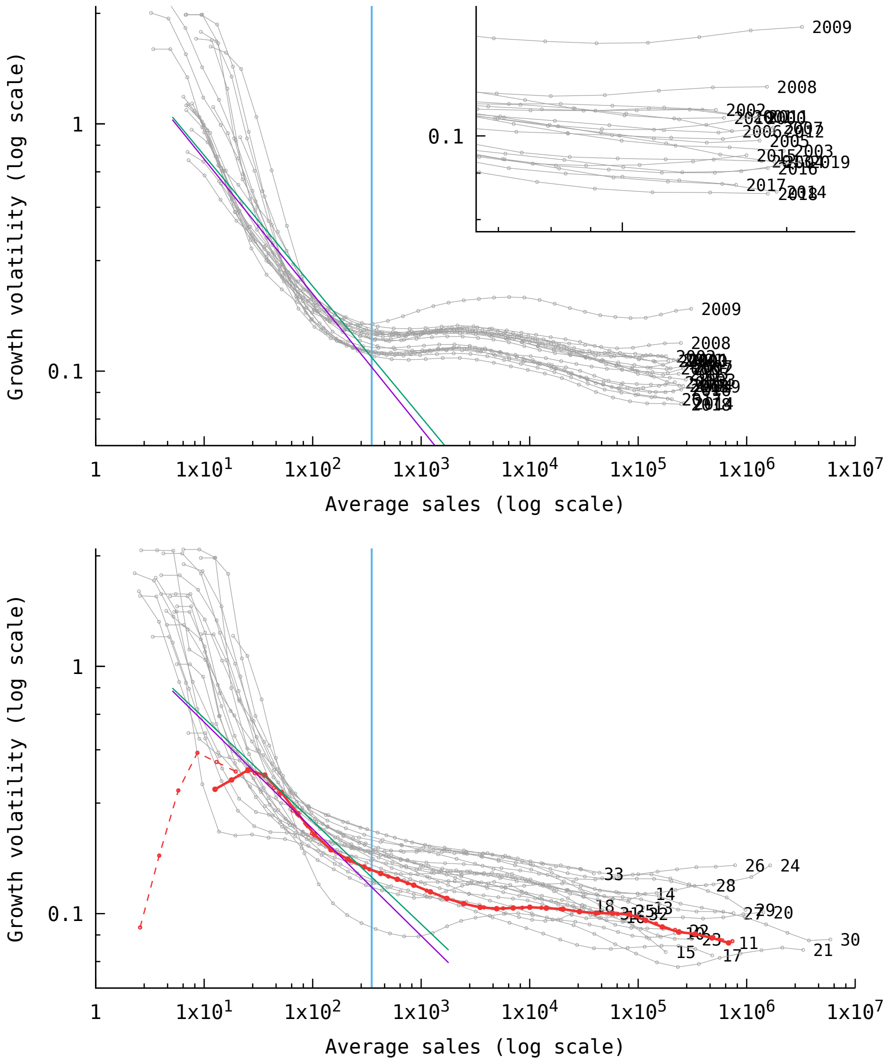


**Figure A.2:** Kernel regression estimate of the size-volatility relationship across years (top panel) and industrial sectors (bottom panel). The inset in the top panel zooms in the scaling of mean absolute deviation for largest firms. Kernel regressions are computed on a 32 points regular grid using an Epanenchnikov kernel and a rule-of-thumb bandwidth selection procedure. Binned relations are built including in the sample non individual firms active in the manufacturing industry in the time window 2000-2019. We report also a linear fit estimated on bins recording sales lower or equal than $350K$ (light-blue solid line) euro. The average of the estimated slopes (dark-violet solid line) is $-0.54$ and $-0.43$ for years and sectors respectively while the estimated slopes (dark-green solid line) pooling observations are $-0.53(0.01)$ and $-0.42(0.01)$ respectively with asymptotic standard error in parenthesis.

[42] Clearly one should expect the same behavior for the standard deviation. However, the mean absolute deviation, being a linear measure of dispersion, is more sensitive to adding zero deviations from the mean.

## A.3 Disaggregating firms by legal types

Figure A.3 shows the size-volatility relation built with the “bins methodology” grouping observations by type of legal unit but pooling across years and manufacturing sectors. While corporations span a significant larger size range than companies three of the four panels (top and bottom-left) show that the size-volatility relation when represented in a double-log scale does not look linear but features a decay in the rate at which growth volatility scales down with size quite similar for these legal types of organization. On the contrary Micro enterprises display a rather peculiar pattern. We believe this behavior is mechanically generated by the “Micro-entreprise regime” and this is why we remove these firms from our sample.

