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Institute of Economics
Scuola Superiore Sant'Anna

Piazza Martiri della Libertà, 33 - 56127 Pisa, Italy
ph. +39 050 88.33.43
institute.economics@sssup.it

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Generalized Optimization Algorithms for Complex Models

Mario Martinoli [1]
Raffaello Seri [2]
Fulvio Corsi [3]

[1] Institute of Economics and L'EMbeDS, Scuola Superiore Sant'Anna, Pisa, Italy
[2] DiECO, University of Insubria, Italy
[3] Department of Economics, University of Pisa, Italy

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Generalized Optimization Algorithms for Complex Models

Mario Martinoli*†

Raffaello Seri‡

Fulvio Corsi§

Abstract

Linking the statistic and the machine learning literature, we provide new general results on the convergence of stochastic approximation schemes and inexact Newton methods. Building on these results, we put forward a new optimization scheme that we call generalized inexact Newton method (GINM). We extensively discuss the theoretical and the computational aspects of the GINM. The results apply to both deterministic and stochastic approximation schemes, and are particular effective in the case in which the objective function to be optimized is highly irregular and/or the stochastic equicontinuity hypothesis is violated. Examples are common in dynamic discrete choice models and complex simulation models characterized by nonlinearities and high levels of heterogeneity. The theory is supported by extensive Monte Carlo experiments.

Keywords: Optimization, stochastic approximation, Newton-Raphson methods, asymptotic convergence; M-estimation; stochastic equicontinuity.
JEL classification: C61; C15; C44

1 Introduction

Several topics discussed in social sciences, hard sciences and life sciences are represented as optimization problems. Increasingly frequently, these maximization (or, equivalently, minimization) problems relate to highly parametrized, nonlinear complex functions $F(\boldsymbol{\theta})$ for $\boldsymbol{\theta} \in \Theta$, where Θ is the parameter space. Most algorithms devised to optimize such a function achieve the goal by constructing a sequence $\left\{\boldsymbol{\theta}^{(i)}\right\}$ in which each value depends on the previous value (or values). Some of these methods, like the Nelder–Mead algorithm (Nelder and Mead, 1965), use no derivatives but approximate the objective function $F(\boldsymbol{\theta})$ through convex hulls or simplexes. However, optimization is more commonly performed through derivative-based algorithms, that we can separate in two main groups according to the number of derivatives that are required. The first group originates from the Newton–Raphson algorithm that requires the computation of the first derivative

*Institute of Economics & L'EMbeDS, Sant'Anna School of Advanced Studies
†Corresponding author. Sant'Anna School of Advanced Studies, Piazza Martiri della Libertà 33, Pisa (Italy). Email: *m.martinoli@santannapisa.it*
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‡DiECO, University of Insubria
§Department of Economics, University of Pisa

(or gradient) and of the second derivative (or Hessian) of the objective function itself. Other widely used examples are the Broyden–Fletcher–Goldfarb–Shanno (BFGS) and Berndt–Hall–Hall–Hausman (BHHH, see Berndt et al., 1974) algorithms that replace the Hessian with different approximations. However, also the gradient is generally approximated, as one of the building blocks of the implementation of these algorithms is the use of numerical differentiation methods exploiting the fact that the difference quotient approaches the derivative when the points entering the quotient are near enough. Therefore, in the literature, some results have appeared considering inexact Newton methods (INM), i.e. a large class of optimization methods encompassing (most of) the previous algorithms and allowing for the simultaneous replacement of the gradient and of the Hessian with approximations (see Nocedal and Wright, 1999, and references therein). The second group of derivative-based optimization algorithms, often called stochastic approximation schemes (SAS), requires only an approximation of the gradient and replaces the Hessian with a deterministic diagonal matrix containing a quantity called step size. The approximation of the gradient can either be unbiased, as in the original paper (Robbins and Sutton, 1951), or biased, as more recently considered (Karimi et al., 2019). It is clear that SAS are a subset of INM, but considering them separately allows for a more precise analysis of the algorithms.

However, most of these optimization methods break down when the function is noisy or non-differentiable. In fact, when the function is "rough" or "rugged", the optimization algorithm can go back and forth without settling on a value. This is generally referred to in the literature under the name of "chattering". This term is used in several branches of Applied Mathematics to denote some phenomena that share the same feature, i.e. an alternating behavior among two or more states. In production engineering, machining vibrations or chatter are self-excited vibrations between the cutting tool and the workpiece "which grow until the tool jumps out of the cutting zone or breaks because of the exponentially growing dynamic displacements between the tool and workpiece" (Altintas, 2012, p. 2, see Tobias and Fishwick, 1958a,b for the first explanation of the phenomenon). In qualitative simulation, chatter or chattering arises when the "behavior [of some variables] is unconstrained except by continuity" (Kuipers et al., 1991, p. 345). In optimal control and dynamic programming, "chattering refers to fast oscillations of the optimal control switching infinitely many times over a finite time interval" (Caponigro et al., 2018, p. 2046; see Zelikin and Borisov, 1994 for applications); Artstein (1989) introduces a theory of chattering systems and Wagner (2014) the concept of policy chattering. In statistics and econometrics, the term is generally used (see McFadden, 1989) for the situation in which an optimization algorithm oscillates around the optimal value.

There are two distinct aspects related to chattering in statistics and econometrics. The first is a theoretical aspect, linked to the limit of the objective function in an optimization problem; the second is a computational problem linked to the type of optimization routine that is used, whether it is derivative-free or derivative-based. Let us start from the second problem. Suppose to consider a derivative-based algorithm. Since, based on the theoretical problem, the function is rugged and nowhere differentiable, the derivatives do not exist. However, as numerical derivatives are generally calculated using the representation of the derivative as a limit of the difference quotient, the algorithms use finite derivatives whose value is random and determined by the roughness of the function at the point. It is not clear how this should converge. Now, suppose to use derivative-free algorithms. In this case, we can imagine that the algorithm converges to the global minimum of the objective function seen in the previous theoretical point. But here the theoretical aspect kicks in: the asymptotic distribution of this theoretical limit is expected to depend dramatically on chattering. This theoretical problem persists even in the absence of computational issues.

In this paper, we consider the issue of finding the zero of an objective function $\boldsymbol{\theta} \mapsto F(\boldsymbol{\theta})$ on the basis of a noisy version of this function, say $\boldsymbol{\theta} \mapsto \hat{F}(\boldsymbol{\theta})$. We give four main contributions to the literature.

First, we provide new general findings that do not depend on a particular choice of the approximations of the gradient and the Hessian. We consider separate results for INM and SAS. In particular, when $\hat{F}(\boldsymbol{\theta})$ is a deterministic approximation of $F(\boldsymbol{\theta})$, we characterize the rate of convergence to the true value $\boldsymbol{\theta}^\star$ and, when $\hat{F}(\boldsymbol{\theta})$ involves a stochastic element, we bound the escape probability, that is the probability that the sequence $\left\{\boldsymbol{\theta}^{(i)}\right\}$ gets out of a neighborhood of $\boldsymbol{\theta}^\star$ in less than n steps. Then, we compute an upper bound on the probability that $\boldsymbol{\theta}^{(i)}$ from SAS never visits a region where the gradient is near to zero as a function of the number of steps. These results have, to the best of our knowledge, never been proved before and, as the study of the convergence properties of these algorithms is still a hot topic, this is, in our view, a substantial contribution to the literature.

Second, we propose a class of algorithms in which the first one or two derivatives of $F(\boldsymbol{\theta})$ are replaced by those of a locally approximating function $\tilde{F}(\boldsymbol{\theta})$. We will call our algorithm generalized inexact Newton method (GINM). In the following, we illustrate the method by applying it to both INM and SAS. We choose P points $\mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right) = \{\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_P\}$ in a neighborhood of $\boldsymbol{\theta}^{(i)}$, and we estimate the values $\widehat{F}(\boldsymbol{\theta})$ for $\boldsymbol{\theta} \in \mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)$. Then, we fit a regression $\widetilde{F}(\boldsymbol{\theta})$ on the values assumed by $\widehat{F}(\boldsymbol{\theta})$ in $\boldsymbol{\theta} \in \mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)$ and we use the estimated coefficients to calculate its first and second derivatives (i.e. only the first derivative for SAS and both the first and second derivatives for INM). We then use these approximated derivatives in the optimization routine (see Section 3.2 for more details). As we claim generality for our procedure, in the paper we also consider the case in which $\widetilde{F}(\boldsymbol{\theta})$ is estimated by a more general polynomial of degree D. It is worth noting that, with respect to the classical implementation of the Newton–Raphson algorithm with numerical derivatives, the technique only requires one further step, i.e. the estimation of a local regression and applies also to non-differentiable approximations $\widehat{F}(\boldsymbol{\theta})$ of the function $F(\boldsymbol{\theta})$.

Third, we also produce some new results on the least squares approximation of a function and of its derivatives in a point $\boldsymbol{\theta}_0$ and in a neighborhood of $\boldsymbol{\theta}_0$ on the basis of the approximated values of the function in a set of points. In particular, we give some bounds on the approximation of the function $\widetilde{F}(\boldsymbol{\theta})$ and its derivatives. These results are of independent interest and can be adapted to different situations.

Fourth, we draw together the previous results and we characterize the deterministic rate of convergence and the escape probability of GINM. These results show that the properties of the algorithm depend on some key quantities: the size of the neighborhood $\mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)$, the degree D of the polynomial $\widetilde{F}(\boldsymbol{\theta})$, the size of the error $\widehat{F}(\boldsymbol{\theta}) - F(\boldsymbol{\theta})$ for $\boldsymbol{\theta} \in \mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)$, and, for the SAS version, the step size. If these quantities are modified during the iterations of the optimization routine, one can get convergence to the optimum of $F(\boldsymbol{\theta})$. We give some results ensuring that this is the case. Moreover, we discuss several aspects connected with the practical implementation of the algorithm.

Despite the results of the paper, and the particular algorithm proposed, apply rather generally, a motivating example is the situation in which the approximation $\widehat{F}(\boldsymbol{\theta})$ is obtained through simulations, as in the method of simulated moments (MSM, see, e.g., McFadden, 1989; Pakes and Pollard, 1989), simulated maximum likelihood (see, e.g., Lee, 1992, 1995) and indirect inference (Gouriéroux et al., 1993; Smith Jr., 1993).

When dealing with simulation-based estimation methods, the use of the derivatives of the approximation $\widehat{F}(\boldsymbol{\theta})$ in an optimization algorithm can be complicated by the fact that the dependence of $\widehat{F}(\boldsymbol{\theta})$ on $\boldsymbol{\theta}$ may not be smooth enough. The reason is that optimization algorithms usually rely on some continuity properties of the mapping $\boldsymbol{\theta} \mapsto \widehat{F}(\boldsymbol{\theta})$.

The condition that is generally used to enforce a sufficient degree of continuity is stochastic equicontinuity (see, e.g., McFadden, 1989, Pakes and Pollard, 1989 and Newey and McFadden, 1994, pp. 2136-2137). This often boils down to the requirement that a simulated process can be expressed in terms of innovations that are drawn once and for all at the beginning of the algorithm and kept constant throughout its execution. It is generally expressed saying that the innovations of the model are recycled for different values of $\boldsymbol{\theta}$. This removes the problem of chattering that was first outlined by McFadden (1989, p. 999). In his contribution, the author specifies that "a simulator must avoid 'chatter' as θ varies; this will generally require that the Monte Carlo random numbers used to construct $f(\theta)$ *not* be redrawn when θ is changed". This concept was then extended by Gouriéroux and Monfort (1996, p. 16). The authors pointed out that "it is *necessary* to keep these basic drawings [i.e. the innovations] fixed when θ changes, in order to have good numerical and statistical properties of the estimators based on these simulations". This was also stressed, among others, by Hall et al. (2012, p. 505) and Kristensen and Shin (2012, p. 78) in more recent contributions. Stochastic equicontinuity ensures that the function is not "rough" or "rugged".

However, this has two drawbacks. First, the recommendations of McFadden (1989) and Gouriéroux and Monfort (1996) do not hold for some complex models such as network models and simulation-based models characterized by strong heterogeneity, in which recycling the drawings is simply not possible. Second, algorithms using recycled innovations only provide the optimum of $\widehat{F}(\boldsymbol{\theta})$, and not of $F(\boldsymbol{\theta})$, and often require modifications of the classical asymptotic results (e.g., an inflation of the covariance matrix of the estimators). To overcome these drawbacks, the researcher can apply GINM in order to recover the optimum of $F(\boldsymbol{\theta})$ and apply the classical asymptotic theory.

The paper is structured as follows. In Section 2 we introduce some notation and preliminary results. In Section 3 we summarize the convergence results of the Newton-based optimization algorithms. In particular, in Section 3.1 we give a general introduction to inexact Newton methods, in Section 3.2 we describe GINM and in Section 3.3 we outline the differences between GINM and other optimization, approximation and estimation methods advanced in the literature. The main theoretical results are contained in Section 4: in Section 4.1 we produce two general convergence results, one for INM and one for biased SAS, as explained above; in Section 4.2 we give some results on the least squares approximation of a function in a point $\boldsymbol{\theta}_0$ and in a neighborhood of $\boldsymbol{\theta}_0$; in Section 4.3 and 4.4, we study the properties of our regression-based inexact Newton method or GINM. In Section 5 we treat the computational aspects of the GINM. Finally, the results of the simulations are exposed in Section 6. In particular, in Section 6.1 we perform an extensive Monte Carlo Experiment in which we estimate the mean of a Gaussian random variable, in the presence of chattering, by varying some quantities of the algorithm. Section 7 concludes. The proofs of the theoretical results are contained in Section 8.

2 Notation and Preliminary Results

In this section, we expose some notation and some preliminary results that will be used throughout the paper.

We use small indexed letters, like c_1, c_2, etc., or C_1, C_2, etc., for constants that are defined inside a theorem but may differ from a theorem to another. We use K_1, K_2, etc., for absolute constants that are different from one place to another and do not appear in the statement of the results but only in the proofs.

We write $\mathbb{N}$ for the positive integers, $\mathbb{N}_0$ for the non-negative integers and $\mathbb{R}$ for the real numbers. We denote sequences indexed by $\mathbb{N}$ or $\mathbb{N}_0$ as $\{a_n\}$. When $n \to \infty$, we use $a_n \simeq b_n$ when $a_n = b_n(1 + o(1))$, $a_n \asymp b_n$ when $b_n/C \leq a_n \leq Cb_n$ for $\infty > C > 0$ and n large enough, $a_n \ll b_n$ when $a_n = o(b_n)$, $a_n \lesssim b_n$

when $a_n \leq b_n (1 + o(1))$.

We use capital bold letters, such as $\mathbf{A}$, to denote matrices and lowercase bold letters, such as $\mathbf{a}$, to denote vectors. Let $\boldsymbol{\iota}$ be a vector of ones, $\mathbf{U}$ a square matrix of ones, $\mathbf{I}$ the identity matrix, $\mathbf{0}$ a matrix or a vector of zeros. The dimensions are generally clear from the context. If a confusion is possible, the dimension will be indicated through an index, as in $\boldsymbol{\iota}_N$. For a vector $\mathbf{a}$, let $\bar{\mathbf{a}}$ be the vector containing the reciprocals of the elements of $\mathbf{a}$. Let $\operatorname{dg}(\mathbf{a})$ be a diagonal matrix having $\mathbf{a}$ on its diagonal. Let $\operatorname{tr}(\mathbf{A})$ be the trace of $\mathbf{A}$, i.e. the sum of the diagonal elements of a square matrix $\mathbf{A}$. For a suitable matrix $\mathbf{A}$, $\mathbf{A}'$ is its transpose, $\mathbf{A}^\star$ its conjugate transpose, $\mathbf{A}^{-1}$ its inverse and $\mathbf{A}^+$ its Moore–Penrose pseudoinverse. The element-wise power of a vector or a matrix is denoted by $\mathbf{A}^{\odot b}$ (so that $\bar{\mathbf{a}} = \mathbf{a}^{\odot(-1)}$), $\mathbf{A}^b$ is the usual power obtained multiplying $\mathbf{A}$ by itself b times and $\mathbf{A}^{\otimes b}$ is the Kronecker multiplication of b copies of $\mathbf{A}$. The element of $\mathbf{A}$ in position (i, j) is denoted as $\mathbf{A}_{ij}$ or $[\mathbf{A}]_{ij}$; the matrix with generic element a_{ij} is denoted $[a_{ij}]$.

When applied to a vector $\mathbf{a}$, the notation $\|\cdot\|_p$ denotes the vector norm defined as $\|\mathbf{a}\|_p := \left(\sum_i |a_i|^p\right)^{\frac{1}{p}}$; when applied to a matrix $\mathbf{A}$, it denotes the matrix norm induced by the vector norm as $\|\mathbf{A}\|_p := \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p}$.

For a matrix $\mathbf{A}$, λ_i is the i-th eigenvalue of $\mathbf{A}$ and σ_i is the i-th singular value of $\mathbf{A}$, i.e. the square root of the i-th non-negative eigenvalue of $\mathbf{A}^\star\mathbf{A}$. The condition number of the matrix is $\kappa(\mathbf{A}) := \frac{\sigma_{\max}(\mathbf{A})}{\sigma_{\min}(\mathbf{A})}$ where $\sigma_{\max}$ ($\sigma_{\min}$) is the largest (smallest) singular value of $\mathbf{A}$. If the matrix is normal, $\kappa(\mathbf{A}) = \frac{|\lambda_{\max}(\mathbf{A})|}{|\lambda_{\min}(\mathbf{A})|}$, where $\lambda_{\max}$ ($\lambda_{\min}$) is the largest (smallest) eigenvalue by modulus of $\mathbf{A}$.

The symbol $\oplus$ denotes the Minkowski sum of sets, i.e. $A \oplus B := \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in A, \mathbf{b} \in B\}$. For a set A, $\mathring{A}$ denotes the interior of A and $\overline{A}$ its closure.

If D is the order of a polynomial, $\mathbb{P}_D$ is the space of polynomials of order D in $\mathbf{x}$.

Consider a function f defined on $\mathbb{R}^n$. Given a multi-index $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}_0^n$, we denote as $|\nu|$ the sum $\nu_1 + \cdots + \nu_n$ and we define the partial derivative:

$$D^\nu f := \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1} \cdots \partial x_n^{\nu_n}} f.$$

For lower-order derivatives, we also write:

$$D^i f := \frac{\partial f}{\partial x_i}, \qquad D^{ij} f := \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

We also use the following simplified notation. One overdot, as in $\dot{f} := \left[D^i f\right]$, denotes the gradient, i.e. the vector containing the first derivatives of a function with respect to the elements of its vector argument, two overdots, as in $\ddot{f} := \left[D^{ij} f\right]$, denote the Hessian, i.e. the matrix containing the second derivatives of a function with respect to the elements of its vector argument.

Let $\overline{\Omega}$ be a compact domain. We define the norms:

$$\|f\|_{\overline{\Omega}} := \sup_{\mathbf{x}\in\overline{\Omega}} |f(\mathbf{x})|$$

and:

$$|f|_q := \max_{|\nu|\leq q} \sup_{\mathbf{x}\in\overline{\Omega}} |D^\nu f(\mathbf{x})|.$$

Let us assume the following property.

R1 There exists a number $\gamma \geq 1$ such that any two points $\mathbf{x}, \mathbf{y} \in \overline{\Omega}$ can be joined by a rectifiable curve $\Gamma \subset \overline{\Omega}$ with length $|\Gamma| \leq \gamma \|\mathbf{x} - \mathbf{y}\|_2$.

A function $f : \overline{\Omega} \to \mathbb{R}$ is of class $\mathcal{C}^q$ in $\overline{\Omega}$ iff functions $D^k f(\mathbf{x})$ and $R_k(\mathbf{x};\mathbf{y})$, with $|k| \le q$, exist in $\overline{\Omega}$ such that the following Taylor's formula holds:

$$D^k f(\mathbf{x}) = \sum_{|s| \le q-|k|} \frac{1}{s!} D^{k+s} f(\mathbf{y}) (\mathbf{x}-\mathbf{y})^s + R_k(\mathbf{x};\mathbf{y})$$

for $\mathbf{x},\mathbf{y} \in \overline{\Omega}$. A function $f : \overline{\Omega} \to \mathbb{R}$ is of class $\mathcal{C}^{q,1}$ in $\overline{\Omega}$ iff f is of class $\mathcal{C}^q$ in $\overline{\Omega}$ and the partial derivatives $D^k f$ of order q are Lipschitz continuous in $\overline{\Omega}$. We define the semi-norm $|\cdot|_{q,1}$ as

$$|f|_{q,1} := \sup \left\{ \frac{\left|D^k f(\mathbf{x}_1) - D^k f(\mathbf{x}_2)\right|}{\|\mathbf{x}_1 - \mathbf{x}_2\|_2} : \mathbf{x}_1, \mathbf{x}_2 \in \overline{\Omega}, \mathbf{x}_1 \neq \mathbf{x}_2, |k| = q \right\}. \tag{2.1}$$

The following lemma, taken from Lemma 1 in Zuppa (2003), collects results from Whitney (1934).

Lemma 1. *Let $\overline{\Omega}$ satisfy assumption **R1** and let f be of class $\mathcal{C}^{q,1}$ in $\overline{\Omega}$. Then, for every $\mathbf{x},\mathbf{y} \in \overline{\Omega}$:*

$$|R_k(\mathbf{x};\mathbf{y})| \le \frac{n^{q-|k|}}{(q-|k|-1)!} \gamma^{q-|k|} \|\mathbf{x}-\mathbf{y}\|_2^{q-|k|+1} |f|_{q,1}$$

where it is intended that $(-1)! = 1$.

We also need some naming conventions concerning the convergence properties of optimization and root-finding algorithms. These are characterized by the construction of a series of values $\boldsymbol{\theta}^{(i)}$ that should approach $\boldsymbol{\theta}^\star$. The convergence properties can be summarized as follows (see, e.g., Dembo et al., 1982, p. 403).

Definition 1. If $\left\{\boldsymbol{\theta}^{(i)}\right\}$ is a sequence converging to $\boldsymbol{\theta}^\star$ and $\|\cdot\|$ is a norm, we say that

1. $\boldsymbol{\theta}^{(i)} \to \boldsymbol{\theta}^\star$ linearly if there is $\mu \in (0,1)$ such that

$$\limsup_{i\to\infty} \frac{\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star\right\|}{\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|} = \mu;$$

2. $\boldsymbol{\theta}^{(i)} \to \boldsymbol{\theta}^\star$ superlinearly if

$$\limsup_{i\to\infty} \frac{\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star\right\|}{\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|} = 0;$$

3. $\boldsymbol{\theta}^{(i)} \to \boldsymbol{\theta}^\star$ with (strong) order at least q, with $q > 1$, if

$$\limsup_{i\to\infty} \frac{\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star\right\|}{\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|^q} < +\infty;$$

   in particular, if $q = 2$, the algorithm is said to be quadratically convergent;

4. $\boldsymbol{\theta}^{(i)} \to \boldsymbol{\theta}^\star$ with weak order at least q, with $q > 1$, if

$$\limsup_{i\to\infty} \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|^{\frac{1}{q^i}} < 1.$$

3 Description of the Algorithm

In this section we describe the algorithm. Let us define the (uncomputable) function $F(\cdot)$, the (computable) function $\hat{F}(\cdot)$ and the function $\tilde{F}(\cdot)$, which is an approximation of $F(\cdot)$ depending on $\widehat{F}(\cdot)$.

3.1 Inexact Newton Algorithms

Suppose that we want to identify $\boldsymbol{\theta}^{\star}$ such that:

$$\dot{F}(\boldsymbol{\theta}^{\star}) \equiv \mathbf{0}$$

where $\dot{F}(\cdot) : \mathbb{R}^K \to \mathbb{R}^K$. Let us start from a point $\boldsymbol{\theta}^{(0)}$ and use the recurrence equation:

$$\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\left(\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{(i)}\right) = -\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)$$

or:

$$\boldsymbol{\theta}^{(i+1)} = \boldsymbol{\theta}^{(i)} - \left[\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right]^{-1} \dot{F}\left(\boldsymbol{\theta}^{(i)}\right).$$

This algorithm is called Newton–Raphson method and is known to be quadratically convergent.

It is sometimes possible to approximate the method by using an inexact Newton method (INM) defined by

$$\boldsymbol{\theta}^{(i+1)} = \boldsymbol{\theta}^{(i)} - \left[\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right]^{-1} \dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right) \tag{3.1}$$

where the functions $\dot{\tilde{F}}(\cdot)$ and $\ddot{\tilde{F}}(\cdot)$ replace $\dot{F}(\cdot)$ and $\ddot{F}(\cdot)$. We can also write this as

$$\begin{aligned}
\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\left(\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{(i)}\right) &= -\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right), \\
\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\left(\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{(i)}\right) &= -\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) + \mathbf{r}^{(i)},
\end{aligned} \tag{3.2}$$

where

$$\mathbf{r}^{(i)} = \dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\left[\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right]^{-1} \dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right).$$

A special case concerns stochastic approximation schemes (SAS). In this case the inverse (approximate) Hessian at step i, $\left[\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right]^{-1}$, is replaced in (3.1) by the step size γ_{i+1}:

$$\boldsymbol{\theta}^{(i+1)} = \boldsymbol{\theta}^{(i)} - \gamma_{i+1}\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right). \tag{3.3}$$

In Dembo et al. (1982), it is shown that the performance of the inexact Newton method in (3.1) depends on the ratio $\frac{\left\|\mathbf{r}^{(i)}\right\|}{\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|}$ and in particular on the fact that

$$\frac{\left\|\mathbf{r}^{(i)}\right\|}{\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|} \leq \eta_i$$

for $\eta_i \geq 0$, where $\{\eta_i\}$ is called a forcing sequence and controls the level of accuracy of the algorithm.

In Ypma (1984), it is shown that one can consider instead

$$\frac{\left\| \left[\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right) \right]^{-1} \mathbf{r}^{(i)} \right\|}{\left\| \left[\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right) \right]^{-1} \dot{F}\left(\boldsymbol{\theta}^{(i)}\right) \right\|} \leq \nu_i$$

for $\nu_i \geq 0$, where $\{\nu_i\}$ plays the same role of $\{\eta_i\}$ above. Morini (1999) generalizes the previous treatments to the case of preconditioning, in which $\left[\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right) \right]^{-1}$ is replaced by a generic matrix $\mathbf{P}^{(i)}$.

The previous results hold under so-called residual control-type conditions, i.e. conditions based on the control of the residual $\mathbf{r}^{(i)}$. However, in the following we will provide direct conditions in terms of the distance $\left\| \boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star} \right\|$.

3.2 Approximating Algorithm

In this section, we describe the simplest versions of the proposed algorithm. For stochastic approximation schemes, since SAS do not need the computation of the Hessian matrix, the algorithm simplifies as follows:

1. for any $i \geq 0$, we select P points $\mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right) = \{\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_P\}$ in a neighborhood of $\boldsymbol{\theta}^{(i)}$;
2. we compute $\hat{F}(\boldsymbol{\theta})$ for $\boldsymbol{\theta} \in \mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)$;
3. we estimate a linear function through the regression:

$$\hat{F}(\boldsymbol{\theta}) = \beta_1 + \boldsymbol{\beta}_2' \boldsymbol{\theta} + \varepsilon$$

   for $\boldsymbol{\theta} \in \mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)$;
4. we define $\tilde{F}(\boldsymbol{\theta}) = \hat{\beta}_1 + \hat{\boldsymbol{\beta}}_2' \boldsymbol{\theta}$ and $\dot{\tilde{F}}(\boldsymbol{\theta}) = \hat{\boldsymbol{\beta}}_2'$, and we replace $\dot{\tilde{F}}(\boldsymbol{\theta})$ in (3.3).

For inexact Newton methods, we have:

1. for any $i \geq 0$, we select P points $\mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right) = \{\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_P\}$ in a neighborhood of $\boldsymbol{\theta}^{(i)}$;
2. we compute $\hat{F}(\boldsymbol{\theta})$ for $\boldsymbol{\theta} \in \mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)$;
3. we estimate a quadratic function through the regression:

$$\hat{F}(\boldsymbol{\theta}) = \beta_1 + \boldsymbol{\beta}_2' \boldsymbol{\theta} + \boldsymbol{\beta}_3' \mathbf{D}_K^{+} (\boldsymbol{\theta} \otimes \boldsymbol{\theta}) + \varepsilon$$

   for $\boldsymbol{\theta} \in \mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)$, where $\mathbf{D}_K^{+}$ is the Moore–Penrose inverse of the duplication matrix (see Magnus and Neudecker, 2019, p. 56);
4. we define $\tilde{F}(\boldsymbol{\theta}) = \hat{\beta}_1 + \hat{\boldsymbol{\beta}}_2' \boldsymbol{\theta} + \hat{\boldsymbol{\beta}}_3' \mathbf{D}_K^{+} (\boldsymbol{\theta} \otimes \boldsymbol{\theta})$, $\dot{\tilde{F}}(\boldsymbol{\theta}) = \hat{\boldsymbol{\beta}}_2' + (\boldsymbol{\theta} \otimes \mathbf{I}_K + \mathbf{I}_K \otimes \boldsymbol{\theta})' \mathbf{D}_K^{+,\prime} \hat{\boldsymbol{\beta}}_3$ and $\ddot{\tilde{F}}(\boldsymbol{\theta}) = 2\mathbf{D}_K^{+,\prime} \hat{\boldsymbol{\beta}}_3$, and we replace them in (3.1).

In the following we will also cover the case in which the linear or quadratic functions are replaced by more general polynomials of degree D.

3.3 Relation with Other Approximation Algorithms

In this section we investigate the relation between the proposed algorithm and other optimization, approximation and estimation methods advanced in the literature.

The area of research that is nearest to our algorithm is the one concerning derivative free optimization (see, e.g., Conn and Toint, 1996; Conn et al., 1997; the latter contains, in Section 2, a history of the method), i.e. optimization methods in which derivatives are not known or computable. A subgroup of these algorithms forms what are called trust-region methods. Our method shares with these ones the fact of identifying a region, that they call trust region and whose radius decreases when the algorithm progresses, containing a set of points that are used to produce an approximation to the objective function. Three crucial differences between our method and these ones are that (i) they do not try to approximate the derivatives but only the function, hence the name of derivative free methods, (ii) the pointset in the trust region recycles points from the previous steps, and (iii) the approximation to the objective function is usually obtained through interpolation.

The approximation step of the algorithm has a relation with spectral (see Boyd, 2001) and pseudospectral (see Fornberg and Sloan, 1994) methods for the computation of derivatives and the solution of partial differential equations. The rationale of the methods is to approximate the function as a sum of smooth basis functions such that the computation of the derivatives of the function can be easily performed. There are some important differences: (i) these methods are generally intended to produce a global approximation to the function, while our method yields a local one; (ii) these methods approximate unknown functions and their derivatives that are then replaced into a partial differential equation, without any direct involvement of data; (iii) the smooth basis functions are generally more complex than plain polynomials. However, the two methods share the idea of approximating a function with a sum of basis functions and using this approximation to compute approximate derivatives. In our case, the choice of the basis function is justified by the fact that we scale differently the points of the pointset $\mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)$ as far as the algorithm progresses.

It also has some contacts with techniques like local polynomial regression (see Fan and Gijbels, 1996) or moving least squares (see, e.g., Lancaster and Salkauskas, 1981). In these techniques, one observes a set of couples $(y_i, \mathbf{x}_i)$ for $i = 1, \ldots, n$, composed of a response $y_i \in \mathbb{R}$ and explanatory variables $\mathbf{x}_i \in \mathbb{R}^k$. For any point $\mathbf{x} \in \mathbb{R}^k$, one can estimate the corresponding expected value of y by weighting the observations $(y_i, \mathbf{x}_i)$ according to the distance between $\mathbf{x}$ and $\mathbf{x}_i$. Our technique differs from these ones because our points are not fixed or predetermined before the estimation is performed, there is no weighting of the observations and we put a particular emphasis on the size of the neighborhood of the pointset $\mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)$.

Another related technique is the generalized finite difference method (see Jensen, 1972; Benito et al., 2001). In this method, a function is locally approximated through its Taylor expansion and the derivatives are estimated, together with the value of the function, by interpolation or weighted least squares estimation using observations on an irregular grid. The aim of this method is the approximation of the derivatives of a function for the solution of partial differential equations, as for spectral and pseudospectral methods, and not the overall approximation of a surface, as for moving least squares. However, the method shares with moving least squares the ability to accommodate irregularly spaced observations. This method is rather similar to our local approximation technique, despite the use is radically different.

Finally, it is worth mentioning the contribution by Forneron (2023). In this work, the author develops a new Gauss–Newton algorithm combining non-smooth moments with smoothed Jacobian estimates. A grid-search step is also added to each iteration to reach global convergence. Our method differs from this as we do not use a grid-search approach, rather we use a set of points to approximate the objective function whose radius reduces as the algorithm progresses. Moreover, our approach is more general as we do not directly focus on estimation of moment condition models.

4 Theoretical Results

In this section we will devise conditions on the proposed algorithm under which $\left\{\boldsymbol{\theta}^{(i)}\right\}$ converges to $\boldsymbol{\theta}^{\star}$ and we will provide results on the convergence rates. We will first consider, in Section 4.1, some general results on optimization algorithms that do not depend on the specific choice of $\dot{\tilde{F}}(\cdot)$ and $\ddot{\tilde{F}}(\cdot)$ outlined in Section 3.2 and may be of independent interest. They generalize some results in the optimization and machine learning literatures. In Section 4.2 we compute upper bounds on the approximation error of $\tilde{F}(\cdot)$, $\dot{\tilde{F}}(\cdot)$ and $\ddot{\tilde{F}}(\cdot)$. In Sections 4.3 and 4.4 we then give some results on the algorithm described in Section 3.2 in conjunction with the results of Section 4.1.

4.1 Optimization Results

We first characterize the properties of the function $F(\cdot)$ in four assumptions.

Opt The function $F : \boldsymbol{\theta} \mapsto F(\boldsymbol{\theta})$ is defined on a compact set $\Theta \subset \mathbb{R}^K$ and the parameter space contains an open neighborhood of a value $\boldsymbol{\theta}^{\star}$ such that $\dot{F}(\boldsymbol{\theta}^{\star}) = \mathbf{0}$.

Lip-1 The function $F : \boldsymbol{\theta} \mapsto F(\boldsymbol{\theta})$ is of class $\mathcal{C}^1$ on $\Theta \subset \mathbb{R}^K$ and $\left\|\dot{F}(\boldsymbol{\theta}_1) - \dot{F}(\boldsymbol{\theta}_2)\right\|_2 \leq L_1 \|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_2$ for $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathring{\Theta}$.

Lip-2 The function $F : \boldsymbol{\theta} \mapsto F(\boldsymbol{\theta})$ is of class $\mathcal{C}^2$ on $\Theta \subset \mathbb{R}^K$ and $\left\|\ddot{F}(\boldsymbol{\theta}_1) - \ddot{F}(\boldsymbol{\theta}_2)\right\|_2 \leq L_2 \|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_2$ for $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathring{\Theta}$.

Hess $\min_{\boldsymbol{\theta}\in\Theta} \lambda_{\min}\left[\ddot{F}(\boldsymbol{\theta})\right] \geq m > 0$.

Remark 1. Assumption **Opt** defines the target value $\boldsymbol{\theta}^{\star}$ as a solution of the first-order conditions of the optimization problem. Assumptions **Lip-**1 and **Lip-**2 require the function to be differentiable, respectively with Lipschitz gradient and Hessian. Assumption **Hess** concerns the smallest eigenvalue of the Hessian. It is equivalent to the requirement that the function is strongly convex (see Bertsekas et al., 2003, p. 72).

Let $\left\{\boldsymbol{\theta}^{(i)}\right\}$ be a sequence in $\Theta \subset \mathbb{R}^K$. Whenever the algorithm is stochastic, we need an assumption quantifying the effect of replacing the first derivative with its approximated value. We define the σ-algebra $\mathcal{F}_i = \sigma\left\{\boldsymbol{\theta}^{(i)}, \boldsymbol{\theta}^{(i-1)}, \dots, \tilde{F}\left(\boldsymbol{\theta}^{(i-1)}\right), \tilde{F}\left(\boldsymbol{\theta}^{(i-2)}\right), \dots \dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i-1)}\right), \dots\right\}$ containing events known prior to step i, including the value of $\boldsymbol{\theta}^{(i)}$ and of all the derivatives that are needed for the algorithm to run. The collection of these σ-algebras for $i \geq 1$ constitutes a filtration.

MaV $\left\|\mathbb{E}\left[\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)|\mathcal{F}_i\right] - \dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 \leq b_i$, $\mathbb{E}\left[\left\|\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right) - \dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2 |\mathcal{F}_i\right] \leq \sigma_i$.

Remark 2. We note that

$$\begin{aligned}\left\|\mathbb{E}\left[\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)|\mathcal{F}_i\right] - \dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 &\leq \mathbb{E}\left[\left\|\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right) - \dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 |\mathcal{F}_i\right] \\ &\leq \left(\mathbb{E}\left[\left\|\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right) - \dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2 |\mathcal{F}_i\right]\right)^{\frac{1}{2}},\end{aligned}$$

so that one could always take $b_i \leq \sigma_i^{\frac{1}{2}}$. The inequality becomes an equality in the deterministic case.

We now provide two sets of results. First, we study the general inexact Newton algorithm in (3.1). We characterize the behavior of $\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^{\star}\right\|_2$ as a function of $\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2$ and we compute a rate of convergence for deterministic algorithms. Moreover, supposing that the algorithm is stochastic, we study the escape probability of the algorithm, i.e. the probability that it gets out of the ball of radius Δ centered in $\boldsymbol{\theta}^{\star}$ in less than n steps. Second, as the previous results do not work well for stochastic approximation schemes, we give analogous results for the particular case in (3.3).

4.1.1 Inexact Newton Method

The first set of results characterizes the behavior of $\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^{\star}\right\|_2$ in terms of $\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2$. It provides an upper bound for deterministic algorithms (or an almost sure upper bound for stochastic algorithms). We do not produce a residual control-type condition (see Section 3.1), rather we directly express the result in terms of the relation between $\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^{\star}\right\|_2$ and $\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2$. We will need the definitions:

$$\begin{aligned}\delta_1^{(i)} &:= \left\|\dot{\hat{F}}\left(\boldsymbol{\theta}^{(i)}\right)-\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2,\\ \delta_2^{(i)} &:= \left\|\ddot{\hat{F}}\left(\boldsymbol{\theta}^{(i)}\right)-\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2.\end{aligned}$$

Theorem 1. *Under* ***Opt*** *and* ***Lip*****-2***, provided* $\delta_2^{(i)}<\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right)$,

$$\begin{aligned}\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^{\star}\right\|_2 \leq \frac{1}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right)-\delta_2^{(i)}}&\left\{\delta_1^{(i)}+\delta_2^{(i)}\frac{\left\|\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right\|_2}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right)}\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2\right.\\ &\left.+\left(\frac{3}{2}-\frac{\delta_2^{(i)}}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right)}\right)L_2\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2^2\right\}.\end{aligned}$$

Remark 3. (i) Using the first inequality in Lemma 3 instead of the second one, one gets a similar inequality for $\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^{\star}\right\|_2$:

$$\begin{aligned}\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^{\star}\right\|_2 \leq \frac{1}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right)-\delta_2^{(i)}}&\left\{\delta_1^{(i)}+\delta_2^{(i)}\frac{L_1}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right)}\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2\right.\\ &\left.+\left(1-\frac{\delta_2^{(i)}}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right)}\right)\frac{3L_2}{2}\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2^2\right\}.\end{aligned}$$

(ii) When $\delta_1^{(i)}\equiv 0$ and $\delta_2^{(i)}\equiv 0$, we recover the quadratic convergence properties of the Newton–Raphson method. Indeed, in that case,

$$\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^{\star}\right\|_2 \leq \frac{3L_2}{2\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right)}\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2^2.$$

In order to retain also for the INM the quadratic convergence of the Newton–Raphson method, we need to have $\delta_1^{(i)}=O\left(\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2^2\right)$ and $\delta_2^{(i)}=O\left(\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2\right)$. Provided $\delta_1^{(i)}=o(1)$, $\delta_2^{(i)}=o(1)$ and

$\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2 = o(1)$, convergence is guaranteed to be linear if

$$\limsup_{i\to\infty} \frac{\delta_1^{(i)}}{\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2} < \lambda_{\min}\left(\ddot{F}(\boldsymbol{\theta}^{\star})\right). \tag{4.1}$$

This shows that there is a difference between $\delta_1^{(i)}$ and $\delta_2^{(i)}$ as far as their impact on convergence rates is concerned.

(iii) This result can be used to study the impact of numerical differentiation in optimization algorithms, given $\delta_1^{(i)}$ and $\delta_2^{(i)}$ are interpreted as the errors arising in numerical differentiation of the first and of the second derivative. Let us suppose to use higher-order finite differences of order $Q \geq 1$ and step size ε. From Hong et al. (2015, p. 251), $\delta_1^{(i)} = O\left(\varepsilon^{2Q}\right)$ and $\delta_2^{(i)} = O\left(\varepsilon^{2Q-1}\right)$. Therefore,

$$\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^{\star}\right\|_2 = O\left(\varepsilon^{2Q} + \varepsilon^{2Q-1}\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2 + \left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2^2\right).$$

The case with $Q = 1$ corresponds to the usual difference quotient. Note that this confirms the statement that "one needs to adjust the step size as a function of the sample size" in (Hong et al., 2015, p. 250). Indeed, provided $\left\|\boldsymbol{\theta}^{(0)}-\boldsymbol{\theta}^{\star}\right\|_2 \leq \Delta$ and ε is small enough, $\limsup_{i\to\infty}\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2 = O\left(\varepsilon^{2Q}\right)$. If the empirical cumulative distribution function $\widehat{\mathbb{P}}_{\mathbf{y}}$ entering into the definition of $\boldsymbol{\theta}^{\star}$ is based on a large sample size, the error $O\left(\varepsilon^{2Q}\right)$ can turn out to be significant with respect to the distance between $\boldsymbol{\theta}^{\star}$ and the minimizer of $F(\cdot)$, that is the pseudo-true value of the optimization problem.

(iv) Using the bound, we can investigate what happens when the Hessian is regularized. In this case, we have $\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right) = \dot{F}\left(\boldsymbol{\theta}^{(i)}\right)$ and $\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right) = \ddot{F}\left(\boldsymbol{\theta}^{(i)}\right) + \lambda_i \mathbf{I}_K$. We then have

$$\begin{aligned}\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^{\star}\right\|_2 \leq & \frac{1}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right)-\lambda_i} \\ & \cdot\left\{\lambda_i \frac{\left\|\ddot{F}(\boldsymbol{\theta}^{\star})\right\|_2}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right)}\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2 + \frac{3L_2}{2}\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2^2\right\}.\end{aligned}$$

The algorithm retains its quadratic convergence if $\lambda_i = O\left(\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2\right)$.