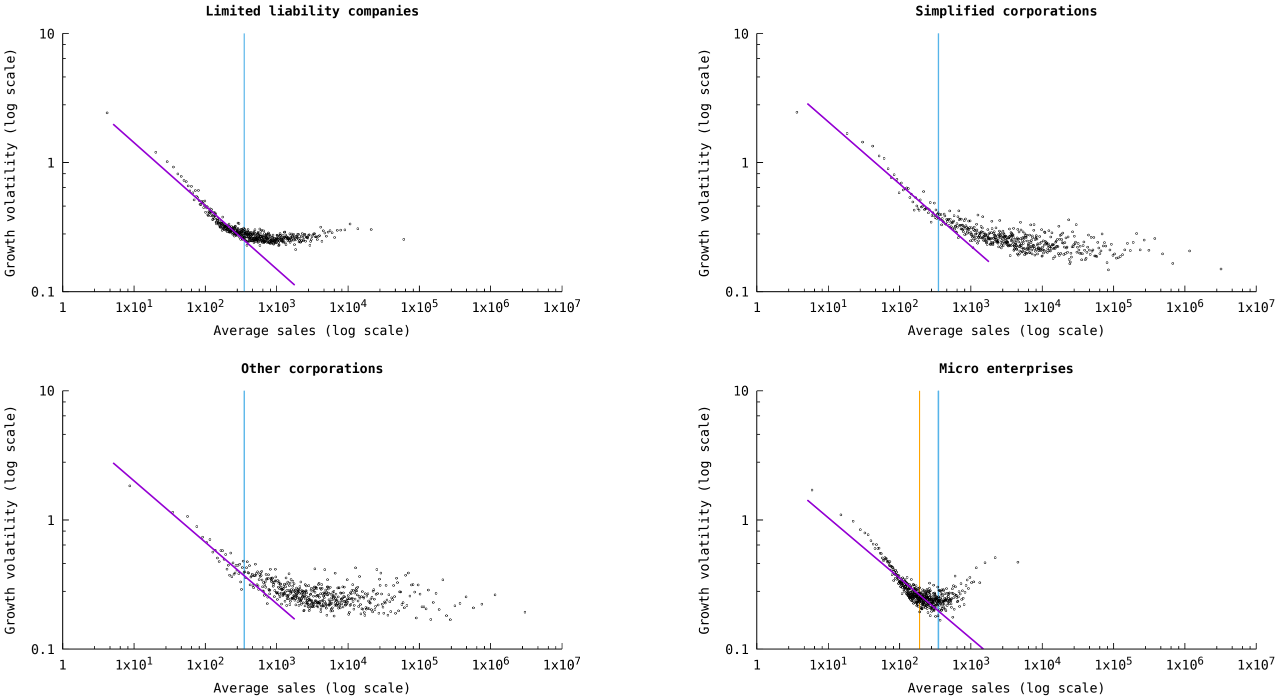


**Figure A.3:** Size-volatility relationship across type of legal unit. Binned relations are built including in the sample firms active in the manufacturing industry in the time window 2000-2019. Pooling observations across years and sectors gives about 1.09M firm-year observations for Limited Liability Companies, 345K for Simplified Corporations, 245K unique firm for Other Corporations, and 475K for Individual Companies. We report also a linear fit (dashed green line) estimated on bins recording sales lower or equal than $250K$ euro. Estimated slopes are $-0.49(0.01)$, $-0.48(0.02)$, $-0.48(0.03)$ and $-0.47(0.01)$ respectively with asymptotic standard errors in parenthesis.

# B Proofs of Propositions

This appendix contains the proofs of the two propositions presented in the main text. They both make use of a preliminary lemma using the law of total variance to decompose the variance in Equation (2) in three components: a component measuring the contribution from the assignment of market opportunities to firms, a component accounting for productivity heterogeneity of firms and a residual term.

**Lemma 1.** *Consider an industry populated by $N$ firms heterogeneous both in size and productivity. Let $H$, $F_s$ and $F_a$ denote the number of market opportunities and the cumulative distribution functions of size and productivity and $M_t$ the total market size. Then*

$$\begin{aligned} V\left[g_i|S_{i,t}\right] = & \frac{1}{S_{i,t}}\frac{H}{M_t}\Big[\Big(1-\frac{S_{i,t}}{M_t}\Big)z^2\big(2E\left[F_a(a_i)\right]-1\big)^2+ \\ & +\Big(1+(H-1)\frac{S_{i,t}}{M_t}\Big)4z^2 V[F_a(a_i)]+ \\ & +4z^2\big(E\left[F_a(a_i)\right]-E\left[F_a^2(a_i)\right]\big)\Big] \quad , \end{aligned} \tag{B.1}$$

*where $g_i$ is defined in equation (2).*

*Proof.* Let us consider a random variable $Y$, which is jointly determined by two other random variables $X_1$ and $X_2$. Bowsher and Swain (2012) show that in this case the law of total variance

$$V(Y) = \underbrace{V[E(Y|X_1)]}_{\text{From } X_1} + \underbrace{E\{V[E(Y|X_1,X_2)|X_1]\}}_{\text{From } X_2} + \underbrace{E[V(Y|X_1,X_2)]}_{\text{Residual variance of } Y}. \tag{B.2}$$

states that the variance $V(Y)$ can be decomposed in a first term that depends on the variability of $X_1$, a second term that depends on $X_2$ and a residual term that does not depend upon $X_1$ and $X_2$. The second term is new vis-à-vis the standard law of total variance based on one random variable only. It is equal to the average difference between the variance of $Y$ conditional on $X_1$ only and the one conditional on both $X_1$ and $X_2$, and is indeed equal to 0 if $Y$ and $X_2$ are independent. The choice of the order for the two components $X_1$ and $X_2$ matters for the resulting decomposition, as it depends on the sequence of conditioning. Following Bowsher and Swain (2012), we decompose the variance of firms' growth rates according to the natural order of our two-steps process. More specifically, $X_1$ represents the number of market opportunities assigned to firms $h_i(H, s_{i,t})$ and $X_2$ the CDF of productivity $F_a(a_i)$.[43] All other variables appearing in Equation (2) (size $S_{i,t}$, total number of market opportunities $H$ and market size $M_t$) are treated as constants. To further simplify the notation, in these proofs we remove the time subscript also in $S_{i,t}$ and $M_t$ since $t$ represents unambiguously the beginning of the period. Moreover we replace $F_a(a_i)$ with $F_{a_i}$ and $h_i(H, s_i)$ with $h_i$. Then using (B.2) and equation (2) in Section 3 one can write

$$V[g_i|S_i] = \frac{1}{S_i^2}\Big[V\big[E\big[\sum_{h=0}^{h_i}\epsilon(F_{a_i})|h_i\big]\big] + E\big[V[E(\sum_{h=0}^{h_i}\epsilon(F_{a_i})|h_i,F_{a_i})|h_i]\big] + E[V(\sum_{h=0}^{h_i}\epsilon(F_{a_i})|h_i,F_{a_i})]\Big] \tag{B.3}$$

Noting that the two conditioning variables $h_i$ and $F_{a_i}$ the first term in (B.3) expands as follows

$$V\big[E[\sum_{h=0}^{h_i}\epsilon(F_{a_i})|h_i]\big] = V\big[h_i E[E(\epsilon(F_{a_i})|F_{a_i})]\big] = V[h_i]E[E(\epsilon(F_{a_i})|F_{a_i})]^2 \quad , \tag{B.4}$$

where we have used the Law of Iterated Expectations to note that $E[E(\epsilon(F_a)|F_a)]$ is constant and the facts that $\epsilon$ are identically distributed by assumption and $F_{a_i}$ is Uniformly distributed because of the Probability Integral Transform theorem. Furthermore since $h_i$ is a Binomial random variable with success probability $s_i = \frac{S_i}{M}$ by assumption,[44] we get

$$V\big[E[\sum_{h=0}^{h_i}\epsilon(F_{a_i})|h_i]\big] = H\frac{S_i}{M}\Big(1-\frac{S_i}{M}\Big)\Big(E[E(\epsilon(F_{a_i})|F_{a_i})]\Big)^2 \quad . \tag{B.5}$$

Recalling that $\epsilon$ are i.i.d, the second terms can be rewritten as follows

$$\begin{aligned} E\Big[V\big[E\big[\sum_{h=0}^{h_i}\epsilon(F_{a_i})|h_i,F_a\big]|h_i\big]\Big] = & E[h_i^2]E\big[V[E(\epsilon(F_{a_i})|F_{a_i})]\big] = \\ = & \big(V[h_i]+E[h_i]^2\big)V[E[\epsilon(F_{a_i})|F_{a_i}]] = \\ = & H\frac{S_i}{M}\Big(1+(H-1)\frac{S_i}{M}\Big)V\big[E[\epsilon(F_{a_i})|F_{a_i}]\big] \quad . \end{aligned} \tag{B.6}$$

[43] Note that we condition on the productivity distribution $F_a$ and not on the overall realization process $\epsilon(F_a)$. This is sufficient to generate the analytical results which are relevant for our purpose. Conversely, the inclusion of the additional conditionality on $\epsilon(F_a)$ would only add another term to the decomposition in Equation (B.2), without any additional analytical gain.