(v) The result can be used to obtain inequalities for the score. Indeed, under **Lip-1,** from the second inequality in Lemma 2,

$$\left\|\dot{F}\left(\boldsymbol{\theta}^{(i+1)}\right)\right\|_2 \leq L_1\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^{\star}\right\|_2.$$

Corollary 1. *Under **Opt** and **Lip**-2, provided* $\delta_2^{(i)} < \lambda_{\min}\left(\ddot{F}(\boldsymbol{\theta}^{\star})\right) - L_2\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2$,

$$\begin{aligned}\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^{\star}\right\|_2 \leq & \frac{1}{\lambda_{\min}\left(\ddot{F}(\boldsymbol{\theta}^{\star})\right) - L_2\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2 - \delta_2^{(i)}} \\ & \cdot\left\{\delta_1^{(i)} + \delta_2^{(i)}\frac{L_1}{\lambda_{\min}\left(\ddot{F}(\boldsymbol{\theta}^{\star})\right) - L_2\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2}\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2\right.\end{aligned}$$

$$+ \left(1 - \frac{\delta_2^{(i)}}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^\star\right)\right) - L_2 \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2}\right) \frac{3L_2}{2} \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2^2 \Bigg\}.$$

Under ***Opt****,* ***Lip****-2 and* ***Hess****, provided* $\delta_2^{(i)} < m$*,*

$$\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star\right\|_2 \leq \frac{\delta_1^{(i)}}{m - \delta_2^{(i)}} + \frac{\delta_2^{(i)} L_1}{m\left(m - \delta_2^{(i)}\right)} \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 + \frac{3L_2}{2m} \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2^2.$$

Remark 4. This corollary starts from the bound in Remark 3 (i). Using the bound in Theorem 1 instead, one can obtain two different bounds. One is obtained from the first one replacing L_1 with $\left\|\ddot{F}\left(\boldsymbol{\theta}^\star\right)\right\|_2$ and $\frac{3}{2}\left(1 - \frac{\delta_2^{(i)}}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^\star\right)\right) - L_2\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2}\right)$ with $\left(\frac{3}{2} - \frac{\delta_2^{(i)}}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^\star\right)\right) - L_2\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2}\right)$. The other is obtained from the second one replacing L_1 with $\left\|\ddot{F}\left(\boldsymbol{\theta}^\star\right)\right\|_2$ and $\frac{3}{2m}$ with $\frac{3m - 2\delta_2^{(i)}}{2m\left(m - \delta_2^{(i)}\right)}$.

The following result for INM shows what happens when $\boldsymbol{\theta}^{(0)}$ is in a neighborhood of $\boldsymbol{\theta}^\star$, i.e. when $\left\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^\star\right\|_2 \leq \Delta$, a situation in line with Dembo et al. (1982, Theorem 2.3).

Theorem 2. *(i) Under* ***Opt****,* ***Lip****-2 and* ***Hess****, suppose that, for any* $i \geq 0$*, there are constants* $1 > c_1 > 0$*,* $c_2 \geq 0$*,* $c_3 > 0$*,* $\Delta > 0$*,* $\xi \geq 0$ *and* $\delta > 0$ *such that*

$$\left\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^\star\right\|_2 \leq \Delta,$$
$$\delta_1^{(i)} + \delta_2^{(i)}\left(1 + \frac{M}{m}\right)\Delta \leq \left(m - \frac{3L_2}{2}\Delta\right)\Delta,$$
$$\delta_2^{(i)} < m,$$
$$\frac{1}{m - \delta_2^{(i)}}\left(\frac{\delta_2^{(i)} M}{m} + \frac{3L_2}{2}\Delta\right) \leq c_1\left(1 + c_2 (i+1)^{-\xi}\right),$$
$$\frac{\delta_1^{(i)}}{m - \delta_2^{(i)}} \leq c_3 (i+1)^{-\delta}(1 + o(1)),$$

where $M = \min\left\{\left\|\ddot{F}\left(\boldsymbol{\theta}^\star\right)\right\|_2, L_1\right\}$*. Then, we have*

$$\left\|\boldsymbol{\theta}^{(n+1)} - \boldsymbol{\theta}^\star\right\|_2 \lesssim \frac{c_3 n^{-\delta}}{c_1 \left|\ln c_1\right|}.$$

(ii) Under ***Opt****,* ***Lip****-2 and* ***Hess****, suppose that, for any* $i \geq 0$*, there are constants* $1 > c_1 > 0$*,* $c_2 \geq 0$*,* $\Delta > 0$ *and* $\xi \geq 0$ *such that*

$$\left\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^\star\right\|_2 \leq \Delta,$$
$$\delta_1^{(i)} \equiv 0,$$
$$\delta_2^{(i)} \leq \frac{m - \frac{3L_2}{2}\Delta}{1 + \frac{M}{m}},$$
$$\frac{1}{m - \delta_2^{(i)}}\left(\frac{\delta_2^{(i)} M}{m} + \frac{3L_2}{2}\Delta\right) \leq c_1\left(1 + c_2 (i+1)^{-\xi}\right),$$

where $M = \min\left\{\left\|\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right\|_2, L_1\right\}$. *We have*

$$\left\|\boldsymbol{\theta}^{(n+1)} - \boldsymbol{\theta}^{\star}\right\|_2 \lesssim \begin{cases} \Delta e^{c_2\zeta(\xi)} c_1^{n+1} \exp\left\{\frac{c_2}{1-\xi} n^{1-\xi}\right\}, & \xi > 0, \xi \neq 1, \\ \Delta e^{c_2\gamma_{(0)}} c_1^{n+1} n^{c_2}, & \xi = 1. \end{cases}$$

Remark 5. The previous results are especially suitable in order to study the properties of deterministic algorithms. In this context, it is important to note that $\delta_1^{(i)}$ and $\delta_2^{(i)}$ play different roles. This is evident from Theorem 2. While $\delta_1^{(i)}$ must converge to 0, $\delta_2^{(i)}$ is not compelled to. In particular, when $\delta_1^{(i)} \neq 0$ and $\delta_2^{(i)} = o(1)$, one could take c_1, c_2 and c_3 such that $c_1 = \frac{3L_2}{2m}\Delta$ and

$$\begin{aligned} \delta_1^{(i)} &\leq mc_3 (i+1)^{-\delta} (1 + o(1)), \\ \delta_2^{(i)} &\leq \frac{m}{1 + \left(\frac{2M+3L_2\Delta}{3L_2\Delta c_2}\right)(i+1)^{\xi}} \lesssim \frac{3mL_2\Delta c_2}{2M + 3L_2\Delta}(i+1)^{-\xi}, \\ \delta_1^{(i)} + \delta_2^{(i)}\left(1 + \frac{M}{m}\right)\Delta &\leq \left(m - \frac{3L_2}{2}\Delta\right)\Delta. \end{aligned}$$

The case $\delta_1^{(i)} \neq 0$ and $\delta_2^{(i)} \neq o(1)$ is mainly associated with stochastic approximation schemes (see (3.3)). In order to simplify, let us take $\delta_2^{(i)} \equiv \delta_2 < m$. We can then take $c_1 = \frac{1}{m-\delta_2}\left(\frac{\delta_2 M}{m} + \frac{3L_2\Delta}{2}\right)$, $c_2 = 0$ and

$$\begin{aligned} \delta_1^{(i)} &\leq \left(m - \frac{3L_2}{2}\Delta - \delta_2\left(1 + \frac{M}{m}\right)\right)\Delta, \\ \delta_1^{(i)} &\leq c_3 (m - \delta_2)(i+1)^{-\delta}(1 + o(1)). \end{aligned}$$

However, we will show in Section 4.1.2 that better results can be obtained with a direct approach.

In the next result, we suppose that the approximation algorithm is stochastic. The following theorem characterizes an analysis of the escape probability of the algorithm. In particular, if we define $B := \boldsymbol{\theta}^{\star} \oplus \Delta\mathsf{B}$, the ball of radius Δ centered in $\boldsymbol{\theta}^{\star}$, it studies the probability that $\boldsymbol{\theta}^{(i)}$ gets out of B in the steps between 1 and $n+1$, i.e. $\mathbb{P}\left\{\max_{1\leq i\leq n+1}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 > \Delta\right\}$.

Theorem 3. *Under* ***Opt****,* ***Lip****-2 and* ***Hess****, provided* $\left\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^{\star}\right\|_2 \leq \Delta < \frac{2(1-\varepsilon)m}{3L_2}$ *for* $0 < \varepsilon < 1$*, we have*

$$\begin{aligned} &\mathbb{P}\left\{\max_{1\leq i\leq n+1}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 > \Delta\right\} \\ &\leq \frac{2\sum_{i=0}^{n}\mathbb{E}\left(\delta_1^{(i)}\right)^2}{\Delta^2\left((1-\varepsilon)m - \frac{3L_2}{2}\Delta\right)^2} \\ &\quad + \left[\frac{2\varepsilon^2M^2 + \left((1-\varepsilon)m - \frac{3L_2}{2}\Delta\right)^2}{\varepsilon^2m^2\left((1-\varepsilon)m - \frac{3L_2}{2}\Delta\right)^2}\right]\sum_{i=0}^{n}\mathbb{E}\left(\delta_2^{(i)}\right)^2 \end{aligned}$$

where $M > 0$ *is a constant such that* $\left\|\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right\|_2 \leq M$*. Moreover, for small enough* $\sum_{i=0}^{n}\mathbb{E}\left(\delta_1^{(i)}\right)^2 = O(1)$

and $\sum_{i=0}^{n} \mathbb{E}\left(\delta_2^{(i)}\right)^2 = O(1)$,

$$\max_{1 \le i \le n+1} \left\| \boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star \right\|_2 = O_{\mathbb{P}}\left(\left(\sum_{i=0}^{n} \mathbb{E}\left(\delta_1^{(i)}\right)^2 \right)^{\frac{1}{2}} \right).$$

Remark 6. It is clear that the probability $\mathbb{P}\left\{\max_{1\le i\le n+1} \left\| \boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star \right\|_2 > \Delta\right\}$ is positive and non-vanishing in almost all cases of interest. The reason is that, if the algorithm approximating the derivatives is stochastic, it cannot be ruled out that it gives rise to extreme values of the derivatives that lead $\boldsymbol{\theta}^{(i)}$ astray. However, the bound reproduces several stylized facts whose comprehension may help in the design of the algorithm. First, when $\Delta \downarrow 0$ while all other quantities are fixed, the bound blows up, as expected, because the probability of getting out of the ball is larger for smaller values of Δ. Second, for fixed or decreasing Δ, the bound converges to 0 if $\sum_{i=0}^{n} \mathbb{E}\left(\delta_1^{(i)}\right)^2 = o\left(\Delta^2\right)$ and $\sum_{i=0}^{n} \mathbb{E}\left(\delta_2^{(i)}\right)^2 = o(1)$. This confirms the different roles played by $\delta_1^{(i)}$ and $\delta_2^{(i)}$ (see Remark 5). Third, the bound increases with M and decreases with m, i.e. it increases with the condition number of the matrix $\ddot{F}(\boldsymbol{\theta}^\star)$. It is well known that the larger is the condition number the more eccentric are the level curves of the objective function, a condition associated with tenuous identification in the econometric literature (see Keane, 1992) and with slow convergence rates of gradient-based optimization algorithms (see, e.g., Alger, 2019, Section 3.1). The convergence rate of the Newton–Raphson algorithm does not depend on the condition number of the Hessian because it is affinely invariant. The presence of both m and M in the bound is due to the fact that this covers both the case in which the Hessian is correctly approximated and the optimization method is affinely invariant and the case in which it is not. This is witnessed by the fact that, if $\delta_2^{(i)} \equiv 0$, the bound only depends on m. The fact that the bound increases with L_2 confirms that the difficulty of the optimization problem plays a role in the escape probability. Fourth, the bound increases in n and it is necessary that $\mathbb{E}\left(\delta_1^{(i)}\right)^2 \downarrow 0$ and $\mathbb{E}\left(\delta_2^{(i)}\right)^2 \downarrow 0$ rapidly enough for it to converge. In order to have $\sum_{i=0}^{\infty} \mathbb{E}\left(\delta_1^{(i)}\right)^2 = O(1)$ and $\sum_{i=0}^{\infty} \mathbb{E}\left(\delta_2^{(i)}\right)^2 = O(1)$, we need $\mathbb{E}\left(\delta_1^{(i)}\right)^2 = o\left(i^{-1}\right)$ and $\mathbb{E}\left(\delta_2^{(i)}\right)^2 = o\left(i^{-1}\right)$. At last, when $\mathbb{E}\left(\delta_1^{(i)}\right)^2$ and $\mathbb{E}\left(\delta_2^{(i)}\right)^2$ decrease for a certain i, the bound decreases too.

4.1.2 Stochastic Approximation Schemes

The previous results are not well suited to study the algorithm in (3.3). This section gives different, specialized results for this kind of algorithm. The first one is the analog of Theorem 1 and Corollary 1.

Theorem 4. *Under **Opt** and **Lip**-2, if* $\gamma_{i+1}\lambda_{\min}\left(\ddot{F}(\boldsymbol{\theta}^\star)\right) < 1$,

$$\left\| \boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star \right\|_2 \le \left(1 - \gamma_{i+1}\lambda_{\min}\left(\ddot{F}(\boldsymbol{\theta}^\star)\right)\right) \left\| \boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star \right\|_2 + \gamma_{i+1}\frac{L_2}{2} \left\| \boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star \right\|_2^2 + \gamma_{i+1}\delta_1^{(i)}.$$

The second result parallels Theorem 2.

Theorem 5. *Suppose that **Opt** and **Lip**-2 hold and* $\frac{L_2}{2}\Delta < \lambda_{\min}\left(\ddot{F}(\boldsymbol{\theta}^\star)\right)$.
(i) Suppose that, for any $i \ge 0$, *there are constants* $\Delta > 0$, $\xi > 1$, $\gamma > 0$, $\delta > 0$, $c_1 > 0$, $c_2 \ge 0$ *and* $c_3 \ge 0$, *such that*

$$\left\| \boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^\star \right\|_2 \le \Delta,$$

$$
\begin{aligned}
\gamma_{i+1}\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) &< 1, \\
\delta_1^{(i)} &\le \left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right)\Delta, \\
\left|\gamma_{i+1} - c_1\left(i+1\right)^{-\gamma}\right| &\le c_2\left(i+1\right)^{-\xi}, \\
\delta_1^{(i)} &\le c_3\left(i+1\right)^{-\delta}\left(1+o\left(1\right)\right).
\end{aligned}
$$

If $\gamma < 1$,

$$
\left\|\boldsymbol{\theta}^{(n+1)} - \boldsymbol{\theta}^{\star}\right\|_2 \lesssim \frac{c_3 e^{2\left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right)c_2\zeta(\xi)} n^{-\delta}}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta}.
$$

If $\gamma = 1$,

$$
\left\|\boldsymbol{\theta}^{(n+1)} - \boldsymbol{\theta}^{\star}\right\|_2 \lesssim \begin{cases} \frac{c_1 c_3 e^{2\left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right)c_2\zeta(\xi)}}{\left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right)c_1 - \delta} n^{-\delta}, & \left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right)c_1 > \delta, \\ c_1 c_3 e^{2\left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right)c_2\zeta(\xi)} n^{-\delta}\ln n, & \left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right)c_1 = \delta, \\ C n^{-\left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right)c_1} & \left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right)c_1 < \delta, \end{cases}
$$

where

$$
\begin{aligned}
C \le{}& c_1 c_3 e^{2\left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right)c_2\zeta(\xi)} \sum_{k=0}^{\infty} \frac{\left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right)^k \left(\frac{2c_1+c_2}{2}\right)^k}{k!} \\
&\cdot \zeta\left(1 + \delta + k - \left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right)c_1\right) \\
&+ \Delta \exp\left\{\left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right)\left(c_2\zeta\left(\xi\right) - c_1\gamma_{(0)}\right)\right\}.
\end{aligned}
$$

If $\gamma > 1$, *there is no guarantee of convergence.*
(ii) Suppose that, for any $i \ge 0$, *there are constants* $\Delta > 0$, $\xi > 1$, $\gamma > 0$, $\delta > 0$, $c_1 > 0$ *and* $c_2 \ge 0$, *such that*

$$
\begin{aligned}
\left\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^{\star}\right\|_2 &\le \Delta, \\
\gamma_{i+1}\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) &< 1, \\
\left|\gamma_{i+1} - c_1\left(i+1\right)^{-\gamma}\right| &\le c_2\left(i+1\right)^{-\xi}, \\
\delta_1^{(i)} &\equiv 0.
\end{aligned}
$$

For $\gamma < 1$,

$$
\left\|\boldsymbol{\theta}^{(n+1)} - \boldsymbol{\theta}^{\star}\right\|_2 \lesssim \exp\left\{\left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right)\left(c_2\zeta\left(\xi\right) - c_1\zeta\left(\gamma\right) - \frac{c_1}{1-\gamma}n^{1-\gamma}\right)\right\},
$$

for $\gamma = 1$,

$$
\left\|\boldsymbol{\theta}^{(n+1)} - \boldsymbol{\theta}^{\star}\right\|_2 \lesssim \Delta \exp\left\{\left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right)\left(c_2\zeta\left(\xi\right) - c_1\gamma_{(0)}\right)\right\} n^{-\left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right)c_1}
$$

and, for $\gamma > 1$, there is no guarantee of convergence.

Remark 7. In the most relevant case, i.e. the one with $\delta_1^{(i)} \neq 0$ and $\gamma < 1$, the rate of convergence only depends on the rate of decrease of $\delta_1^{(i)}$, in the sense that $\left\|\boldsymbol{\theta}^{(n+1)} - \boldsymbol{\theta}^\star\right\|_2 = O\left(\delta_1^{(n)}\right)$. If $\gamma < 1$, the rate is therefore independent of the value of γ.

When the algorithm is stochastic, the following corollary provides a result on the escape probability analogous to Theorem 3.

Corollary 2. *Under **Opt** and **Lip**-2, we have*

$$\mathbb{P}\left\{\max_{1\le i\le n+1}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 > \Delta\right\} \le \frac{\sum_{i=0}^n \mathbb{E}\left(\delta_1^{(i)}\right)^2}{\Delta^2\left[\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^\star\right)\right) - \frac{L_2}{2}\Delta\right]^2}$$

provided $\gamma_{i+1} < \lambda_{\min}^{-1}\left(\ddot{F}\left(\boldsymbol{\theta}^\star\right)\right)$ for any i and $\Delta < \frac{2\lambda_{\min}\left(\ddot{F}(\boldsymbol{\theta}^\star)\right)}{L_2}$. Moreover,

$$\max_{1\le i\le n+1}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 = O_{\mathbb{P}}\left(\left(\sum_{i=0}^n \mathbb{E}\left(\delta_1^{(i)}\right)^2\right)^{\frac{1}{2}}\right).$$

Remark 8. The bound on $\mathbb{P}\left\{\max_{1\le i\le n+1}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 > \Delta\right\}$ does not depend on the learning sequence $\{\gamma_i\}$. This implies that

$$\sup_{\left\{\gamma_i:\gamma_{i+1}<\lambda_{\min}^{-1}\left(\ddot{F}(\boldsymbol{\theta}^\star)\right)\right\}} \mathbb{P}\left\{\max_{1\le i\le n+1}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 > \Delta\right\} \le \frac{\sum_{i=0}^n \mathbb{E}\left(\delta_1^{(i)}\right)^2}{\Delta^2\left[\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^\star\right)\right) - \frac{L_2}{2}\Delta\right]^2}.$$

The result sounds surprising, but it should be remarked that, once it is ascertained that the exponent γ of the learning sequence is smaller than 1, also the convergence rate of Theorem 5 does not feature γ.

The next theorem is inspired by some results in Ghadimi and Lan (2013); Karimi et al. (2019).

Theorem 6. *Under **Opt**, **Lip**-1 and **MaV**, suppose that $\max_{\boldsymbol{\theta}\in\Theta}\left\|\dot{F}\left(\boldsymbol{\theta}\right)\right\|_2 \le c_1 < \infty$ and $1 - \gamma_{i+1}L_1 \ge c_2 > 0$. Moreover, let $\boldsymbol{\theta}^{(0)}$ be fixed and $F\left(\boldsymbol{\theta}^\star\right) > -\infty$. Then, the following results hold:*
(i) We have

$$\min_{0\le i\le n}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 = O_{\mathbb{P}}\left(\left(\frac{1 + \sum_{i=0}^n \gamma_{i+1}b_i + \sum_{i=0}^n \gamma_{i+1}^2\sigma_i}{\sum_{i=0}^n \gamma_{i+1}}\right)^{\frac{1}{2}}\right).$$

(ii) We have

$$\frac{1}{n}\sum_{i=0}^n\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 = O_{\mathbb{P}}\left\{\frac{1}{n}\left(\sum_{i=0}^n \gamma_{i+1}^{-1}\right)^{\frac{1}{2}}\left(1 + \sum_{i=0}^n \gamma_{i+1}b_i + \sum_{i=0}^n \gamma_{i+1}^2\sigma_i\right)^{\frac{1}{2}}\right\}.$$

*(iii) If also **Hess** holds, the previous results hold respectively with $\min_{0\le i\le n}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2$ and $\frac{1}{n}\sum_{i=0}^n\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2$ replacing $\min_{0\le i\le n}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2$ and $\frac{1}{n}\sum_{i=0}^n\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2$.*

Remark 9. (i) The deterministic case can be recovered taking $b_i = \delta_1^{(i)}$ and $\sigma_i = \left(\delta_1^{(i)}\right)^2$. Following the proof of the theorem, if $\gamma_{i+1} \sim i^{-\gamma}$ and $\delta_1^{(i)} \sim i^{-\delta}$, one gets the following convergence rates:

$$\min_{0\le i\le n}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 = \begin{cases} O\left(1\right), & \gamma>1,\\ O\left(\ln^{-\frac{1}{2}} n\right), & \gamma=1,\\ O\left(n^{-\frac{\delta}{2}}\right), & \gamma<1, \gamma+\delta<1,\\ O\left(n^{-\frac{\delta}{2}}\ln^{\frac{1}{2}} n\right), & \gamma<1,\gamma+\delta=1,\\ O\left(n^{\frac{\gamma-1}{2}}\right), & \gamma<1,\gamma+\delta>1,\end{cases}$$

and

$$\frac{1}{n}\sum_{i=0}^{n}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 = \begin{cases} O\left(n^{\frac{\gamma-1}{2}}\right), & \gamma>1,\\ O\left(1\right), & \gamma=1,\\ O\left(n^{-\frac{\delta}{2}}\right), & \gamma<1, \gamma+\delta<1,\\ O\left(n^{-\frac{\delta}{2}}\ln^{\frac{1}{2}} n\right), & \gamma<1,\gamma+\delta=1,\\ O\left(n^{\frac{\gamma-1}{2}}\right), & \gamma<1,\gamma+\delta>1.\end{cases}$$

(ii) In the stochastic case, we note that

$$\begin{aligned}\mathbb{E}\left\|\mathbb{E}\left[\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)|\mathcal{F}_i\right]-\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 &\le \mathbb{E}\left\|\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)-\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 \le \mathbb{E}\delta_1^{(i)},\\ \mathbb{E}\mathbb{E}\left(\left\|\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)-F\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2|\mathcal{F}_i\right) &= \mathbb{E}\left(\delta_1^{(i)}\right)^2.\end{aligned}$$

(iii) The first result states that the probability that the optimization algorithm never visits a region where the score is near to zero tends to decrease with the number of steps of the algorithm, i.e.

$$\mathbb{P}\left\{\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 \ge \varepsilon, 0\le i\le n\right\} = O\left(\frac{1+\sum_{i=0}^{n}\gamma_{i+1}b_i+\sum_{i=0}^{n}\gamma_{i+1}^2\sigma_i}{\varepsilon^2\sum_{i=0}^{n}\gamma_{i+1}}\right).$$

(iv) From Remark 2, it is clear that, if $b_i \asymp i^{-\beta}$ and $\sigma \asymp i^{-\sigma}$, one can always take $\beta \ge \frac{\sigma}{2}$. In the extreme case in which $b_i \equiv 0$,

$$\min_{0\le i\le n}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 = \begin{cases} O_{\mathbb{P}}\left(1\right), & \gamma>1,\\ O_{\mathbb{P}}\left(\ln^{-\frac{1}{2}} n\right), & \gamma=1,\\ O_{\mathbb{P}}\left(n^{\frac{\gamma-1}{2}}\right), & \frac{1-\sigma}{2}<\gamma<1,\\ O_{\mathbb{P}}\left(n^{\frac{\gamma-1}{2}}\ln^{\frac{1}{2}} n\right), & \gamma=\frac{1-\sigma}{2},\\ O_{\mathbb{P}}\left(n^{-\frac{\gamma+\sigma}{2}}\right), & \gamma<\frac{1-\sigma}{2}.\end{cases}$$

In the other extreme case, in which $\beta = \frac{\sigma}{2}$,

$$\min_{0\le i\le n}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 = \begin{cases} O_{\mathbb{P}}\left(1\right), & \gamma>1,\\ O_{\mathbb{P}}\left(\ln^{-\frac{1}{2}} n\right), & \gamma=1,\\ O_{\mathbb{P}}\left(n^{\frac{\gamma-1}{2}}\right), & 1-\frac{\sigma}{2}<\gamma<1,\\ O_{\mathbb{P}}\left(n^{\frac{\gamma-1}{2}}\ln^{\frac{1}{2}} n\right), & \gamma=1-\frac{\sigma}{2},\\ O_{\mathbb{P}}\left(n^{-\frac{\sigma}{4}}\right), & \gamma<1-\frac{\sigma}{2}.\end{cases}$$

(v) Most optimization methods have an objective function of the form:

$$F(\boldsymbol{\theta}) := \frac{1}{N}\sum_{k=1}^{N} F_k(\boldsymbol{\theta})$$

where each $F_k(\cdot)$ is identically distributed. The gradient descent (GD) algorithm is defined as

$$\boldsymbol{\theta}^{(i+1)} = \boldsymbol{\theta}^{(i)} - \gamma_{i+1}\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) = \boldsymbol{\theta}^{(i)} - \gamma_{i+1}\frac{1}{N}\sum_{k=1}^{N}\dot{F}_k\left(\boldsymbol{\theta}^{(i)}\right),$$

but it is sometimes replaced by the stochastic gradient descent (SGD) algorithm

$$\boldsymbol{\theta}^{(i+1)} = \boldsymbol{\theta}^{(i)} - \gamma_{i+1}\dot{\hat{F}}\left(\boldsymbol{\theta}^{(i)}\right) = \boldsymbol{\theta}^{(i)} - \gamma_{i+1}\dot{F}_K\left(\boldsymbol{\theta}^{(i)}\right),$$

where K is randomly uniformly drawn from $\{1, 2, \ldots, N\}$. For the GD algorithm, it is clear that $b_i \equiv 0$ and $\sigma_i \equiv 0$, and

$$\min_{0\le i\le n}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 \le \min_{0\le i\le n}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 = \begin{cases} O_{\mathbb{P}}(1), & \gamma > 1,\\ O_{\mathbb{P}}\left(\ln^{-\frac{1}{2}} n\right), & \gamma = 1,\\ O_{\mathbb{P}}\left(n^{\frac{\gamma-1}{2}}\right), & \gamma < 1,\end{cases}$$

and

$$\frac{1}{n}\sum_{i=0}^{n}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 \le \frac{1}{n}\sum_{i=0}^{n}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 = O_{\mathbb{P}}\left(n^{\frac{\gamma-1}{2}}\right).$$

For the SGD algorithm, $b_i \equiv 0$ and σ_i is $O(1)$. Therefore,

$$\min_{0\le i\le n}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 \le \min_{0\le i\le n}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 = \begin{cases} O_{\mathbb{P}}(1), & \gamma > 1,\\ O_{\mathbb{P}}\left(\ln^{-\frac{1}{2}} n\right), & \gamma = 1,\\ O_{\mathbb{P}}\left(n^{\frac{\gamma-1}{2}}\right), & \frac{1}{2} < \gamma < 1,\\ O_{\mathbb{P}}\left(n^{-\frac{1}{4}}\ln^{\frac{1}{2}} n\right), & \gamma = \frac{1}{2},\\ O_{\mathbb{P}}\left(n^{-\frac{\gamma}{2}}\right), & 0 < \gamma < \frac{1}{2},\end{cases}$$

and

$$\frac{1}{n}\sum_{i=0}^{n}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 \le \frac{1}{n}\sum_{i=0}^{n}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 = \begin{cases} O_{\mathbb{P}}\left(n^{\frac{\gamma-1}{2}}\right), & \gamma > \frac{1}{2},\\ O_{\mathbb{P}}\left(n^{-\frac{1}{4}}\ln^{\frac{1}{2}} n\right), & \gamma = \frac{1}{2},\\ O_{\mathbb{P}}\left(n^{-\frac{\gamma}{2}}\right), & 0 < \gamma < \frac{1}{2}.\end{cases}$$

The following result gives a convergence rate for the stochastic case.

Theorem 7. *Under* ***Opt****,* ***Lip****-1,* ***MaV****,* ***Hess****, suppose that*

$$\Delta + \Delta\gamma_{i+1}^2 L_1^2 + \gamma_{i+1}(1 + \gamma_{i+1}L_1)\, b_i > 2\gamma_{i+1}\Delta m,$$

$$\left|\gamma_{i+1} - c_1(i+1)^{-\gamma}\right| \le c_2(i+1)^{-\xi},$$

$$\left|b_i - c_3(i+1)^{-\beta}\right| \le c_4(i+1)^{-\zeta},$$

$$\sigma_i \le c_5(i+1)^{-\sigma}(1 + o(1))$$

for $i \geq 0$, with $\beta \geq 0$, $\zeta \geq 0$, $\sigma \geq 0$, $1 < \xi$ and $1 < \gamma + \zeta$. Then, the following rates of convergence hold:

- *If $1 > \gamma > 0$,*

$$\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2 = O_{\mathbb{P}}\left(i^{-\frac{(\gamma+\sigma)\wedge\beta}{2}}\right).$$

- *If $\gamma = 1$,*

$$\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2 = \begin{cases} O_{\mathbb{P}}\left(i^{-\frac{(\sigma+1)\wedge\beta}{2}}\right), & 2mc_1 > (\sigma+1)\wedge\beta, \\ O_{\mathbb{P}}\left(i^{-mc_1}\ln^{\frac{1}{2}} i\right), & 2mc_1 = (\sigma+1)\wedge\beta, \\ O_{\mathbb{P}}\left(i^{-mc_1}\right), & 2mc_1 < (\sigma+1)\wedge\beta. \end{cases}$$

- *If $\gamma > 1$, then $\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2 = O_{\mathbb{P}}(1)$ and the algorithm is not guaranteed to converge.*

Remark 10. Let us consider again the GD and SGD algorithms. For GD, we can take $\beta = \sigma = \infty$. Therefore, for $\gamma > 1$ the rate of convergence is $O_{\mathbb{P}}(1)$, for $\gamma = 1$ it is $O_{\mathbb{P}}\left(i^{-mc_1}\right)$ and for $1 > \gamma > 0$ it is faster than polynomial. In particular, following the proof, we get

$$\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2 = O_{\mathbb{P}}\left(\exp\left\{-\frac{mc_1}{1-\gamma} i^{1-\gamma} + O\left(i^{1-2\gamma}\right)\right\}\right).$$

For SGD, we can take $\beta = \infty$ and $\sigma = 0$. Then,

$$\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2 = \begin{cases} O_{\mathbb{P}}(1), & \gamma > 1, \\ O_{\mathbb{P}}\left(i^{-\frac{1}{2}}\right), & \gamma = 1, 2mc_1 > 1, \\ O_{\mathbb{P}}\left(i^{-\frac{1}{2}}\ln^{\frac{1}{2}} i\right), & \gamma = 1, 2mc_1 = 1, \\ O_{\mathbb{P}}\left(i^{-mc_1}\right), & \gamma = 1, 2mc_1 < 1, \\ O_{\mathbb{P}}\left(i^{-\frac{\gamma}{2}}\right), & 1 > \gamma > 0. \end{cases}$$

We can thus represent the convergence rates of SGD as in Figure 4.1. It is apparent that the convergence rate of $\left\|\boldsymbol{\theta}^{(n)} - \boldsymbol{\theta}^{\star}\right\|_2$ is faster for larger $\gamma < 1$. The discrepancy between the rates of $\left\|\boldsymbol{\theta}^{(n)} - \boldsymbol{\theta}^{\star}\right\|_2$ and $\min_{0\leq i\leq n}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2$, and the discontinuity in the rate of $\min_{0\leq i\leq n}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2$ are probably artifacts of the method of proof.

4.2 Local Approximation by Least Squares

In this section, we characterize some results on the least squares approximation of a function in a point $\boldsymbol{\theta}_0$ and in a neighborhood of $\boldsymbol{\theta}_0$, on the basis of the approximate values of the function in a set of points.

Consider a point $\boldsymbol{\theta}_0 \in \Theta \subset \mathbb{R}^K$. Let $\mathcal{P}(\boldsymbol{\theta}_0) := \{\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_P\}$ be a set of P points. In most cases of interest, $\boldsymbol{\theta}_0$ will be quite near to $\mathcal{P}(\boldsymbol{\theta}_0)$ but this is not necessary. Moreover, in some cases $\boldsymbol{\theta}_0$ will belong to $\mathcal{P}(\boldsymbol{\theta}_0)$. Let $\rho := \max_j \|\boldsymbol{\theta}_j - \boldsymbol{\theta}_0\|_2$ be the radius of the smallest closed ball centered in $\boldsymbol{\theta}_0$ and containing $\mathcal{P}(\boldsymbol{\theta}_0)$. This means that $\mathcal{P}(\boldsymbol{\theta}_0) \subset \boldsymbol{\theta}_0 \oplus \rho\mathsf{B}$. We suppose that the points in $\mathcal{P}(\boldsymbol{\theta}_0)$ are a dilated version of a set of points $\mathcal{P}_0(\boldsymbol{\theta}_0) := \{\boldsymbol{\theta}_{0,1}, \dots, \boldsymbol{\theta}_{0,P}\}$ with $\rho_0 := \max_j \|\boldsymbol{\theta}_{0,j} - \boldsymbol{\theta}_0\|_2$, i.e.

$$\boldsymbol{\theta}_j := \boldsymbol{\theta}_0 + h\left(\boldsymbol{\theta}_{0,j} - \boldsymbol{\theta}_0\right)$$

where $h = \frac{\rho}{\rho_0}$, in most cases of interest, will be supposed to converge to 0. We define also $\widetilde{\rho} := \min_j \|\boldsymbol{\theta}_j - \boldsymbol{\theta}_0\|_2$;

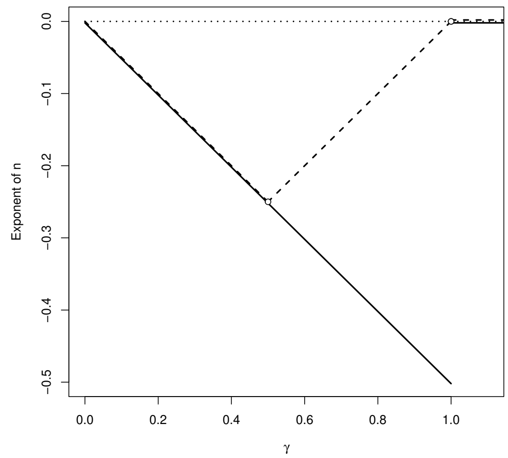

Figure 4.1: Exponent $f(\gamma)$ of n as a function of γ for $\left\|\boldsymbol{\theta}^{(n)}-\boldsymbol{\theta}^{\star}\right\|_2 = O_{\mathbb{P}}\left(n^{f(\gamma)}\right)$ (solid line) from Theorem 7, $\min_{0\leq i\leq n}\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2 = O_{\mathbb{P}}\left(n^{f(\gamma)}\right)$ (dashed line) from Theorem 6, and $\max_{0\leq i\leq n}\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2 = O_{\mathbb{P}}\left(n^{f(\gamma)}\right)$ (dotted line) from Corollary 2; the empty points represent the presence of logarithmic terms or the coexistence of different convergence rates not depending on γ.

this is 0 if $\boldsymbol{\theta}_0 \in \mathcal{P}(\boldsymbol{\theta}_0)$. Next, we suppose without loss of generality that the point $\boldsymbol{\theta}_0$ is in the origin, so that all points are replaced by $\boldsymbol{\theta} \mapsto \boldsymbol{\theta} - \boldsymbol{\theta}_0$.

The function $F(\boldsymbol{\theta}_j)$ is contaminated by an error ε_j, i.e. one observes the value $\hat{F}(\boldsymbol{\theta}_j) = F(\boldsymbol{\theta}_j) + \varepsilon_j$ instead of $F(\boldsymbol{\theta}_j)$. In this section we keep the treatment quite general and we do not specify a source for ε_j. We then provide an approximation to the function $F(\cdot)$. For a generic $\boldsymbol{\theta} \in \rho\mathsf{B}$, we build the vector of regressors $\mathbf{x}_D(\boldsymbol{\theta})$ containing all the monomials of elements of $\boldsymbol{\theta}$ up to order D. As usual, the order of multiplication in the monomials does not matter, so that the number of monomials in K variables exactly of order d is $\binom{d+K-1}{d}$ and up to order D is $\binom{D+K}{D}$. We then take, as approximating function, a polynomial of order D in the elements of $\boldsymbol{\theta}$, $\boldsymbol{\beta}'\mathbf{x}_D(\boldsymbol{\theta})$.

Example 1. When using a first-order polynomial, we can write

$$\beta_1 + \boldsymbol{\beta}_2'\boldsymbol{\theta} = \boldsymbol{\beta}'\mathbf{x}_1(\boldsymbol{\theta})$$

where $\boldsymbol{\beta} = \left[\beta_1, \boldsymbol{\beta}_2'\right]'$ and $\mathbf{x}_1(\boldsymbol{\theta}) = \left[1, \boldsymbol{\theta}'\right]'$. For a second-order polynomial, we can write

$$\beta_1 + \boldsymbol{\beta}_2'\boldsymbol{\theta} + \boldsymbol{\beta}_3'\mathbf{D}_K^+(\boldsymbol{\theta}\otimes\boldsymbol{\theta}) = \boldsymbol{\beta}'\mathbf{x}_2(\boldsymbol{\theta})$$

where $\mathbf{D}_K^+$ was defined in Section 3.1, $\boldsymbol{\beta} = \left[\beta_1, \boldsymbol{\beta}_2', \boldsymbol{\beta}_3'\right]'$ and $\mathbf{x}_2(\boldsymbol{\theta}) = \left[1, \boldsymbol{\theta}', \left(\mathbf{D}_K^+(\boldsymbol{\theta}\otimes\boldsymbol{\theta})\right)'\right]'$.

Remark 11. In principle, one could substitute the least squares approximation with other methods, e.g., least absolute shrinkage and selection operator (Lasso). However, whereas OLS produces an efficient estimate of $\boldsymbol{\beta}$, Lasso puts forward a biased estimate of $\boldsymbol{\beta}$.

Let $\mathbf{y}$ be the vector whose generic j-th element is $\hat{F}(\boldsymbol{\theta}_j)$. For each j, we build the vector of regressors $\mathbf{x}_D(\boldsymbol{\theta}_j)$ from $\boldsymbol{\theta}_j$. Let $\mathbf{X}$ be the matrix of regressors obtained stacking the rows $\mathbf{x}_D'(\boldsymbol{\theta}_j)$. Let $\mathbf{X}_0$ be the same matrix for the point set $\mathcal{P}_0(\boldsymbol{\theta}_0)$. The OLS estimator of $\boldsymbol{\beta}$ is

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}.$$

For a generic $\boldsymbol{\theta}$, we define the OLS predictor as

$$\tilde{F}(\boldsymbol{\theta}) = \mathbf{x}'_D(\boldsymbol{\theta})\hat{\boldsymbol{\beta}} = \mathbf{x}'_D(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}.$$

For the statement of the results, let $\boldsymbol{\varepsilon}$ be the vector whose j-th element is $\varepsilon_j = \hat{F}(\boldsymbol{\theta}_j) - F(\boldsymbol{\theta}_j)$. We note that, despite we do not explicit it, $\boldsymbol{\varepsilon}$ may depend on the set of points $\mathcal{P}(\boldsymbol{\theta}_0)$ in which the function is computed.

We will need an assumption concerning the function F.

Fun-d The function $F : \boldsymbol{\theta} \mapsto F(\boldsymbol{\theta})$ is of class $\mathcal{C}^{d,1}$ on a compact set $\Theta \subset \mathbb{R}^K$. The parameter space contains an open neighborhood of a value $\boldsymbol{\theta}^\star$ such that $\dot{F}(\boldsymbol{\theta}^\star) = \mathbf{0}$.

Assumption **Fun-**d requires the function to be differentiable up to order d with Lipschitz derivative. In this paper, we will mainly use **Fun-**1 and **Fun-**2 but some results are stated in more general form.

We present four different results: the first one concerns the quality of the approximation of the function in a single point, the second one deals with the approximation of the function and its derivatives in a neighborhood of a point, the third one regards the gradient and the Hessian, and the last one provides upper bounds for the bias and the variance of the approximation when the error affecting the function is stochastic.