[44] Then its mean and variance are equal to $H\frac{S_i}{M}$ and $H\frac{S_i}{M}\left(1-\frac{S_i}{M}\right)$ respectively.

Similarly, the third term develops as follows

$$E[V(\sum_{h=0}^{h_i}\epsilon(F_{a_i})|h_i,F_{a_i})] = E[h_i]E(V(\epsilon(F_{a_i})|F_{a_i})) = \\ = H\frac{S_i}{M}E[V(\epsilon(F_{a_i})|F_{a_i})] \quad . \tag{B.7}$$

Substituting (B.5), (B.6) and (B.7) in (B.3) gives

$$\begin{aligned} V(g_i|S_i) = & \frac{1}{S_i}\frac{H}{M}\{\big(1-\frac{S_i}{M}\big)(E[E(\epsilon(F_{a_i})|F_{a_i})])^2 + \\ & + \big[1+(H-1)\frac{S_i}{M}\big]V[E(\epsilon(F_{a_i})|F_{a_i})] + \\ & + E[V(\epsilon(F_{a_i})|F_{a_i})]\} \end{aligned} \tag{B.8}$$

Finally we compute the conditional expectation of $\epsilon(F_a)$ given $F_a$ and its variance

$$\begin{aligned} E(\epsilon(F_{a_i})|F_{a_i}) = & zF_{a_i} - z(1-F_{a_i}) = z(2F_{a_i}-1) \\ E(\epsilon(F_{a_i})^2|F_{a_i}) = & z^2F_{a_i} + (-z)^2(1-F_{a_i}) = z^2 \\ V(\epsilon(F_{a_i})|F_{a_i}) = & E(\epsilon(F_{a_i})^2|F_{a_i}) - E(\epsilon(F_{a_i})|F_{a_i})^2 = \\ = & z^2 - (z(2F_{a_i}-1))^2 = 4z^2(F_{a_i} - F_{a_i}^2) \quad , \end{aligned} \tag{B.9}$$

and substitute them in (B.8) to obtain (B.1). We do not further elaborate on the variance terms, as treating $F_a$ as a random variable depends on its relationship with firm size. □

**Proposition 1.** *1 Let $H$ denote the number of market opportunities available in an industry of size $M$. Suppose that business opportunities are assigned and realized according to hypotheses (hp1) and (hp2), and assume that firm productivity $a_i$ is perfectly correlated with size $S_i$, i.e. $\rho = corr(a_i, S_i) = 1$. Then, in the limit as $M \to \infty$ and $H \to \infty$ such that $\frac{H}{M} = k < \infty$, the standard deviation of firm growth rates, $\sqrt{V(g_i \mid S_i)}$, decays as a log-linear function of $S_i$ with a constant slope equal to $-0.5$.*

*Proof.* When the rank correlation between a firm's size and productivity is equal to 1, there exists a deterministic and strictly monotonic relation between the two variables $a_i = f(S_i)$ with $f' > 0$. This relation is in general non-linear and implies that the ranks of $S_i$ and $a_i$ are perfectly aligned.[45] As a consequence, for a given $S_i$, $F_a$ is not considered anymore as a random variable and the second term in Equation (B.1) vanishes since term $V(F_{a_i})$ is zero. Therefore, we can write

$$V\big[g_i|S_i\big] = \frac{1}{S_i}\frac{H}{M}\Big(\Big(1-\frac{S_i}{M}\Big)z^2(2F_{a_i}-1)^2 + 4z^2(F_{a_i}-F_{a_i}^2)\Big) \quad . \tag{B.10}$$

In the limit for $H, M \to \infty$ such that $\frac{H}{M} = k < \infty$, $(1-S/M)$ is approximately 1 when $S$ is small with respect to $M$. When $S$ is very large and close to $M$, $(1-S/M)$ is approximately 0, and the last expression converges to $1/Sk4z^2(F-F^2)$. In both cases, this proves the proposition. □

**Proposition 2.** *Let $H$ denote the number of market opportunities and $M$ the total market size, with $\lim_{M,H\to\infty} H/M = k < \infty$. Suppose that market opportunities are assigned and realized according to assumptions (hp1) and (hp2), and that firm productivity $a_i$ is imperfectly correlated with firm size $S_i$. Then, in the limit as $M, H \to \infty$, the standard deviation of firm growth rates, $\sqrt{V(g_i \mid S_i)}$, exhibits a non-linear decay with firm size: for small firms it declines approximately as a power law with exponent $-0.5$, whereas for large firms it flattens and converges to a horizontal asymptote. An increase in industry turbulence $k$ shifts upwards the size-volatility schedule and reduces the volatility gap between small and large firms.*