Theorem 8. *Suppose **Fun-**d holds. If $\boldsymbol{\theta}_0 \in \mathcal{P}(\boldsymbol{\theta}_0)$,*

$$\left|\tilde{F}(\boldsymbol{\theta}_0) - F(\boldsymbol{\theta}_0)\right| \leq P^{\frac{1}{2}} \frac{K^D}{(D-1)!} \rho^{D+1} |F|_{D,1} + \rho^{-D} \frac{P^{-\frac{1}{2}} \rho_0^D \|\boldsymbol{\varepsilon}\|_2}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}}.$$

If $\boldsymbol{\theta}_0 \notin \mathcal{P}(\boldsymbol{\theta}_0)$,

$$\begin{aligned}\left|\tilde{F}(\boldsymbol{\theta}_0) - F(\boldsymbol{\theta}_0)\right| &\leq P^{\frac{1}{2}} \frac{K^D}{(D-1)!} \rho^{D+1} |F|_{D,1} + \widetilde{\rho}\,|F|_{0,1} \\ &\quad + \rho^{-D} \left\{ \widetilde{\rho} \left(\frac{1-\widetilde{\rho}^{2D}}{1-\widetilde{\rho}^2} \right)^{\frac{1}{2}} \|F\|_{\mathcal{P}(\boldsymbol{\theta}_0)} + P^{-\frac{1}{2}} \|\boldsymbol{\varepsilon}\|_2 \right\} \frac{\rho_0^D}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}}.\end{aligned}$$

Remark 12. (i) When $\mathcal{P}(\boldsymbol{\theta}_0) \equiv \mathcal{P}_0(\boldsymbol{\theta}_0)$ and $\rho = \rho_0$, we have

$$\begin{aligned}\left|\tilde{F}(\boldsymbol{\theta}_0) - F(\boldsymbol{\theta}_0)\right| &\leq P^{\frac{1}{2}} \frac{K^D}{(D-1)!} \rho^{D+1} |F|_{D,1} + \widetilde{\rho}\,|F|_{0,1} \\ &\quad + \left\{ \widetilde{\rho} \left(\frac{1-\widetilde{\rho}^{2D}}{1-\widetilde{\rho}^2} \right)^{\frac{1}{2}} \|F\|_{\mathcal{P}(\boldsymbol{\theta}_0)} + P^{-\frac{1}{2}} \|\boldsymbol{\varepsilon}\|_2 \right\} \frac{1}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}}.\end{aligned}$$

(ii) When $\boldsymbol{\theta}_0 \notin \mathcal{P}(\boldsymbol{\theta}_0)$ and $\widetilde{\rho}$ is small,

$$\left|\tilde{F}(\boldsymbol{\theta}_0) - F(\boldsymbol{\theta}_0)\right| \lesssim P^{\frac{1}{2}} \frac{K^D}{(D-1)!} \rho^{D+1} |F|_{D,1} + \rho^{-D} \frac{P^{-\frac{1}{2}} \rho_0^D \|\boldsymbol{\varepsilon}\|_2}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}}$$

$$+\widetilde{\rho}\left\{|F|_{0,1}+\rho^{-D}\frac{\rho_0^D\,\|F\|_{\mathcal{P}(\boldsymbol{\theta}_0)}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}}\right\}$$

and, as expected, the bound is only slightly larger than the one for $\boldsymbol{\theta}_0 \in \mathcal{P}(\boldsymbol{\theta}_0)$.
(iii) Considering only the most important asymptotic parameters, for $\rho \downarrow 0$ and $\widetilde{\rho} \downarrow 0$, we get

$$\left|\tilde{F}(\boldsymbol{\theta}_0)-F(\boldsymbol{\theta}_0)\right| = O\left(\rho^{D+1}+\widetilde{\rho}+\rho^{-D}\left(\widetilde{\rho}+\|\boldsymbol{\varepsilon}\|_2\right)\right).$$

It is clear that $\widetilde{\rho} \leq \rho$, but the choice of a $\boldsymbol{\theta}_0$ not belonging to $\mathcal{P}(\boldsymbol{\theta}_0)$ may affect negatively the rate of convergence to 0. Indeed, if $\widetilde{\rho}=0$,

$$\left|\tilde{F}(\boldsymbol{\theta}_0)-F(\boldsymbol{\theta}_0)\right| = O\left(\rho^{D+1}+\rho^{-D}\|\boldsymbol{\varepsilon}\|_2\right)$$

while, if $\widetilde{\rho}$ and ρ converge to zero at the same rate,

$$\left|\tilde{F}(\boldsymbol{\theta}_0)-F(\boldsymbol{\theta}_0)\right| = O\left(\rho^{-D+1}+\rho^{-D}\|\boldsymbol{\varepsilon}\|_2\right).$$

Now we turn to the uniform approximation in a neighborhood of $\boldsymbol{\theta}_0$. Define the constant

$$C_D(\mathcal{P}_0) := \sup_{p\in\mathbb{P}_D}\frac{\|p(\boldsymbol{\theta})\|_{\rho_0\mathsf{B}}}{\|p(\boldsymbol{\theta})\|_{\mathcal{P}_0(\mathbf{0})}}.$$

Calvi and Levenberg (2008, p. 85) discuss its meaning and its relation with other properties of the polynomials. Other properties are collected in Bos et al. (2011b,a). The constant does not depend on the position of $\boldsymbol{\theta}_0$, but only on the configuration of points $\mathcal{P}_0(\mathbf{0})$ and on its radius ρ_0, and the degree of the polynomial D. The existence of a finite $C_D(\mathcal{P}_0)$ is equivalent to the fact that $\mathcal{P}_0(\mathbf{0})$ is a $\mathbb{P}_D$-determining class for $\rho_0\mathsf{B}$, i.e. the fact that any polynomial of degree D is zero on $\mathcal{P}_0(\mathbf{0})$ implies that it is zero on $\rho_0\mathsf{B}$.

Theorem 9. *Suppose **Fun-**D holds and let $\boldsymbol{\theta}_0$ be a point in Θ. Then, for $S \leq D$,*

$$\begin{aligned}
&\max_{|k|=S}\left\|D^k\tilde{F}(\boldsymbol{\theta})-D^kF(\boldsymbol{\theta})\right\|_{\boldsymbol{\theta}_0\oplus\rho\mathsf{B}}\\
&\leq \rho_0^D\rho^{-D}\frac{P^{-\frac{1}{2}}S!\sqrt{\sum_{d=S}^{D}\binom{d}{S}^2\rho^{2(d-S)}}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}}\max_{\boldsymbol{\theta}\in\boldsymbol{\theta}_0\oplus\rho\mathsf{B}}\|\boldsymbol{\varepsilon}\|_2\\
&\quad+\left[\frac{\left(2+C_D(\mathcal{P}_0)\left(1+P^{\frac{1}{2}}\right)\right)(D!)^2K^S}{((D-S)!)^2(D-1)!}+\frac{1}{(D-S-1)!}\right]K^{D-S}\rho^{D-S+1}|F|_{D,1}
\end{aligned}$$

where it is intended that $(-1)!=1$.

Remark 13. (i) By restricting our attention to the most important asymptotic parameters, i.e. ρ and $\max_{\boldsymbol{\theta}\in\boldsymbol{\theta}_0\oplus\rho\mathsf{B}}\|\boldsymbol{\varepsilon}\|_2$, we get

$$\max_{|k|=S}\left\|D^k\tilde{F}(\boldsymbol{\theta})-D^kF(\boldsymbol{\theta})\right\|_{\boldsymbol{\theta}_0\oplus\rho\mathsf{B}} = O\left(\rho^{D-S+1}+\rho^{-D}\max_{\boldsymbol{\theta}\in\boldsymbol{\theta}_0\oplus\rho\mathsf{B}}\|\boldsymbol{\varepsilon}\|_2\right).$$

Note that this implies, as expected, a worse rate of convergence for higher derivatives. If we suppose that all other parameters are fixed, we need $\rho \downarrow 0$, $\rho^{-D}\max_{\boldsymbol{\theta}\in\boldsymbol{\theta}_0\oplus\rho\mathsf{B}}\|\boldsymbol{\varepsilon}\|_2 \downarrow 0$ in order to have convergence to 0.

Indeed, when $\rho \downarrow 0$ and $S \leq D$, $\sum_{d=S}^{D} \binom{d}{S}^2 \rho^{2(d-S)} = 1 + o(1)$.
(ii) For $S = 0$, we get

$$\begin{aligned}
&\left\| \tilde{F}(\boldsymbol{\theta}) - F(\boldsymbol{\theta}) \right\|_{\boldsymbol{\theta}_0 \oplus \rho \mathsf{B}} \\
\leq & \frac{\left(3 + C_D(\mathcal{P}_0)\left(1 + P^{\frac{1}{2}}\right)\right) K^D}{(D-1)!} \rho^{D+1} |F|_{D,1} \\
&+ \rho^{-D} \frac{P^{-\frac{1}{2}} \rho_0^D}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}} \sqrt{\frac{1-\rho^{2(D+1)}}{1-\rho^2}} \max_{\boldsymbol{\theta} \in \boldsymbol{\theta}_0 \oplus \rho \mathsf{B}} \|\boldsymbol{\varepsilon}\|_2 .
\end{aligned}$$

This compares favorably with the bound of Theorem 8 for $\boldsymbol{\theta}_0 \in \mathcal{P}(\boldsymbol{\theta}_0)$,

$$\left| \tilde{F}(\boldsymbol{\theta}_0) - F(\boldsymbol{\theta}_0) \right| \leq P^{\frac{1}{2}} \frac{K^D}{(D-1)!} \rho^{D+1} |F|_{D,1} + \rho^{-D} \frac{P^{-\frac{1}{2}} \rho_0^D}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}} \|\boldsymbol{\varepsilon}\|_2$$

that gives a bound for the approximation in a single point. The two main sources of disagreement between the two formulas are the terms $3 + C_D(\mathcal{P}_0)\left(1 + P^{\frac{1}{2}}\right)$ and $\sqrt{\frac{1-\rho^{2(D+1)}}{1-\rho^2}}$ in the first formula that are replaced by $P^{\frac{1}{2}}$ and 1 in the second formula. The term $\sqrt{\frac{1-\rho^{2(D+1)}}{1-\rho^2}}$ takes into account the size of the neighborhood: indeed, it converges to 1 when $\rho \downarrow 0$ and diverges as ρ^D when $\rho \to \infty$. The term $3 + C_D(\mathcal{P}_0)\left(1 + P^{\frac{1}{2}}\right)$ is more difficult to characterize and compare to $P^{\frac{1}{2}}$. Calvi and Levenberg (2008, Section 3.1) discuss the case in which $C_D(\mathcal{P}_0)$ is bounded from above and, as a result, the two terms have the same asymptotic behavior in P. Other results are contained in Bos et al. (2011b,a), where several examples of meshes with slowly increasing values of $C_D(\mathcal{P}_0)$ are proposed. However, note that P is rarely an asymptotic parameter in what follows.
(iii) Given the relevance of the approximation of the first derivative in Theorem 1, it may be interesting to choose ρ in such a way to minimize $\max_{|k|=1} \sup_{\boldsymbol{\theta} \in \rho \mathsf{B}} \left| D^k \tilde{F}(\boldsymbol{\theta}) - D^k F(\boldsymbol{\theta}) \right|$. This means that $\rho^\star \sim \max_{\boldsymbol{\theta} \in \boldsymbol{\theta}_0 \oplus \rho \mathsf{B}} \|\boldsymbol{\varepsilon}\|_2^{\frac{1}{2D}}$ and, if $\max_{\boldsymbol{\theta} \in \boldsymbol{\theta}_0 \oplus \rho \mathsf{B}} \|\boldsymbol{\varepsilon}\|_2 \downarrow 0$,

$$\max_{|k|=1} \left\| D^k \tilde{F}(\boldsymbol{\theta}) - D^k F(\boldsymbol{\theta}) \right\|_{\boldsymbol{\theta}_0 \oplus \rho \mathsf{B}} = O\left(\max_{\boldsymbol{\theta} \in \boldsymbol{\theta}_0 \oplus \rho \mathsf{B}} \|\boldsymbol{\varepsilon}\|_2^{\frac{1}{2}} \right).$$

The second derivative has a worse convergence rate, namely

$$\max_{|k|=2} \left\| D^k \tilde{F}(\boldsymbol{\theta}) - D^k F(\boldsymbol{\theta}) \right\|_{\boldsymbol{\theta}_0 \oplus \rho \mathsf{B}} = O\left(\max_{\boldsymbol{\theta} \in \boldsymbol{\theta}_0 \oplus \rho \mathsf{B}} \|\boldsymbol{\varepsilon}\|_2^{\frac{D-1}{2D}} \right).$$

Under **Lip-**1 and **Lip-**2, similar results hold for the gradient and the Hessian. In the following, $\left[D^i \tilde{F}(\boldsymbol{\theta}) - D^i F(\boldsymbol{\theta})\right]$ denotes the gradient of $\tilde{F}(\boldsymbol{\theta}) - F(\boldsymbol{\theta})$ and $\left[D^{ij} \tilde{F}(\boldsymbol{\theta}) - D^{ij} F(\boldsymbol{\theta})\right]$ the Hessian of the same function. The advantage of these results is that they are less affected by the curse of dimensionality in K, the number of parameters.

Corollary 3. *Let $\boldsymbol{\theta}_0$ be a point in Θ. Then, under **Lip-**1, for $S = D = 1$,*

$$\sup_{\boldsymbol{\theta} \in \boldsymbol{\theta}_0 \oplus \rho \mathsf{B}} \left\| \left[D^i \tilde{F}(\boldsymbol{\theta}) - D^i F(\boldsymbol{\theta}) \right] \right\|_2 \leq \frac{\rho_0 K^{\frac{1}{2}} \max_{\boldsymbol{\theta} \in \boldsymbol{\theta}_0 \oplus \rho \mathsf{B}} \|\boldsymbol{\varepsilon}\|_2}{P^{\frac{1}{2}} \rho \sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}}$$

$$+\frac{1}{2}\left(4+C_1\left(\mathcal{P}_0\right)\left(1+P^{\frac{1}{2}}\right)\right)L_1\rho.$$

*Under **Lip**-2, for $S=1$ and $D=2$,*

$$\sup_{\boldsymbol{\theta}\in\boldsymbol{\theta}_0\oplus\rho\mathsf{B}}\left\|\left[D^i\tilde{F}\left(\boldsymbol{\theta}\right)-D^iF\left(\boldsymbol{\theta}\right)\right]\right\|_2\leq\frac{\rho_0^2K^{\frac{1}{2}}\sqrt{1+4\rho^2}\max_{\boldsymbol{\theta}\in\boldsymbol{\theta}_0\oplus\rho\mathsf{B}}\left\|\boldsymbol{\varepsilon}\right\|_2}{P^{\frac{1}{2}}\rho^2\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}}+\frac{1}{6}\left(11+4C_2\left(\mathcal{P}_0\right)\left(1+P^{\frac{1}{2}}\right)\right)L_2\rho^2$$

and, for $S=D=2$,

$$\sup_{\boldsymbol{\theta}\in\boldsymbol{\theta}_0\oplus\rho\mathsf{B}}\left\|\left[D^{ij}\tilde{F}\left(\boldsymbol{\theta}\right)-D^{ij}F\left(\boldsymbol{\theta}\right)\right]\right\|_2\leq\frac{2\rho_0^2K\max_{\boldsymbol{\theta}\in\boldsymbol{\theta}_0\oplus\rho\mathsf{B}}\left\|\boldsymbol{\varepsilon}\right\|_2}{P^{\frac{1}{2}}\rho^2\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}}+\frac{1}{3}\left(3+4K^{\frac{1}{2}}+2K^{\frac{1}{2}}C_2\left(\mathcal{P}_0\right)\left(1+P^{\frac{1}{2}}\right)\right)L_2\rho.$$

At last, we provide a version of Theorem 9 for the mean and the variance of the approximation error, when this is stochastic.

Theorem 10. *Let $\boldsymbol{\theta}_0$ be a point in Θ. Then, under **Fun**-D, for $S\leq D$,*

$$\max_{|k|=S}\left\|\mathbb{E}D^k\tilde{F}\left(\boldsymbol{\theta}\right)-D^kF\left(\boldsymbol{\theta}\right)\right\|_{\boldsymbol{\theta}_0\oplus\rho\mathsf{B}}$$
$$\leq\rho^{-D}\frac{P^{-\frac{1}{2}}\rho_0^DS!\sqrt{\sum_{d=S}^D\binom{d}{S}^2\rho^{2(d-S)}}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}}\max_{\boldsymbol{\theta}\in\boldsymbol{\theta}_0\oplus\rho\mathsf{B}}\left\|\mathbb{E}\boldsymbol{\varepsilon}\right\|_2$$
$$+\left[\frac{\left(2+C_D\left(\mathcal{P}_0\right)\left(1+P^{\frac{1}{2}}\right)\right)\left(D!\right)^2K^S}{\left(\left(D-S\right)!\right)^2\left(D-1\right)!}+\frac{1}{\left(D-S-1\right)!}\right]K^{D-S}\rho^{D-S+1}\left|F\right|_{D,1}$$

where it is intended that $(-1)!=1$, and

$$\max_{|k|=S}\left\|\mathbb{E}\left|D^k\tilde{F}\left(\boldsymbol{\theta}\right)-D^kF\left(\boldsymbol{\theta}\right)-\mathbb{E}\left(D^k\tilde{F}\left(\boldsymbol{\theta}\right)-D^kF\left(\boldsymbol{\theta}\right)\right)\right|^2\right\|_{\boldsymbol{\theta}_0\oplus\rho\mathsf{B}}$$
$$\leq\rho^{-2D}\frac{P^{-1}\rho_0^{2D}\left(S!\right)^2}{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}\left(\sum_{d=S}^D\binom{d}{S}^2\rho^{2(d-S)}\right)\max_{\boldsymbol{\theta}\in\boldsymbol{\theta}_0\oplus\rho\mathsf{B}}\mathbb{E}\left\|\boldsymbol{\varepsilon}-\mathbb{E}\boldsymbol{\varepsilon}\right\|_2^2.$$

4.3 Results Specific to the Approximating Algorithm

In this section we apply the previous results to our algorithm. In particular, we will replace the generic point $\boldsymbol{\theta}_0\in\Theta$ with a point of the sequence $\left\{\boldsymbol{\theta}^{(i)}\right\}$. Moreover, we will suppose that all the expectations are conditional on the σ-algebra $\mathcal{F}_i$. As the process $\left\{\boldsymbol{\theta}^{(i)}\right\}$ is Markov, any expectation at step i depends only on the previous point $\boldsymbol{\theta}^{(i-1)}$ in the sequence.

We need two assumptions quantifying the effect of replacing the true probability $\mathbb{P}_{\mathbf{z}(\cdot)}$ with an approximated version $\widehat{\mathbb{P}}_{\mathbf{z}(\cdot)}$. The first one is a deterministic bound, the second one concerns the stochastic case.

AUB The element $\hat{F}(\boldsymbol{\theta}) - F(\boldsymbol{\theta})$ is such that, for a sequence $\{a_N\}$,

$$\max_{\boldsymbol{\theta}_j \in \mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)} \left|\hat{F}(\boldsymbol{\theta}) - F(\boldsymbol{\theta})\right| \leq a_N, \quad \mathbb{P}\text{-as.}$$

MaV2 $\max_{\boldsymbol{\theta}_j \in \mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)} \left|\mathbb{E}\left(\hat{F}(\boldsymbol{\theta}_j)|\mathcal{F}_i\right) - F(\boldsymbol{\theta}_j)\right| \leq B_i$, $\max_{\boldsymbol{\theta}_j \in \mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)} \mathbb{E}\left(\left(\hat{F}(\boldsymbol{\theta}_j) - F(\boldsymbol{\theta}_j)\right)^2 |\mathcal{F}_i\right) \leq \Sigma_i$.

Remark 14. (i) The bounds in Assumption **MaV2** could be replaced with unconditional bounds like $\max_{\boldsymbol{\theta}\in\Theta}\left|\mathbb{E}\hat{F}(\boldsymbol{\theta}) - F(\boldsymbol{\theta})\right| \leq B_i$ and $\max_{\boldsymbol{\theta}\in\Theta} \mathbb{E}\left(\hat{F}(\boldsymbol{\theta}) - F(\boldsymbol{\theta})\right)^2 \leq \Sigma_i$. The dependence on i of the bounds comes from the fact that the function $F(\boldsymbol{\theta})$ is estimated by $\hat{F}(\boldsymbol{\theta})$ on the basis of a number of simulated observations depending on the step i.
(ii) The bounds in Assumptions **AUB** and **MaV2** can generally be obtained using the functional differentiability of $f\left(\widehat{\mathbb{P}}_{\mathbf{y}}, \cdot\right)$. A short explanation can be useful. Let ϕ be a function of a probability measure, defined in a neighborhood of $\mathbb{P}$. Define also the empirical process $\mathbb{G}_n := \sqrt{n}\left(\widehat{\mathbb{P}}_n - \mathbb{P}\right)$ based on n observations. Then an informal reasoning, that can be made rigorous along the lines of van der Vaart (1998, Chapter 20), leads to the development

$$\phi\left(\widehat{\mathbb{P}}_n\right) - \phi(\mathbb{P}) = \frac{1}{\sqrt{n}}\phi'_{\mathbb{P}}(\mathbb{G}_n) + \frac{1}{2n}\phi''_{\mathbb{P}}(\mathbb{G}_n) + \ldots$$

where $\phi_{\mathbb{P}}^{(n)}$ are functional derivatives of order n, and the function $\phi'_{\mathbb{P}}(\cdot)$ is generally linear. Now, this implies that

$$\mathbb{E}\left[\phi\left(\widehat{\mathbb{P}}_n\right) - \phi(\mathbb{P})\right] \simeq \begin{cases} \frac{1}{2n}\mathbb{E}\phi''_{\mathbb{P}}(\mathbb{G}_n), & \mathbb{E}\widehat{\mathbb{P}}_n = \mathbb{P}, \\ \frac{1}{\sqrt{n}}\phi'_{\mathbb{P}}(\mathbb{E}\mathbb{G}_n), & \mathbb{E}\widehat{\mathbb{P}}_n \neq \mathbb{P}, \end{cases}$$

and

$$\mathbb{V}\left(\phi\left(\widehat{\mathbb{P}}_n\right) - \phi(\mathbb{P})\right) \simeq \frac{1}{n}\mathbb{V}\left(\phi'_{\mathbb{P}}(\mathbb{G}_n)\right).$$

These results can be used to characterize B_i and Σ_i using the identifications

$$\begin{aligned} \phi\left(\widehat{\mathbb{P}}_{\mathbf{z}(\boldsymbol{\theta}_j)}\right) &= \hat{F}(\boldsymbol{\theta}_j) = f\left(\widehat{\mathbb{P}}_{\mathbf{y}}, \widehat{\mathbb{P}}_{\mathbf{z}(\boldsymbol{\theta}_j)}\right), \\ \phi\left(\mathbb{P}_{\mathbf{z}(\boldsymbol{\theta}_j)}\right) &= F(\boldsymbol{\theta}_j) = f\left(\widehat{\mathbb{P}}_{\mathbf{y}}, \mathbb{P}_{\mathbf{z}(\boldsymbol{\theta}_j)}\right). \end{aligned}$$

Moreover, under suitable assumptions, a functional Law of the Iterated Logarithm yields

$$\limsup_{N\to\infty} \sqrt{\frac{n}{\ln\ln n}} \left|\phi\left(\widehat{\mathbb{P}}_n\right) - \phi(\mathbb{P})\right| = \limsup_{N\to\infty} \sqrt{\frac{n}{\ln\ln n}} \left|\phi'_{\mathbb{P}}\left(\widehat{\mathbb{P}}_n - \mathbb{P}\right)\right| \leq c, \quad \mathbb{P}\text{-as.}$$

As a result, in most cases,

$$\begin{aligned} B_i &= O\left(N^{-\frac{1}{2}}\right), \\ \Sigma_i &= O\left(N^{-1}\right), \\ a_N &= \sqrt{\frac{N}{\ln\ln N}}. \end{aligned}$$

The following corollary provides formulas for the quantities involved in some of the previous convergence results.

Corollary 4. *Under **AUB** and **Fun-**D,*

$$\delta_1^{(i)} \leq \rho_i^{-D} \frac{\rho_0^D K^{\frac{1}{2}} \sqrt{\sum_{d=1}^D d^2 \rho_i^{2(d-1)}}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}} a_N + \rho_i^D \left[\frac{D^2 K \left(2 + C_D\left(\mathcal{P}_0\right)\left(1 + P^{\frac{1}{2}}\right)\right)}{D-1} + 1\right] \frac{K^{D-\frac{1}{2}} \left|F\right|_{D,1}}{(D-2)!},$$

$$\delta_2^{(i)} \leq \rho_i^{-D} \frac{2\rho_0^D K \sqrt{\sum_{d=2}^D \binom{d}{2}^2 \rho_i^{2(d-2)}}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}} a_N + \rho_i^{D-1} \left[\frac{D^2 (D-1) K^2 \left(2 + C_D\left(\mathcal{P}_0\right)\left(1 + P^{\frac{1}{2}}\right)\right)}{D-2} + 1\right] \frac{K^{D-1} \left|F\right|_{D,1}}{(D-3)!}.$$

*Under **MaV2** and **Fun-**D,*

$$\mathbb{E}\left(\delta_1^{(i)}\right)^2 \leq \left\{\rho_i^{-D} \frac{K^{\frac{1}{2}} \rho_0^D \sqrt{\sum_{d=1}^D d^2 \rho_i^{2(d-1)}}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}} \left(\Sigma_i^{\frac{1}{2}} + B_i\right) + \rho_i^D \left[\frac{D^2 K \left(2 + C_D\left(\mathcal{P}_0\right)\left(1 + P^{\frac{1}{2}}\right)\right)}{(D-1)!} + \frac{1}{(D-2)!}\right] K^{D-\frac{1}{2}} \left|F\right|_{D,1}\right\}^2,$$

$$\mathbb{E}\left(\delta_2^{(i)}\right)^2 \leq \left\{2\rho_i^{-D} \frac{K \rho_0^D \sqrt{\sum_{d=2}^D \binom{d}{2}^2 \rho_i^{2(d-2)}}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}} \left(\Sigma_i^{\frac{1}{2}} + B_i\right) + \rho_i^{D-1} \left[\frac{D^2 (D-1) K^2 \left(2 + C_D\left(\mathcal{P}_0\right)\left(1 + P^{\frac{1}{2}}\right)\right)}{(D-2)!} + \frac{1}{(D-3)!}\right] K^{D-1} \left|F\right|_{D,1}\right\}^2$$

and

$$b_i = \rho_i^{-D} \frac{K^{\frac{1}{2}} \rho_0^D \sqrt{\sum_{d=1}^D d^2 \rho_i^{2(d-1)}}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}} B_i + \rho_i^D \left[\frac{D^2 K \left(2 + C_D\left(\mathcal{P}_0\right)\left(1 + P^{\frac{1}{2}}\right)\right)}{(D-1)!} + \frac{1}{(D-2)!}\right] K^{D-\frac{1}{2}} \left|F\right|_{D,1},$$

$$\sigma_i = \left\{\rho_i^{-D} \frac{K^{\frac{1}{2}} \rho_0^D \sqrt{\sum_{d=1}^D d^2 \rho_i^{2(d-1)}}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}} \left(\Sigma_i^{\frac{1}{2}} + B_i\right) + \rho_i^D \left[\frac{D^2 K \left(2 + C_D\left(\mathcal{P}_0\right)\left(1 + P^{\frac{1}{2}}\right)\right)}{(D-1)!} + \frac{1}{(D-2)!}\right] K^{D-\frac{1}{2}} \left|F\right|_{D,1}\right\}^2.$$

Remark 15. From the previous formulas, for $D=1$,

$$\left(\mathbb{E}\left(\delta_1^{(i)}\right)^2\right)^{\frac{1}{2}} \leq \frac{K^{\frac{1}{2}}\rho_0\left(\Sigma_i^{\frac{1}{2}}+B_i\right)}{\rho_i\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}} + \rho_i\left[K\left(2+C_1\left(\mathcal{P}_0\right)\left(1+P^{\frac{1}{2}}\right)\right)+1\right]K^{\frac{1}{2}}\left|F\right|_{1,1},$$

$$b_i = \frac{K^{\frac{1}{2}}\rho_0 B_i}{\rho_i\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}} + \rho_i\left[K\left(2+C_1\left(\mathcal{P}_0\right)\left(1+P^{\frac{1}{2}}\right)\right)+1\right]K^{\frac{1}{2}}\left|F\right|_{1,1},$$

$$\sigma_i^{\frac{1}{2}} = \frac{K^{\frac{1}{2}}\rho_0\left(\Sigma_i^{\frac{1}{2}}+B_i\right)}{\rho_i\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}} + \rho_i\left[K\left(2+C_1\left(\mathcal{P}_0\right)\left(1+P^{\frac{1}{2}}\right)\right)+1\right]K^{\frac{1}{2}}\left|F\right|_{D,1},$$

and, for $D=2$,

$$\begin{aligned}
\left(\mathbb{E}\left(\delta_1^{(i)}\right)^2\right)^{\frac{1}{2}} &\leq \frac{K^{\frac{1}{2}}\rho_0^2\sqrt{1+4\rho_i^2}\left(\Sigma_i^{\frac{1}{2}}+B_i\right)}{\rho_i^2\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}} \\
&\quad + \rho_i^2\left[4K\left(2+C_2\left(\mathcal{P}_0\right)\left(1+P^{\frac{1}{2}}\right)\right)+1\right]K^{\frac{3}{2}}\left|F\right|_{2,1}, \\
\left(\mathbb{E}\left(\delta_2^{(i)}\right)^2\right)^{\frac{1}{2}} &\leq \frac{2K\rho_0^2\left(\Sigma_i^{\frac{1}{2}}+B_i\right)}{\rho_i^2\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}} \\
&\quad + \rho_i\left[4K^2\left(2+C_2\left(\mathcal{P}_0\right)\left(1+P^{\frac{1}{2}}\right)\right)+1\right]K\left|F\right|_{2,1}, \\
b_i &= \frac{K^{\frac{1}{2}}\rho_0^D\sqrt{1+4\rho_i^2}B_i}{\rho_i^2\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}} \\
&\quad + \rho_i^2\left[4K\left(2+C_2\left(\mathcal{P}_0\right)\left(1+P^{\frac{1}{2}}\right)\right)+1\right]K^{\frac{3}{2}}\left|F\right|_{2,1}, \\
\sigma_i^{\frac{1}{2}} &= \frac{K^{\frac{1}{2}}\rho_0^2\sqrt{1+4\rho_i^2}\left(\Sigma_i^{\frac{1}{2}}+B_i\right)}{\rho_i^2\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}} \\
&\quad + \rho_i^2\left[4K\left(2+C_2\left(\mathcal{P}_0\right)\left(1+P^{\frac{1}{2}}\right)\right)+1\right]K^{\frac{3}{2}}\left|F\right|_{2,1}.
\end{aligned}$$

Slightly better formulas for $D\in\{1,2\}$ can be obtained using Corollary 3.

The next two results give deterministic convergence rates for the INM and SAS algorithms.

Theorem 11. *For the Inexact Newton Method, under* ***Opt****,* ***Fun-****D*, ***Hess*** *and* ***AUB****, suppose that, for any* $i\geq 0$*, there exist* $C_1>0$*,* $C_2>0$*,* $\frac{m}{2K|F|_{2,1}}>\Delta>0$*,* $\rho>0$ *and* $\alpha>0$*, such that*

$$\begin{aligned}
\left\|\boldsymbol{\theta}^{(0)}-\boldsymbol{\theta}^{\star}\right\|_2 &\leq \Delta, \\
\delta_1^{(i)}+\delta_2^{(i)}\left(1+\frac{M}{m}\right)\Delta &\leq \left(m-\frac{3L_2}{2}\Delta\right)\Delta, \\
\rho_i &\leq C_1\left(i+1\right)^{-\rho} \\
a_N &\leq C_2\left(i+1\right)^{-\alpha}
\end{aligned}$$

where $M = \min\left\{\left\|\ddot{F}\left(\boldsymbol{\theta}^\star\right)\right\|_2, \sqrt{K}\left|F\right|_{1,1}\right\}$. *We have*

$$\left\|\boldsymbol{\theta}^{(n+1)} - \boldsymbol{\theta}^\star\right\|_2 = O\left(n^{-(\alpha - D\rho)\wedge(D\rho)}\right).$$

Theorem 12. *For the Stochastic Approximation Scheme, suppose that Assumptions **Opt**, **Fun-**D and **AUB** hold and* $K\Delta\left|F\right|_{2,1} < \lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^\star\right)\right)$. *Moreover, for any* $i \geq 0$, *there are constants* $\Delta > 0$, $1 > \gamma > 0$, $\xi > 1$, $\rho > 0$, $\alpha > 0$, $C_1 > 0$, $C_2 > 0$, $C_3 > 0$ *and* $C_4 > 0$, *such that*

$$\begin{aligned}
\left\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^\star\right\|_2 &\leq \Delta, \\
\gamma_{i+1}\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^\star\right)\right) &< 1, \\
\left|\gamma_{i+1} - C_1\left(i+1\right)^{-\gamma}\right| &\leq C_2\left(i+1\right)^{-\xi}. \\
\rho_i &\leq C_3\left(i+1\right)^{-\rho}, \\
a_N &\leq C_4\left(i+1\right)^{-\alpha}, \\
\delta_1^{(i)} &\leq \left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^\star\right)\right) - K\Delta\left|F\right|_{2,1}\right)\Delta.
\end{aligned}$$

Then,

$$\left\|\boldsymbol{\theta}^{(n+1)} - \boldsymbol{\theta}^\star\right\|_2 = O\left(n^{-(\alpha - D\rho)\wedge(D\rho)}\right).$$

Remark 16. Theorems 11 and 12 can be used to study the convergence of a Gaussian quadrature (e.g., Monte Carlo integration), which is known to converge exponentially fast (see, e.g., Trefethen, 2008).

The following results provide upper bounds on the escape probabilities of INM and SAS algorithms, and a convergence rate for the SAS algorithm.

Theorem 13. *Suppose that* $\rho_i \downarrow 0$ *and* $B_i^2 = O\left(\Sigma_i\right)$. *Then, the escape probability can be characterized as follows:*

- *For the Inexact Newton Method, under **Opt**, **Fun-**D and **Hess**, provided* $\left\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^\star\right\|_2 \leq \Delta < \frac{2(1-\varepsilon)m}{3K\left|F\right|_{2,1}}$ *for* $0 < \varepsilon < 1$, *we have*

$$\mathbb{P}\left\{\max_{1\leq i\leq n+1}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 > \Delta\right\} \leq C\sum_{i=0}^{n}\left(\rho_i^{-2D}\Sigma_i + \rho_i^{2D-2}\right)$$

  where the constant C *depends upon* Δ, ε, m, M, $\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)$, K, P, D, ρ_0, $C_D\left(\mathcal{P}_0\right)$, $\left|F\right|_{2,1}$ *and* $\left|F\right|_{D,1}$.

- *For the Stochastic Approximation Scheme, under **Opt** and **Fun-**D, provided* $\gamma_{i+1} < \lambda_{\min}^{-1}\left(\ddot{F}\left(\boldsymbol{\theta}^\star\right)\right)$ *for any* i *and* $\Delta < \frac{2\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^\star\right)\right)}{K\left|F\right|_{2,1}}$, *we have*

$$\mathbb{P}\left\{\max_{1\leq i\leq n+1}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 > \Delta\right\} \leq C\sum_{i=0}^{n}\left(\rho_i^{-2D}\Sigma_i + \rho_i^{2D}\right)$$

  where the constant C *depends upon* Δ, $\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^\star\right)\right)$, $\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)$, K, P, D, ρ_0, $C_D\left(\mathcal{P}_0\right)$, $\left|F\right|_{2,1}$ *and* $\left|F\right|_{D,1}$.

Table 1: Escape probabilities for INM with $D = 2$ and SAS with $D = 1$ and $D = 2$.

Method	Rate	Conditions for convergence	Fastest rate	Conditions for fastest rate
INM with $D = 2$	$\sum_{i=0}^{n} i^{-(\nu-4\rho)\wedge(2\rho)}$	$\frac{1}{2} < \rho < \frac{\nu-1}{4}$	$\sum_{i=0}^{n} i^{-\frac{\nu}{3}} = \sum_{i=0}^{n} i^{-2\rho}$	$\nu = 6\rho$
SAS with $D = 1$	$\sum_{i=0}^{n} i^{-(\nu-2\rho)\wedge(2\rho)}$	$\frac{1}{2} < \rho < \frac{\nu-1}{2}$	$\sum_{i=0}^{n} i^{-\frac{\nu}{2}} = \sum_{i=0}^{n} i^{-2\rho}$	$\nu = 4\rho$
SAS with $D = 2$	$\sum_{i=0}^{n} i^{-(\nu-4\rho)\wedge(4\rho)}$	$\frac{1}{4} < \rho < \frac{\nu-1}{4}$	$\sum_{i=0}^{n} i^{-\frac{\nu}{2}} = \sum_{i=0}^{n} i^{-4\rho}$	$\nu = 8\rho$

Remark 17. Suppose that $N_i \asymp i^{\nu}$, $\rho_i \asymp i^{-\rho}$ and $\Sigma_i \asymp N_i^{-1}$. Then, we get the results in Table 1: "Rate" denotes the general rate, "Conditions for convergence" denotes the conditions under which the escape probability does not diverge, "Fastest rate" is the fastest possible rate, obtained balancing the two terms, and "Conditions for fastest rate" denotes the conditions under which the fastest rate is achieved.

Theorem 14. *For the Stochastic Approximation Scheme, under* ***Opt****,* ***Fun-****D,* ***MaV2****,* ***Hess****, suppose that* $N_i \asymp i^{\nu}$*,* $\rho_i \asymp i^{-\rho}$*,* $B_i \asymp N_i^{-\frac{1}{2}}$ *and* $\Sigma_i \asymp N_i^{-1}$*. If* $\Delta := \left\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^{\star}\right\|_2$*, suppose that*

$$\Delta + \Delta\gamma_{i+1}^2 K |F|_{1,1}^2 + \gamma_{i+1}\left(1 + \gamma_{i+1}\sqrt{K}|F|_{1,1}\right) b_i > 2\gamma_{i+1}\Delta m,$$
$$\left|\gamma_{i+1} - c_1 (i+1)^{-\gamma}\right| \leq c_2 (i+1)^{-\xi},$$

for $i \geq 0$*, with* $1 < \xi$*. If* $\nu > 2D\rho$*,* $\gamma + \frac{\nu}{2} > 1 + D\rho$*, and* $\gamma + D\rho > 1$*,*

$$\left\|\boldsymbol{\theta}^{(n+1)} - \boldsymbol{\theta}^{\star}\right\|_2 = \begin{cases} O_{\mathbb{P}}\left(n^{-\left(\frac{\nu}{4}-\frac{D\rho}{2}\right)\wedge\left(\frac{D\rho}{2}\right)}\right), & 1 > \gamma > 0, \\ O_{\mathbb{P}}\left(n^{-\left(\frac{\nu}{4}-\frac{D\rho}{2}\right)\wedge\left(\frac{D\rho}{2}\right)}\right), & \gamma = 1, 2mc_1 > \left(\frac{\nu}{2} - D\rho\right)\wedge(D\rho), \\ O_{\mathbb{P}}\left(n^{-mc_1}\ln^{\frac{1}{2}} n\right), & \gamma = 1, 2mc_1 = \left(\frac{\nu}{2} - D\rho\right)\wedge(D\rho), \\ O_{\mathbb{P}}\left(n^{-mc_1}\right), & \gamma = 1, 2mc_1 < \left(\frac{\nu}{2} - D\rho\right)\wedge(D\rho), \\ O_{\mathbb{P}}(1), & \gamma > 1. \end{cases}$$

Remark 18. For $0 < \gamma < 1$, the role of ρ in the convergence rate is ambiguous. Note that this is far from unexpected, as neighborhoods $\mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)$ that shrink too rapidly or not rapidly enough may both lead to problems of convergence. Let us take, for simplicity, the case $D = 1$. We first note that the results only hold for $1 - \gamma < \rho < \frac{\nu}{2} + \gamma - 1$. The exponent of n^{-1} in $O_{\mathbb{P}}\left(n^{-\left(\frac{\nu}{4}-\frac{\rho}{2}\right)\wedge\frac{\rho}{2}}\right)$ is $\left(\frac{\nu}{4} - \frac{\rho}{2}\right)\wedge\frac{\rho}{2}$. It is increasing in ν, thus suggesting that convergence is faster when N_i increases more steeply, but its behavior in ρ is not monotonic. It is increasing in ρ for $\rho < \frac{\nu}{4}$ and decreasing for $\rho > \frac{\nu}{4}$. This implies that both small and large values of ρ may lead to slower convergence rates.

4.4 Computation of the Hessian matrix at the optimal point

The algorithm of Section 4.2 can be used to calculate an estimate of the Hessian at the optimal point $\boldsymbol{\theta}^{\star}$. This may be useful both for INM and, especially, for SAS. The result is a simple consequence of Theorem 9.

Theorem 15. *Under* ***Opt*** *and* ***Fun-****D, for* $D \geq 2$*,*

$$\left\|\ddot{\hat{F}}(\boldsymbol{\theta}^{\star}) - \ddot{F}(\boldsymbol{\theta}^{\star})\right\|_F$$

$$\leq \rho^{-D} \frac{\rho_0^D P^{-\frac{1}{2}} \sqrt{\sum_{d=2}^D d^2 (d-1)^2 \rho^{2(d-2)}}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}} \max_{\boldsymbol{\theta}\in\boldsymbol{\theta}^\star \oplus \rho \mathsf{B}} \|\boldsymbol{\varepsilon}\|_2$$

$$+ \rho^{D-1} \left[\frac{\left(2 + C_D(\mathcal{P}_0)\left(1 + P^{\frac{1}{2}}\right)\right) D^2 (D-1) K^2}{(D-2)!} + \frac{1}{(D-3)!} \right] K^{D-2} |F|_{D,1}$$

where it is intended that $(-1)! = 1$.

Remark 19. If there is no error in the computation of the objective function, $\boldsymbol{\varepsilon} \equiv \mathbf{0}$ and

$$\left\| \ddot{\hat{F}}(\boldsymbol{\theta}^\star) - \ddot{F}(\boldsymbol{\theta}^\star) \right\|_F$$

$$\leq \rho^{D-1} \left[\frac{\left(2 + C_D(\mathcal{P}_0)\left(1 + P^{\frac{1}{2}}\right)\right) D^2 (D-1) K^2}{(D-2)!} + \frac{1}{(D-3)!} \right] K^{D-2} |F|_{D,1}.$$

If $D = 2$, then

$$\left\| \ddot{\hat{F}}(\boldsymbol{\theta}^\star) - \ddot{F}(\boldsymbol{\theta}^\star) \right\|_F \leq \rho \left[4K^2 \left(2 + C_2(\mathcal{P}_0)\left(1 + P^{\frac{1}{2}}\right)\right) + 1 \right] |F|_{2,1}.$$

5 Computational Aspects

Now we turn to the computational aspects of the algorithm.

5.1 Computation of the Design Matrix

We recall the definitions we introduced in Section 4.2. $\mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right) = \{\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_P\}$ is a set of P points in a neighborhood of $\boldsymbol{\theta}^{(i)}$. Let $\rho := \max_{\boldsymbol{\theta}_j \in \mathcal{P}_i(\boldsymbol{\theta}^{(i)})} \left\| \boldsymbol{\theta}_j - \boldsymbol{\theta}^{(i)} \right\|_2$ be the radius of the smallest closed ball centered in $\boldsymbol{\theta}^{(i)}$ and containing $\mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)$. Each point $\boldsymbol{\theta}_j \in \mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)$ corresponds to a vector $\mathbf{x}_D(\boldsymbol{\theta}_j)$.

Example 2. As an example, if $K = 1$ and $D = 2$, $\boldsymbol{\theta}_j = \theta_j$ and $\mathbf{x}_2(\boldsymbol{\theta}_j) = \left(1, \theta_j, \theta_j^2\right)'$.

We consider estimation of the regression

$$\hat{F}(\boldsymbol{\theta}_j) = \boldsymbol{\beta}' \mathbf{x}_D(\boldsymbol{\theta}_j) + \varepsilon_j$$

where $\mathbf{x}_D(\boldsymbol{\theta}_j)$ is the vector based on $\boldsymbol{\theta}_j$.

First of all, we remark that the result of the regression is invariant with respect to translations, i.e. the predictor does not change if we estimate the regression in which the data are first centered in $\boldsymbol{\theta}^{(i)}$. We adopt the transformation $\boldsymbol{\theta}_j \to \boldsymbol{\theta}_j - \boldsymbol{\theta}^{(i)}$, we build $\mathbf{x}_D\left(\boldsymbol{\theta}_j - \boldsymbol{\theta}^{(i)}\right)$ on the basis of $\boldsymbol{\theta}_j - \boldsymbol{\theta}^{(i)}$ and we estimate the regression as

$$\hat{F}(\boldsymbol{\theta}_j) = \tilde{\boldsymbol{\beta}}' \mathbf{x}_D\left(\boldsymbol{\theta}_j - \boldsymbol{\theta}^{(i)}\right) + \varepsilon_j.$$

The two regressions are observationally equivalent, provided the set of polynomials that are used is downward closed (see Migliorati, 2015), a condition that is always verified in our examples. For this reason, in the following we will always suppose that the points have been recentered in $\boldsymbol{\theta}^{(i)}$ and we will write, with a slight abuse of notation, $\mathbf{x}_D(\boldsymbol{\theta}_j)$ for $\mathbf{x}_D\left(\boldsymbol{\theta}_j - \boldsymbol{\theta}^{(i)}\right)$. This has another advantage, as it makes easier to compute the derivatives.