*Proof.* When the rank correlation between a firm's size and productivity is positive but lower than 1, there is no deterministic functional relation between the two variables. Hence $F_a$ is not given, and the term including $V[F_a]$ in (B.1) does not vanish anymore. In this scenario the behavior of $V[g_i|S_i]$ depends on what happens in the limit $H, M \to \infty$ to the first two term in (B.1) and in particular to $(1-\frac{S_i}{M})$ and $1+(H-1)\frac{S_i}{M})$. When $S_i$ is very small, $S_i << \frac{1}{k}$, both $(1-\frac{S_i}{M})$ and $1+(H-1)\frac{S_i}{M})$ are approximately 1 and we get as in the case of perfect correlation $\lim_{H,M\to\infty} V(g_{i,t}|S_i) = \frac{1}{S_i}kz^2$. When $S_i$ is very large, $S_i >> \frac{1}{k}$, the market share $\frac{S_i}{M}$ is close to 1 and then $(1-\frac{S_i}{M})$ to 0. Differently from the previous case, $(1+(H-1)\frac{S_i}{M})$ is approximated by $S_i\frac{H}{M}$ that is $S_ik$. Noting that all the moments of $F_{a_i}$ are bounded since $F_a$ is a cumulative distribution function, one obtains $\lim_{H,M\to\infty} V(g_{i,t}|S_i) = 4k^2z^2V[F_{a_i}]$ proving the first part of the proposition.

[45] Note that the relation can be linear $a_i = q + mS_i$ even when the variance of $S_i$ is infinite because of the heavy tail of its distribution. The point is that it cannot be quantified it using the Pearson correlation that does not exist.

To complete the proof we compute the logarithm of the growth rate variance in Equation (B.1):

$$\begin{aligned}\log V(g_i|S_i) = \log \frac{H}{M} - \log S_i + \\ \log\left(\gamma + S_i \frac{-E[E[\epsilon(F_{a_i})|F_{a_i}]]^2 + (H-1)V[E[\epsilon(F_{a_i})|F_{a_i}]]}{M}\right) \quad ,\end{aligned} \tag{B.11}$$

where $\gamma = (E[E(\epsilon(F_{a_i})|F_{a_i}])^2 + V[E[\epsilon(F_{a_i})|F_{a_i}]] + E[V[\epsilon(F_a)|F_a]]$. When the size-productivity correlation is perfect $V(E(\epsilon(F_a)|F_a))$ is zero and in (B.11) the number of business opportunities $H$ affects only the constant, shifting up and down the whole schedule. To investigate the case when the size-productivity correlation is not perfect, note that $k$ enters quadratically in the variance for large firms, but not for small firms. This implies that, as k increases, the logarithm of volatility rises twice as fast for large firms compared to small ones. □

# C Simulation algorithms

In this appendix we report the meta code for the two algorithms we use to numerically simulate Proposition 1 and Proposition 2 in Section 3. Algorithm 1 contains four blocks. In the first one "Parameters" we set the model parameters: the number of firms $N$, the exponent of the Pareto distribution $\alpha$ we use for drawing a firm's size, the industry turbulence $k$ measuring the strength of the allocation and riallocation processes and the fixed size of a business opportunity $z$. The second one "Inizialization", initializes the model by drawing a size for each firm and summing them to compute market size $M$ from which we get the number of business opportunities $H$. The third block "Monte Carlo simulation" is the main simulation loop where we randomly assign business opportunities to firms based on their market share. Each business opportunities is then randomly transformed into a positive or negative growth shock based on a firm's ranking in the productivity ladder which in this scenario maps one to one on that of size. Using growth shocks we compute the growth rates for each firm. In the last block "Post-simulation analysis" we compute a firm's growth volatility across the $R$ iterations in the Monte Carlo simulation $\sigma_i$ and regress both parametrically and non-parametrically its log on log size.

**Algorithm 1** Simulation structure (Scenario I)

***Parameters***

Set number of firms $N$, Pareto exponent $\alpha$, industry turbulence $k$, shock size $z$

***Initialization***

Draw firm sizes $\{S_i\}_{i=1}^N$ from a Pareto($\alpha$) distribution with $x_{\min} = 1$
Sort firms in increasing order of size
Compute total market size $M \leftarrow \sum_i S_i$
Set number of market opportunities $H \leftarrow \lfloor k \cdot M \rfloor$

***Monte Carlo Simulation***

**for** $r = 1$ to $R$ **do**
  **for** $i = 1$ to $N$ **do**
    Assign $h_i \sim \text{Binomial}(H, S_i/M)$
    Compute $F_a(a_i)$ as ECDF of $\{S_j\}$
    Draw $\text{pos}_i \sim \text{Binomial}(h_i, F_a(a_i))$
    Compute $\text{neg}_i \leftarrow h_i - \text{pos}_i$
    Compute growth rate $g_i^{(r)} \leftarrow z \cdot (\text{pos}_i - \text{neg}_i)/S_i$

***Post-Simulation Analysis***

**for** $i = 1$ to $N$ **do**
  Compute volatility $\sigma_i \leftarrow$ std. dev. of $\{g_i^{(r)}\}_{r=1}^R$
Fit OLS regression: $\log(\sigma_i) = \beta_0 + \beta_1 \log(S_i) + \varepsilon_i$
Estimate non-parametric regression of $\log(\sigma_i)$ on $\log(S_i)$

Algorithm 2 differs from the previous one in only one aspect, the decoupling of the size and productivity ladders. Our procedure to implement this decoupling is very simple. We set first a perturbation factor $pf$ which is a number in between 0 and 1, identifying a share of the initial rankings in the size ladder. Next, we randomly select $N$ times $pf$ size ranking, reshuffling their positions generating a new ladder we assign to productivity. In this procedure Scenario I is reproduced setting the perturbation factor equal to 0.

**Algorithm 2** Productivity rank perturbation function (Scenario II)

***Parameters***

Set number of firms $N$, perturbation fraction $pf \in [0, 1]$

***Input***

Firm sizes $\{S_i\}_{i=1}^N$

***Procedure***

Compute base ranks $initial.ranks \leftarrow rank(-\{S_i\})$
Compute number of firms to perturb $n_{perturb} \leftarrow \lfloor pf \cdot N \rfloor$
Randomly select indices to perturb $perturb.ids \leftarrow sample(1{:}N, n_{perturb})$
Shuffle selected ranks $new.ranks \leftarrow sample(initial.ranks[perturb.ids])$

***Output***

Perturbed productivity ranks $new.ranks$

‘

# D Simulation results

In this appendix we report a few additional simulation results to complete those displayed in Section 3. We start considering Proposition 1 showing in Figure D.1 the size-volatility relation in case $\alpha = 2$, i.e. considering a firm size distribution with both finite mean and variance. We do not observe qualitative differences with respect to those reported in Figure 3.

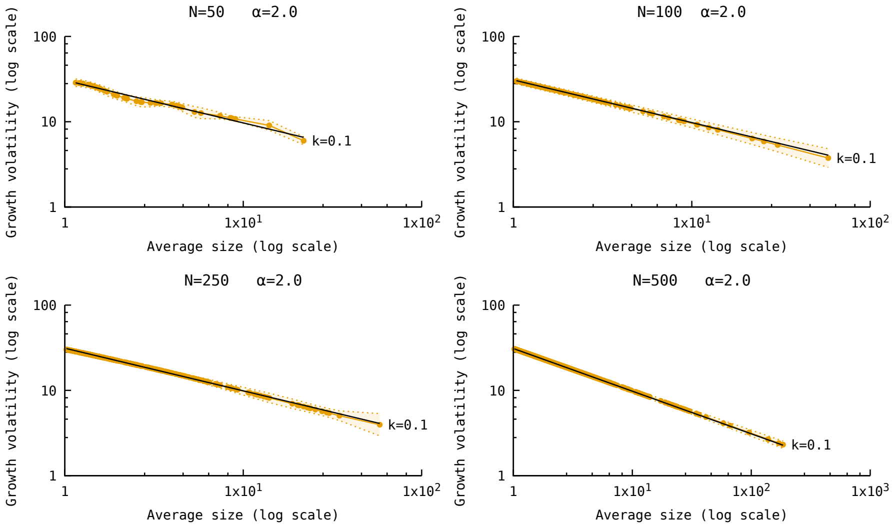


**Figure D.1:** Local-linear kernel estimator of the size-volatility relation for industries populated with a different number of firms ($N$) whose size is drawn from Pareto distributions with shape parameter $\alpha$ equal to 2 together with a 99% asymptotic confidence band( points) and the corresponding $-0.5$ benchmark (solid black line). Local-linear kernel estimator is obtained using the whole sample of artificial data generated in one single Monte Carlo iteration and is evaluated for the plot trimming the bottom and top 1%. Bandwidth is least-squares cross-validated. Bottom-right panel reports the average slope of the size-volatility relation across 100 Monte Carlo iterations together with a 99% confidence band. Industry turbulence $k$ is set to 0.1.