Example 3. If $K = 1$ and $D = 2$, $\mathbf{x}_D\left(\boldsymbol{\theta}_j - \boldsymbol{\theta}^{(i)}\right) = \left(1, \theta_j - \theta^{(i)}, \left(\theta_j - \theta^{(i)}\right)^2\right)'$.

A second problem is the fact that the design matrix tends to a singular matrix. The design matrix $\mathbf{X}$ is obtained by stacking the generic vectors $\mathbf{x}_D(\boldsymbol{\theta}_j)$ for $j = 1, \ldots, P$, i.e.

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_D'(\boldsymbol{\theta}_1) \\ \mathbf{x}_D'(\boldsymbol{\theta}_2) \\ \vdots \\ \mathbf{x}_D'(\boldsymbol{\theta}_P) \end{bmatrix}.$$

The point $\boldsymbol{\theta}^{(i)}$ is associated to a vector $\mathbf{x}_D\left(\boldsymbol{\theta}^{(i)}\right) = \mathbf{e}_1$, but also the other points in $\mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)$ are associated to vectors that converge towards $\mathbf{e}_1$. We must find a way to make the distance between the vectors $\mathbf{x}_D(\boldsymbol{\theta}_j)$ and $\mathbf{x}_D(\boldsymbol{\theta}_k)$, for $1 \leq j \neq k \leq P$, more salient. As above, we suppose that $\mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)$ is a dilated copy of the pointset $\mathcal{P}_0\left(\boldsymbol{\theta}^{(i)}\right) := \{\boldsymbol{\theta}_{0,1}, \ldots, \boldsymbol{\theta}_{0,P}\}$ with $\rho_0 := \max_j \left\|\boldsymbol{\theta}_{0,j} - \boldsymbol{\theta}^{(i)}\right\|_2$. Therefore,

$$\mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right) = \left\{\boldsymbol{\theta}^{(i)} + \frac{\rho}{\rho_0}\left(\boldsymbol{\theta}_{0,j} - \boldsymbol{\theta}^{(i)}\right) : \boldsymbol{\theta}_{0,j} \in \mathcal{P}_0\left(\boldsymbol{\theta}^{(i)}\right)\right\} = \left\{\frac{\rho}{\rho_0}\boldsymbol{\theta}_{0,j} : \boldsymbol{\theta}_{0,j} \in \mathcal{P}_0\left(\boldsymbol{\theta}^{(i)}\right)\right\}$$

where the latter equality comes from the fact that we suppose that $\boldsymbol{\theta}^{(i)} \equiv \mathbf{0}$. Let us define $h := \frac{\rho}{\rho_0}$. This means that $\mathbf{x}_D(\boldsymbol{\theta}_j)$ for $\boldsymbol{\theta}_j \in \mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)$ can be written as

$$\mathbf{x}_D(\boldsymbol{\theta}_j) = \mathbf{x}_D(h\boldsymbol{\theta}_{0,j}) = \mathbf{x}_D(\boldsymbol{\theta}_{0,j}) \odot \mathbf{h}$$

where $\mathbf{h}$ is the vector containing the powers of h according to the following rule: the k-th element of $\mathbf{h}$ has the same power of the k-th element of $\mathbf{x}_D(\boldsymbol{\theta}_{0,j})$.

Example 4. If $K = 1$ and $D = 2$, $\mathbf{h} = \left(1, h, h^2\right)'$. Therefore:

$$\mathbf{x}_D(\boldsymbol{\theta}_j) = \left(1, \theta_j - \theta^{(i)}, \left(\theta_j - \theta^{(i)}\right)^2\right)' = \left(1, h\left(\theta_{0,j} - \theta^{(i)}\right), h^2\left(\theta_{0,j} - \theta^{(i)}\right)^2\right)' = \mathbf{x}_D(\boldsymbol{\theta}_{0,j}) \odot \mathbf{h}.$$

If we define the design matrix $\mathbf{X}_0$ by stacking the generic vectors $\mathbf{x}_D(\boldsymbol{\theta}_{0,j})$ for $j = 1, \ldots, P$, we have

$$\mathbf{X} = \mathbf{X}_0 \mathrm{dg}(\mathbf{h}) = \mathbf{X}_0 \odot (\boldsymbol{\iota}_N \mathbf{h}') = \begin{bmatrix} \mathbf{x}_D'(\boldsymbol{\theta}_{0,1}) \odot \mathbf{h}' \\ \mathbf{x}_D'(\boldsymbol{\theta}_{0,2}) \odot \mathbf{h}' \\ \vdots \\ \mathbf{x}_D'(\boldsymbol{\theta}_{0,P}) \odot \mathbf{h}' \end{bmatrix}.$$

The OLS estimator is $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$. Therefore,

$$\begin{aligned}
\hat{\boldsymbol{\beta}} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} \\
&= \left(\mathrm{dg}(\mathbf{h})\,\mathbf{X}_0'\mathbf{X}_0 \mathrm{dg}(\mathbf{h})\right)^{-1}\mathrm{dg}(\mathbf{h})\,\mathbf{X}'\mathbf{y} \\
&= \mathrm{dg}\left(\overline{\mathbf{h}}\right)\left(\mathbf{X}_0'\mathbf{X}_0\right)^{-1}\mathrm{dg}\left(\overline{\mathbf{h}}\right)\mathrm{dg}(\mathbf{h})\,\mathbf{X}_0'\mathbf{y} \\
&= \mathrm{dg}\left(\overline{\mathbf{h}}\right)\left(\mathbf{X}_0'\mathbf{X}_0\right)^{-1}\mathbf{X}_0'\mathbf{y}
\end{aligned}$$

where $\overline{\mathbf{h}}$ is the elementwise reciprocal of $\mathbf{h}$.

It is worth noting that:

- if the points are kept in the same relative position with respect to $\boldsymbol{\theta}^{(i)}$ when i changes, the design matrix $\mathbf{X}_0$, as well as the term $\left(\mathbf{X}_0'\mathbf{X}_0\right)^{-1}\mathbf{X}_0'$, can be computed once and for all and stored for future uses;
- the OLS estimator can be computed by multiplying the diagonal matrix $\mathrm{dg}\left(\overline{\mathbf{h}}\right)$, the pre-computed $\left(\mathbf{X}_0'\mathbf{X}_0\right)^{-1}\mathbf{X}_0'$, and the vector $\mathbf{y}$ of the function evaluated in the points $\mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right) = \{\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_P\}$.

5.2 Computation of the Hessian Matrix

We stated in Section 3.2 that we can estimate the Hessian through the quantity $\ddot{\tilde{F}}(\boldsymbol{\theta})$. This is true only if the matrix is positive (semi)definite. If this is not the case, the solution is to compute a positive definite (or semidefinite) matrix that is near to $\ddot{\tilde{F}}(\boldsymbol{\theta})$. We start remarking that it is not possible to compute the nearest positive definite (pd) matrix as the set of pd matrices is not closed (we will be back to this later). Therefore, we will compute the nearest positive semidefinite (psd) matrix. This clearly depends on the concept of "distance" between two matrices. When the distance is computed through the Frobenius norm, there is a unique nearest positive semidefinite matrix, for which an algorithm has been given in Higham (1988). When the distance is given by the spectral norm, there are several nearest matrices; an algorithm for computing one of them has been provided in Halmos (1972) and discussed in Higham (1988). The following proposition summarizes the relevant results taken from Halmos (1972) and Higham (1988).

Proposition 1. *Let $\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}'$, with $\mathbf{U}'\mathbf{U} = \mathbf{I}$ and diagonal $\boldsymbol{\Lambda} = \mathrm{diag}\left(\lambda_1, \lambda_2, \ldots\right)$, be the spectral decomposition of $\ddot{\tilde{F}}(\boldsymbol{\theta})$. Then the nearest psd matrix in the Frobenius norm is defined by $\mathbf{U}\boldsymbol{\Lambda}_+\mathbf{U}'$, where $\boldsymbol{\Lambda}_+$ is the diagonal matrix in which negative elements of $\boldsymbol{\Lambda}$ are replaced by 0. The Frobenius distance between the two matrices is:*

$$\left\|\ddot{\tilde{F}}(\boldsymbol{\theta}) - \mathbf{U}\boldsymbol{\Lambda}_+\mathbf{U}'\right\|_F = \sqrt{\sum_{j:\lambda_j<0} \lambda_j^2}.$$

The matrix $\mathbf{U}\boldsymbol{\Lambda}_+\mathbf{U}'$ is also a nearest psd matrix in the spectral norm (despite not the only one) and it has distance:

$$\left\|\ddot{\tilde{F}}(\boldsymbol{\theta}) - \mathbf{U}\boldsymbol{\Lambda}_+\mathbf{U}'\right\|_2 = \max\left\{0, -\min_j \lambda_j\right\}.$$

Remark 20. (i) The two distances appearing in the statement of the proposition can be used to check for the adequacy of the matrix $\ddot{\tilde{F}}(\boldsymbol{\theta})$ as an approximation of the Hessian. If this distance is too large it may be a good idea to increase the number of simulations or move the points.
(ii) In case the matrix $\mathbf{U}\boldsymbol{\Lambda}_+\mathbf{U}'$ is psd, its inverse has to be replaced by the Moore–Penrose inverse that is given by $\left(\mathbf{U}\boldsymbol{\Lambda}_+\mathbf{U}'\right)^{\dagger} = \mathbf{U}\boldsymbol{\Lambda}_+^{\dagger}\mathbf{U}'$.

If a pd matrix is required, a solution is to regularize the matrix $\mathbf{U}\boldsymbol{\Lambda}_+\mathbf{U}'$. In this case, one may replace $\boldsymbol{\Lambda}_+$ with $\boldsymbol{\Lambda}_+ + \lambda\mathbf{I}$ for a small $\lambda > 0$:

$$\ddot{\tilde{F}}(\boldsymbol{\theta}) = \mathbf{U}\boldsymbol{\Lambda}_+\mathbf{U}' + \lambda\mathbf{I} = \mathbf{U}\left(\boldsymbol{\Lambda}_+ + \lambda\mathbf{I}\right)\mathbf{U}'.$$

In this case the final Hessian matrix is a linear combination of (a modification of) the Hessian matrix from the algorithm outlined in Section 3.2 and a diagonal matrix. The limiting case in which only the diagonal part is retained corresponds to the Stochastic Approximation Scheme case covered in Section 4.1.2.

5.3 Computational Complexity

It seems very difficult to compare on the same ground our method and the classical optimization algorithms based on numerical differentiation. Indeed, our algorithm is designed for noisy cases where the performances of the latter are expected to be bad.

However, we can still compute the number of function evaluations required by the two methods. The Newton–Raphson algorithm requires the computation of K first derivatives, K second non-mixed derivatives and $K(K-1)/2$ second mixed derivatives. Let us first consider the case of centered-difference approximations (see Eberly, 2020) and compute the number of evaluations necessary for numerical differentiation of order Q. For the first derivatives, we use $2Q$ evaluations, $4Q^2$ for the second mixed derivatives and $2Q+1$ for the second non-mixed derivatives. The final value is

$$K + 4KQ + 2K(K-1)Q^2.$$

Using the simplest possible value, i.e. $Q = 1$, the number of evaluations is:

$$1 + 2K + 2K^2.$$

The number of evaluations can be reduced by exploiting different numerical differentiation formulas and using the fact that some functions appear more than once in the formulas for the first two derivatives (see Monahan, 2011, pp. 200-203). In this way, the leading term can be reduced to K^2 or even $\frac{K^2}{2}$. On the other hand, our method requires P evaluations. The polynomial of degree D in K variables is obtained as the sum of $\frac{(K+D)!}{K!D!}$ monomials, therefore $P \geq \frac{(K+D)!}{K!D!}$. If we only use $D = 2$, $P \geq \frac{(K+2)(K+1)}{2}$. Therefore, our method competes favorably with the classical method as far as the number of evaluations is concerned.

As to the accuracy, we can see what happens when our method is applied to non-noisy data by taking $a_N \equiv 0$ in Assumption **AUB**. In this case, Theorem 9 yields $\delta_1^{(i)} = O\left(\rho^D\right)$ and $\delta_2^{(i)} = O\left(\rho^{D-1}\right)$. On the other hand, classical numerical differentiation yields $\delta_1^{(i)} = O\left(\varepsilon^{2Q}\right)$ and $\delta_2^{(i)} = O\left(\varepsilon^{2Q-1}\right)$.

As to the computational complexity of our method, we consider the case $D = 2$ and we adopt the real-number model of computation. We can reason as follows:[1]

1. populating the matrix $\mathbf{X}_0$ requires $O\left(PK^2\right)$ evaluations;
2. computing $\mathbf{X}_0'\mathbf{X}_0$ requires $O\left(PK^4\right)$ operations;
3. the inversion of $\mathbf{X}_0'\mathbf{X}_0$ requires $O\left(K^6\right)$ operations;
4. multiplying $\left(\mathbf{X}_0'\mathbf{X}_0\right)^{-1}$ and $\mathbf{X}_0'$ requires $O\left(PK^4\right)$ operations;
5. populating $\mathbf{y}$ requires $O(P\xi)$ operations, where ξ is the computational complexity of each function evaluation;
6. the multiplication of $\left(\mathbf{X}_0'\mathbf{X}_0\right)^{-1}\mathbf{X}_0'$, $\mathbf{y}$ and $\mathrm{dg}\left(\overline{\mathbf{h}}\right)$ requires $O\left(PK^2\right)$ operations;
7. populating the putative Hessian matrix $2\mathbf{D}_K^{+,\prime}\hat{\boldsymbol{\beta}}_3$, computing its eigendecomposition and inverting the Hessian matrix $\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)$ requires $O\left(K^3\right)$ operations;

[1] For multiplication and inversion of matrices we adopt the computational complexity of the classical algorithms; see Seri (2022, p. 6) for more details.

8. the other steps, like computing $\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)$ and updating $\boldsymbol{\theta}^{(i+1)} = \boldsymbol{\theta}^{(i)} - \left[\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right]^{-1} \dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)$, require fewer operations than the others.

Steps 1-4 can be performed just once. The other steps have to be repeated for I iterations. Taking into account that $P \geq \frac{(K+2)(K+1)}{2}$, the final computational complexity is

$$O\left(P\left[K^4 + I\left(\xi + K^2\right)\right]\right).$$

6 Applications

In this section we provide an application of our techniques in order to verify the finite-sample properties of GINM. In particular, we estimate the mean μ of a Gaussian random variable when stochastic equincontinuity is violated. The optimization procedure follows the algorithm described in Section 3.2.

6.1 Monte Carlo Experiment

The first example concerns the estimation of the mean μ of a Gaussian random variable with known variance. Although this is a trivial case, it allows to investigate in detail the properties of our algorithm. In the next lines, we describe the Monte Carlo experiment:

1. we simulate a sample of N independent random variables distributed as $\mathcal{N}(0,1)$, and we compute its empirical mean (i.e. the pseudo-true value $\boldsymbol{\theta}^\star$);
2. we select P points in a neighborhood of $\boldsymbol{\theta} = \mu$ and, for each point:
   (a) we simulate a sample of N independent random variables of the following data generating process $\mathbf{z}(\boldsymbol{\theta}) = \mu + \sigma\boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon} \sim \mathcal{N}(0,1)$ and $\sigma \equiv 1$;
   (b) we compute the empirical mean of $\mathbf{z}(\boldsymbol{\theta})$ and the objective function $\hat{F}(\boldsymbol{\theta}_j)$ via the Method of Simulated Moments;
   (c) we build the regression design matrix and we compute the scaling vector $\boldsymbol{\rho}$;
   (d) we compute the OLS estimator $\hat{\boldsymbol{\beta}}$ as explained in Section 5;
   (e) we calculate $\tilde{F}(\boldsymbol{\theta})$, using $\hat{\boldsymbol{\beta}}$ as predictor, the first derivative $\dot{\tilde{F}}(\boldsymbol{\theta})$ and the Hessian $\ddot{\tilde{F}}(\boldsymbol{\theta})$;
   (f) we substitute $\dot{\tilde{F}}(\boldsymbol{\theta})$ and $\ddot{\tilde{F}}(\boldsymbol{\theta})$ in the optimization routine and we compute $\widehat{\mu}$;
3. we repeat steps 2a-2g n times for fixed n.

For our purposes, we vary the sample size of the benchmark data S, the sample size of the simulated data N_i, the mesh of the grid ρ_i, the learning rate γ_i used in the stochastic approximation scheme and the number of iterations n, and we fix the other quantities. The values taken by these quantities are the following: $S = \{10, 100, 1000, 10000\}$, $N_i = 10 \cdot i^\nu$ for $\nu = \{1/4, 3/8, 1/2, 5/8, 3/4\}$, $\rho_i = i^{-\rho}$ with $\rho = \{0.1, 0.5, 0.9\}$, $\gamma_i = i^{-\gamma}$ for $\gamma = \{0.4, \ldots, 1.5\}$ taken on an equispaced grid of cardinality card $= 23$, and $n = \{10, 20, 40, 80, 160, 320, 640\}$. To test the validity of our procedure, we use as starting value $\boldsymbol{\theta}^{(0)} = 50$, which is very far from the true value. We replicate the experiment $R = 50000$ times, and we compute the sample mean squared error (MSE) of the estimator of μ across Monte Carlo runs.

The results of the Monte Carlo experiment for different combinations of ν, ρ, γ, n and S, are depicted in Figure 6.1, Figure 6.2, Figure 6.3 and Figure 6.4.

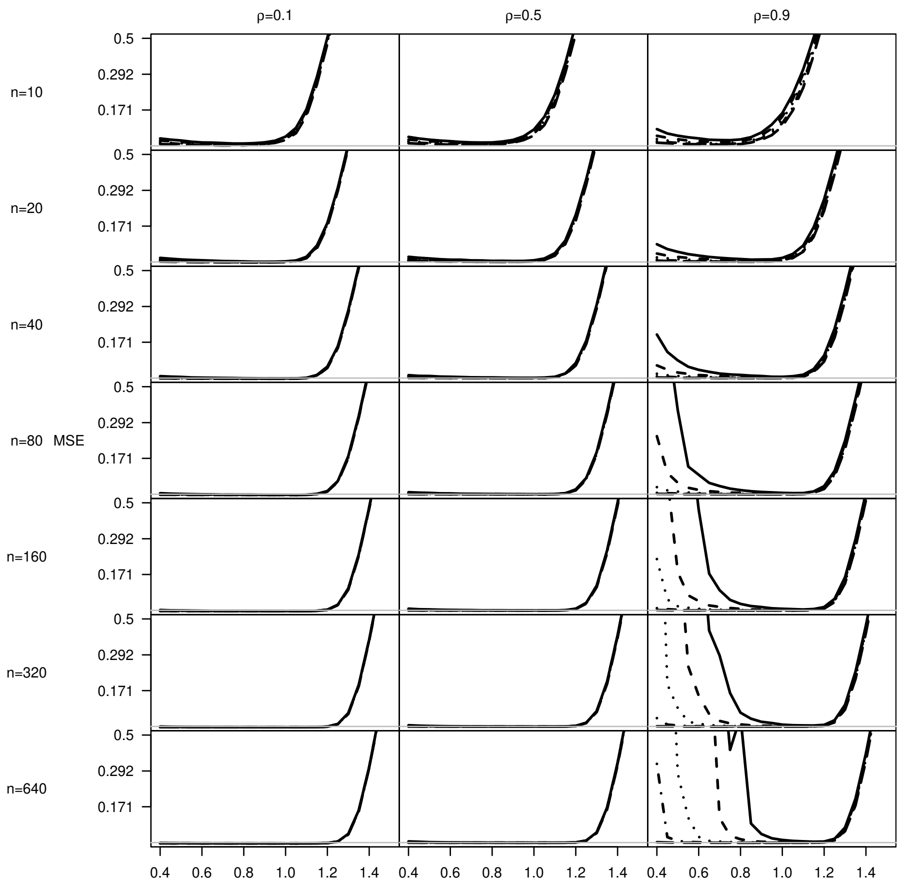

Figure 6.1: Behavior of the MSE of the estimator of μ for $S = 10$, $N_i = 10 \times i^{\frac{1}{4}}$ (solid line), $N_i = 10 \times i^{\frac{3}{8}}$ (dashed line), $N_i = 10 \times i^{\frac{1}{2}}$ (dotted line), $N_i = 10 \times i^{\frac{5}{8}}$ (dot-dashed line), $N_i = 10 \times i^{\frac{3}{4}}$ (long-dashed line), and different combinations of $\rho_i = i^{-\rho}$, $\gamma_i = i^{-\gamma}$ and n, with respect to its Cramér—Rao Lower Bound (grey horizontal line).