We then consider the effect of increasing the market turbulence parameter $k$ to 0.2 and 0.5 (cfr. Figure D.2) and once again the qualitative behavior of the size-volatility relation remains the same, except for a shift upwards of the entire schedule that looks increasing in $k$. See Proposition 2 for a detailed discussion about this feature.

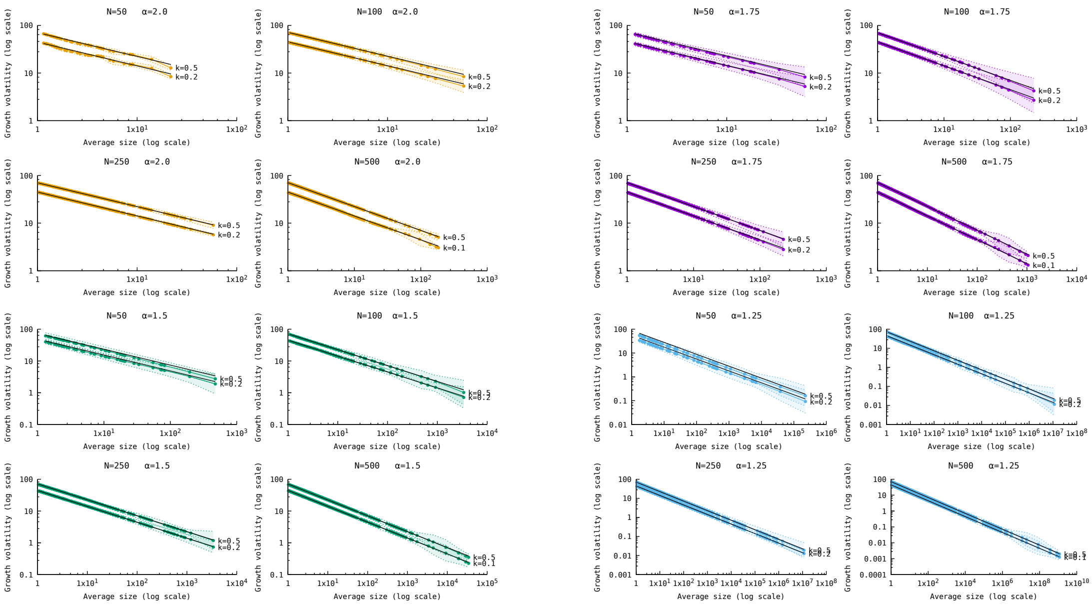


**Figure D.2:** Local-linear kernel estimator of the size-volatility relation for industries populated with a different number of firms ($N$) whose size is drawn from Pareto distributions with diverse shape parameter together with a 99% asymptotic confidence band ($\alpha = 2.0$ gold points, $\alpha = 1.75$ dark-violet points, $\alpha = 1.5$ dark-green points and $\alpha = 1.25$ light-blue points) and the corresponding $-0.5$ benchmark (solid black line). Local-linear kernel estimator is obtained using the whole sample of artificial data generated in one single Monte Carlo iteration and is evaluated for the plot trimming the bottom and top 1%. Bandwidth is least-squares cross-validated. Bottom-right panel reports the average slope of the size-volatility relation across 100 Monte Carlo iterations together with a 99% confidence band. Industry turbulence $k$ is set to 0.2 and 0.5 in all panels.

Next, we move to the second scenario of imperfect correlation between size and productivity and repeat the exercises shown in Figure 5 and Figure 6 also for $\alpha$ equal 2 and 1.5. Figure D.3 confirms that results shown in Section 3 concerning Proposition 2 are by no means peculiar and emerge with only quantitative differences here.

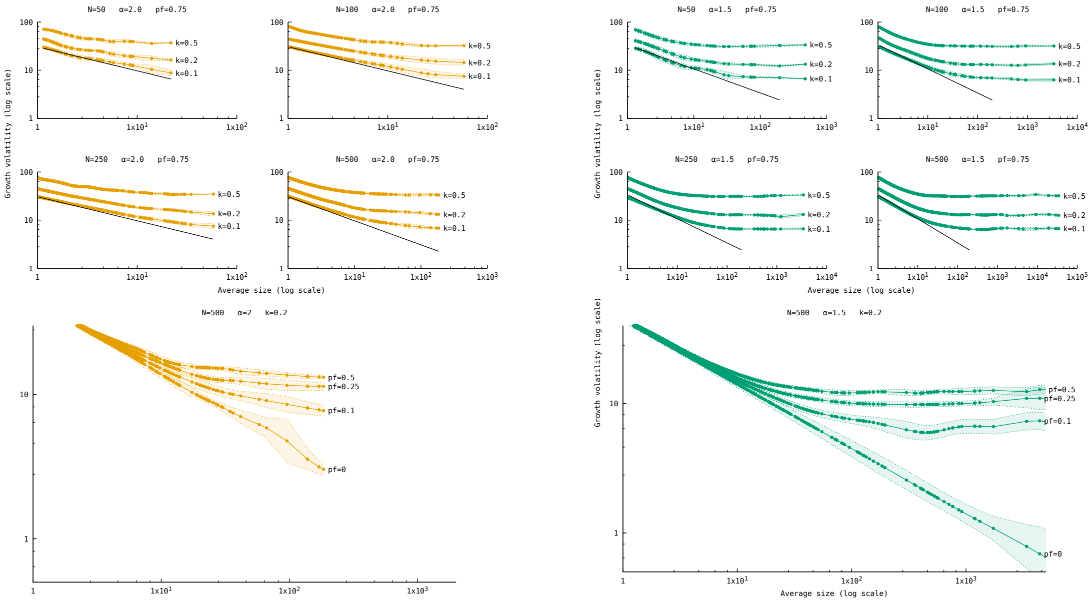


**Figure D.3:** Local-linear kernel estimates of the size-volatility relation for industries populated by a different number of firms ($N$) whose size is drawn from Pareto distributions with diverse shape parameter together with a 99% asymptotic confidence band ($\alpha = 2.0$ gold points, $\alpha = 1.75$ dark-violet points, $\alpha = 1.5$ dark-green points and $\alpha = 1.25$ light-blue points) and the corresponding $-0.5$ benchmark (solid black line). Local-linear kernel estimator is obtained using the whole sample of artificial data generated in one single Monte Carlo iteration and is evaluated for the plot trimming the bottom and top 1%. Bandwidth is least-squares cross-validated. Bottom-right panel reports the average slope of the size-volatility relation across 100 Monte Carlo iterations together with a 99% confidence band. Industry turbulence $k$ is set to 0.2 and 0.5 in all panels.

# E Robustness checks

| Dependent variable $r_{j,t}$ | | (1) | (2) | (3) | (4) | (5) |
|---|---|---|---|---|---|---|
| $\rho_{j,t}$ | | | | | | |
| | mfp | -0.103*** | -0.537*** | -0.145* | -0.129*** | -0.145*** |
| | | (0.038) | (0.153) | (0.077) | (0.039) | (0.044) |
| $k_{j,t}$ | | | | | | |
| | proxy.1 | 0.314*** | -0.936** | 0.277** | 0.180* | 0.284** |
| | | (0.086) | (0.445) | (0.131) | (0.087) | (0.133) |
| $\mathbf{X}_{j,t}$ controls | | Yes | Yes | Yes | Yes | Yes |
| Industry FE | | Yes | Yes | Yes | Yes | Yes |
| Year FE | | Yes | Yes | Yes | Yes | Yes |
| Observations | | 440 | 440 | 440 | 440 | 440 |
| Adjusted R-squared | | 0.525 | 0.333 | 0.470 | 0.554 | 0.479 |

**Table E.1:** Estimates of the regression (3) using French industry-year data. With respect to the baseline reported in Table 3 in column (1) the slope of the size-volatility relation $r_{j,t}$ is estimated using the bin methodology with 50 bins, in column (2) $r_{j,t}$ is replaced with with a simple correlation between the (log) size and the (log) growth volatility, in column (3) the multi-factor productivity index is estimated according to Wooldridge (2009), in column (4) we weight observations by sector industry shares, and in column (5) we compute $\rho_{j,t}$ as the Pearson correlation index. Errors are clustered at the industry level.

Table E.1 shows that our main regression results survive to a battery of robustness checks where we explore different definitions of our dependent and independent variables and where we weight our observations. In column (1) we increase to 50 the number of bins used to estimate the slope of the size-volatility relation in each industry-year. In column (2) we replace as a dependent variable the slope of the size-volatility relation with a simple correlation between the (log) size and the (log) growth volatility. In column (3) instead of following Levinsohn and Petrin (2003) as corrected by Ackerberg et al. (2015) we estimate the multi-factor productivity index as in Wooldridge (2009). In column (4) each observations is weighted by sector industry shares computed year by year. In column (5), even with the caveat that size must have in this case a finite variance, we replace the Spearman rank with the Pearson linear correlation. Across all specifications, the empirical results remain consistent with the model's theoretical predictions: stronger correlation between size and productivity is associated with a steeper decline in volatility, while greater market turbulence tends to compress the size-volatility gradient.