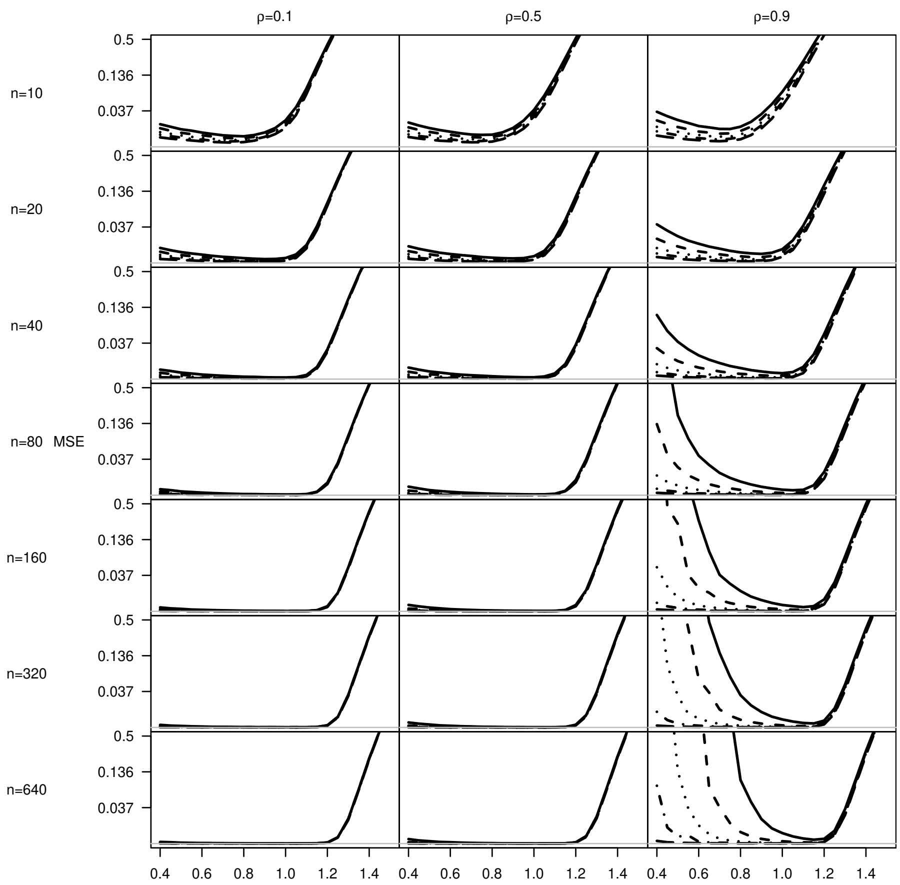

Figure 6.2: Behavior of the MSE of the estimator of μ for $S = 100$, $N_i = 10 \times i^{\frac{1}{4}}$ (solid line), $N_i = 10 \times i^{\frac{3}{8}}$ (dashed line), $N_i = 10 \times i^{\frac{1}{2}}$ (dotted line), $N_i = 10 \times i^{\frac{5}{8}}$ (dot-dashed line), $N_i = 10 \times i^{\frac{3}{4}}$ (long-dashed line), and different combinations of $\rho_i = i^{-\rho}$, $\gamma_i = i^{-\gamma}$ and n, with respect to its Cramér–Rao Lower Bound (grey horizontal line).

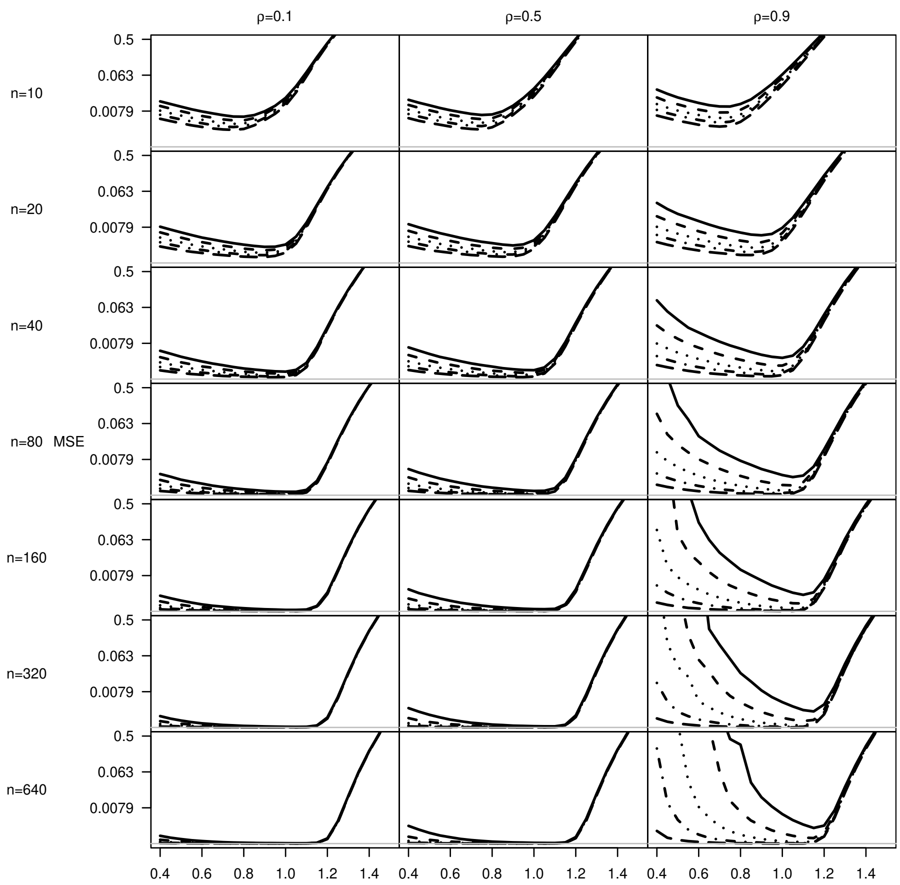

Figure 6.3: Behavior of the MSE of the estimator of μ for $S = 1000$, $N_i = 10 \times i^{\frac{1}{4}}$ (solid line), $N_i = 10 \times i^{\frac{3}{8}}$ (dashed line), $N_i = 10 \times i^{\frac{1}{2}}$ (dotted line), $N_i = 10 \times i^{\frac{5}{8}}$ (dot-dashed line), $N_i = 10 \times i^{\frac{3}{4}}$ (long-dashed line), and different combinations of $\rho_i = i^{-\rho}$, $\gamma_i = i^{-\gamma}$ and n, with respect to its Cramér–Rao Lower Bound (grey horizontal line).

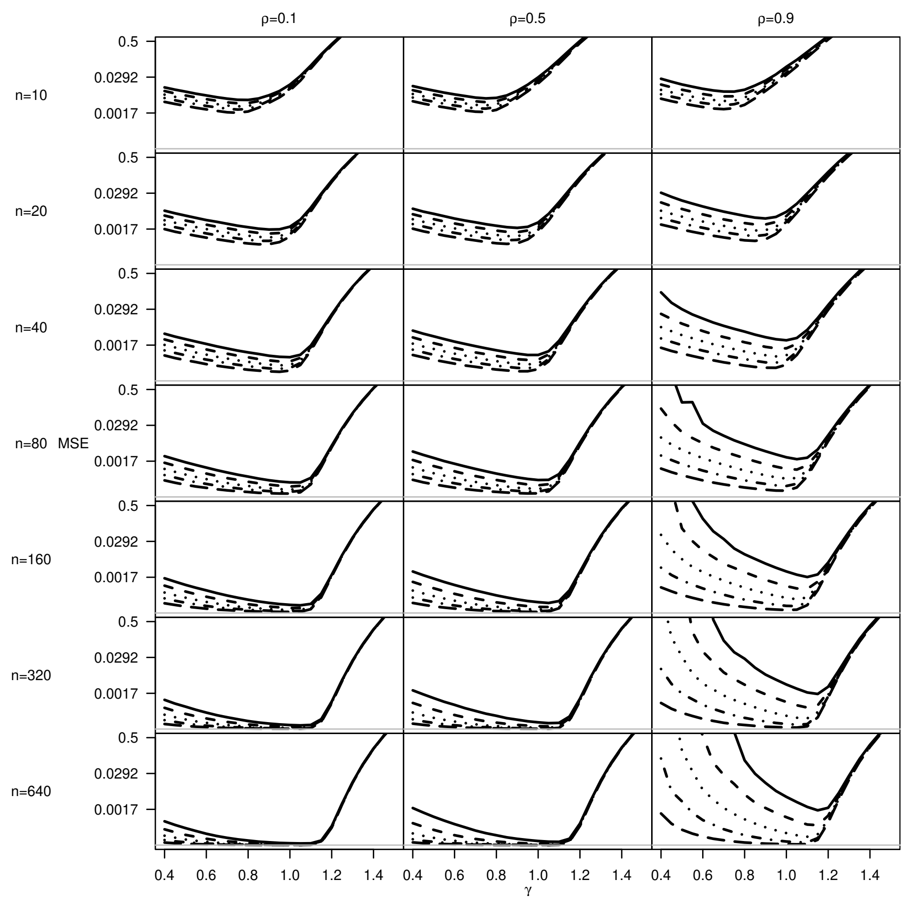

Figure 6.4: Behavior of the MSE of the estimator of μ for $S = 10000$, $N_i = 10 \times i^{\frac{1}{4}}$ (solid line), $N_i = 10 \times i^{\frac{3}{8}}$ (dashed line), $N_i = 10 \times i^{\frac{1}{2}}$ (dotted line), $N_i = 10 \times i^{\frac{5}{8}}$ (dot-dashed line), $N_i = 10 \times i^{\frac{3}{4}}$ (long-dashed line), and different combinations of $\rho_i = i^{-\rho}$, $\gamma_i = i^{-\gamma}$ and n, with respect to its Cramér–Rao Lower Bound (grey horizontal line).

We can state some considerations based on the outcomes of the simulation study. The first comment concerns the impact of the number of steps n of the algorithm on the MSE; the second regards the role of N_i and ρ_i; the third concerns the behavior of the learning parameter γ.

First of all, let us consider the role of the number of steps n of the algorithm. In most cases, the MSE approaches the Cramér–Rao Lower Bound (CRLB) as $n \to \infty$. Theorem 14 predicts that convergence is not guaranteed for $\gamma > 1$, $\gamma \leq 1 - \left(\frac{\nu}{2} - \rho\right) \vee \rho$, and for $\nu > 2\rho$. The figures are coherent with this fact: convergence seems to fail for large ρ especially for small γ, when the second condition boils down to $\gamma \leq 1-\rho$ and is verified. For small S, a fairly low number of iterations n can be sufficient to reach a value of the MSE near the CRLB. However, this is partly an artifact of the logarithmic ordinate axis.

Second, the dependence of the MSE on N_i is monotonic. Indeed, when N_i grows faster or, equivalently, ν is larger, the MSE approaches monotonically from above the CRLB. This is at odds with the dependence of the MSE on ρ_i and on the number of steps n of the algorithm. Indeed, for $\rho_i = i^{-0.1}$ and $\rho_i = i^{-0.5}$, the MSE is monotonically decreasing in n. However, this behavior breaks down for $\rho_i = i^{-0.9}$: (i) for larger ν, the MSE decreases as n increases; (ii) for smaller ν, an increasing n may lead to a higher MSE, especially for small values of γ. Otherwise stated, for small n, say $n < 80$, the optimization routine acts as in the case of $\rho_i = \rho^{-0.1}$ and $\rho_i = \rho^{-0.5}$, while, for $n \geq 80$, the effect of N_i is predominant, as the optimization scheme seems to reach the CRLB only when N_i grows fast enough. This is in line with the results in Table 1 and in Theorem 14: first, from Table 1 the escape probability for SAS with $D = 1$ diverges when ρ is large and ν is small, i.e. when ρ_i decreases slowly (as in $\rho_i = i^{-0.9}$) and N_i diverges slowly (as in $N_i = 10 \times i^{\frac{1}{4}}$); second, the convergence rate in Theorem 14 is $O_{\mathbb{P}}\left(n^{-\left(\frac{\nu}{4}-\frac{\rho}{2}\right)\wedge\frac{\rho}{2}}\right)$, it worsens when ρ is large and ν is small and no convergence is guaranteed when $\nu > 2\rho$. In this case, a larger n has a negative effect on the convergence rate.

Third, for $\gamma = 1$, Theorem 14 predicts a discontinuity in the behavior of the MSE, as for $\gamma > 1$ the algorithm is not guaranteed to converge. However, the simulation experiment tells a more nuanced story. The discontinuity seems to depend on n. In particular, as n increases the algorithm converges also for values of γ larger than 1.

Given all the above, we can claim that the Monte Carlo experiment seems to confirm the theoretical results.

7 Conclusions

In this paper, we provide some new, general results of interest for both the econometric and the machine learning literature.

In the first part of the paper, we study some conditions under which the sequence of values produced by an optimization algorithm, $\left\{\boldsymbol{\theta}^{(i)}\right\}$, converges to the optimum of the function, $\boldsymbol{\theta}^{\star}$. These assumptions do not depend on the specific choice of the approximated gradient and Hessian and are of general interest as they expand some results in the optimization and machine learning literature. The main results of this part of the paper concern the analysis of the algorithms as a function of the number of steps: (i) rigorous results on the convergence rates of INM and SAS; (ii) upper bounds on the escape probabilities of INM and SAS; (iii) upper bounds on the probability that SAS don't visit a region where the score is near to zero.

Subsequently, we propose a special version of the inexact Newton method (see Dembo et al., 1982), i.e. GINM, which has been thought to estimate complex and intractable objective functions (e.g., discontinuous, non-differentiable, non-convex criterion function), even when the (stochastic) equicontinuity hypothesis is violated. We start by selecting P points $\mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)=\{\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_P\}$ in a neighborhood of $\boldsymbol{\theta}^{(i)}$, we compute the values $\widehat{F}(\boldsymbol{\theta})$ for $\boldsymbol{\theta}\in\mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)$, and we estimate through a regression a locally approximating function $\widetilde{F}(\cdot)$ based on these points. We distinguish two cases: (i) when dealing with SAS, we estimate $\widetilde{F}(\cdot)$ using a linear regression; (ii) when dealing with INM, we exploit a quadratic regression for the estimation of $\widetilde{F}(\cdot)$. Once $\widetilde{F}(\boldsymbol{\theta})$ has been estimated, we calculate its first and second derivatives (i.e. only the the first derivative for SAS and both the first and second derivative for INM) and we substitute them in the optimization routine to find $\boldsymbol{\theta}^{\star}$.

After discussing the general construction of the optimization algorithms, we show their asymptotic properties by providing some upper bounds on the approximation error of $\tilde{F}(\cdot)$, $\dot{\tilde{F}}(\cdot)$ and $\ddot{\tilde{F}}(\cdot)$. Moreover, since most optimization methods rely on the construction of a series of values $\boldsymbol{\theta}^{(i)}$ that should approach $\boldsymbol{\theta}^{\star}$, we provide some rigorous bounds for $\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^{\star}\right\|_2$. These bounds can be applied also in the study of the impact of numerical differentiation in optimization algorithms. Combining the latter results and the bounds on the (approximated) objective function and its derivatives, we prove several convergence results. We stress that, provided some quantities characterizing the algorithm are chosen judiciously, the limit of the sequence $\left\{\boldsymbol{\theta}^{(i)}\right\}$ is the minimum of $F(\cdot)$ and is independent of $\widehat{F}(\cdot)$, at odds with what happens with classical simulation-based estimation algorithms. The computational aspects of the GINM are also treated. The GINM is finally used in an extensive MC experiment. The outcomes of the MC experiment confirm the theoretical results.

8 Proofs

In this section we use the definition $\boldsymbol{\eta}^{(i)} := \dot{\hat{F}}\left(\boldsymbol{\theta}^{(i)}\right) - \dot{F}\left(\boldsymbol{\theta}^{(i)}\right)$.

8.1 Preliminary Lemmas

We will need some preliminary lemmas. The first concerns some properties of the derivatives of F, the second provides a bound on the size of the vector $\mathbf{r}^{(i)} := \dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\left[\ddot{\hat{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right]^{-1}\dot{\hat{F}}\left(\boldsymbol{\theta}^{(i)}\right)$, the third yields a solution to the recursive inequality $x_{n+1} \leq a_n + b_n x_n$, the fourth, fifth and sixth provide bounds for generalized harmonic numbers and related sequences, the seventh majorizes a sum with an integral, the eighth gives an asymptotic expression for a series, the ninth characterizes the behavior of $\lambda_{\min}\left(\mathbf{X}'\mathbf{X}\right)$ when the points in $\mathcal{P}_i\left(\boldsymbol{\theta}^{(i)}\right)$ shrink towards $\boldsymbol{\theta}^{(i)}$, and the last provides upper bounds on the norm of vectors of monomials.

Lemma 2. *Under **Lip**-1, we have*

$$\left|F\left(\boldsymbol{\theta}_1\right) - F\left(\boldsymbol{\theta}_2\right) - \left(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\right)'\dot{F}\left(\boldsymbol{\theta}_1\right)\right| \leq \frac{L_1}{2}\left\|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\right\|_2^2,$$

$$\left\|\dot{F}\left(\boldsymbol{\theta}\right)\right\|_2 \leq L_1\left\|\boldsymbol{\theta} - \boldsymbol{\theta}^\star\right\|_2.$$

*Under **Lip**-2, the following inequalities hold:*

$$\left\|\dot{F}\left(\boldsymbol{\theta}_1\right) - \dot{F}\left(\boldsymbol{\theta}_2\right) - \ddot{F}\left(\boldsymbol{\theta}_1\right)\left(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\right)\right\|_2 \leq \frac{L_2}{2}\left\|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\right\|_2^2,$$

$$\left\|\dot{F}\left(\boldsymbol{\theta}\right)\right\|_2 \leq \left\|\ddot{F}\left(\boldsymbol{\theta}^\star\right)\right\|_2\left\|\boldsymbol{\theta} - \boldsymbol{\theta}^\star\right\|_2 + \frac{L_2}{2}\left\|\boldsymbol{\theta} - \boldsymbol{\theta}^\star\right\|_2^2,$$

$$\left|\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}\right)\right) - \lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^\star\right)\right)\right| \leq L_2\left\|\boldsymbol{\theta} - \boldsymbol{\theta}^\star\right\|_2$$

and

$$\left|F\left(\boldsymbol{\theta}_2\right) - F\left(\boldsymbol{\theta}_1\right) - \left(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1\right)'\dot{F}\left(\boldsymbol{\theta}_1\right) - \frac{\left(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1\right)'\ddot{F}\left(\boldsymbol{\theta}_1\right)\left(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1\right)}{2}\right|$$
$$\leq \frac{L_2}{6}\left\|\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1\right\|_2^3.$$

Proof. We start from the first inequality. First of all, we define $g\left(t\right) := F\left(\boldsymbol{\theta}_1 + t\left(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1\right)\right)$ for $t \in [0,1]$. Then, from the differentiability of F, we have

$$g\left(1\right) = g\left(0\right) + g'\left(0\right) + \int_0^1 \left(g'\left(t\right) - g'\left(0\right)\right)\mathrm{d}t$$

or

$$F\left(\boldsymbol{\theta}_2\right) = F\left(\boldsymbol{\theta}_1\right) + \left(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1\right)'\dot{F}\left(\boldsymbol{\theta}_1\right) + \int_0^1 \left(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1\right)'\left(\dot{F}\left(\boldsymbol{\theta}_1 + t\left(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1\right)\right) - \dot{F}\left(\boldsymbol{\theta}_1\right)\right)\mathrm{d}t.$$

From **Lip-**1**,** this can be written as

$$\left|F\left(\boldsymbol{\theta}_2\right) - F\left(\boldsymbol{\theta}_1\right) - \left(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1\right)'\dot{F}\left(\boldsymbol{\theta}_1\right)\right|$$
$$\leq \left\|\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1\right\|_2 \int_0^1 \left\|\dot{F}\left(\boldsymbol{\theta}_1 + t\left(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1\right)\right) - \dot{F}\left(\boldsymbol{\theta}_1\right)\right\|_2 \mathrm{d}t$$

$$\leq L_1 \left\|\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1\right\|_2^2 \int_0^1 t\mathrm{d}t = \frac{L_1}{2} \left\|\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1\right\|_2^2.$$

The second one is trivially true from **Lip-**1.

In what follows, we suppose that **Lip-**2 holds true.

The third inequality can be proved along the lines of the first one, defining $g(t) := \boldsymbol{\lambda}'\dot{F}\left(\boldsymbol{\theta}_1 + t\left(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1\right)\right)$ for $t \in [0,1]$ and $\boldsymbol{\lambda} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$.

For the fourth inequality, we write

$$\dot{F}(\boldsymbol{\theta}) = \ddot{F}(\boldsymbol{\theta}^\star)(\boldsymbol{\theta} - \boldsymbol{\theta}^\star) + \left\{\dot{F}(\boldsymbol{\theta}) - \dot{F}(\boldsymbol{\theta}^\star) - \ddot{F}(\boldsymbol{\theta}^\star)(\boldsymbol{\theta} - \boldsymbol{\theta}^\star)\right\},$$

$$\left\|\dot{F}(\boldsymbol{\theta})\right\|_2 \leq \left\|\ddot{F}(\boldsymbol{\theta}^\star)\right\|_2 \left\|\boldsymbol{\theta} - \boldsymbol{\theta}^\star\right\|_2 + \left\|\dot{F}(\boldsymbol{\theta}) - \dot{F}(\boldsymbol{\theta}^\star) - \ddot{F}(\boldsymbol{\theta}^\star)(\boldsymbol{\theta} - \boldsymbol{\theta}^\star)\right\|_2. \tag{8.1}$$

Using the third inequality, we get the final result.

For the fifth inequality, a consequence of Courant–Fischer theorem is that, for Hermitian $\mathbf{A}$ and $\mathbf{B}$, $\left|\lambda_{\min}(\mathbf{A}) - \lambda_{\min}(\mathbf{B})\right| \leq \left\|\mathbf{A} - \mathbf{B}\right\|_2$. Therefore,

$$\left|\lambda_{\min}\left(\ddot{F}(\boldsymbol{\theta}^\star)\right) - \lambda_{\min}\left(\ddot{F}(\boldsymbol{\theta})\right)\right| \leq \left\|\ddot{F}(\boldsymbol{\theta}) - \ddot{F}(\boldsymbol{\theta}^\star)\right\|_2.$$

Combining this with **Lip-**2, we get the result.

At last, as above, we define $g(t) := F\left(\boldsymbol{\theta}_1 + t\left(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1\right)\right)$ for $t \in [0,1]$. Then, the first-order Taylor expansion with integral remainder yields

$$\begin{aligned}
g(t) &= g(0) + g'(0)\,t + \int_0^t g''(s)(t-s)\,\mathrm{d}s \\
&= g(0) + g'(0)\,t + \int_0^t g''(t-u)\,u\mathrm{d}u, \\
g(1) &= g(0) + g'(0) + \frac{g''(0)}{2} + \int_0^1 \left[g''(1-u) - g''(0)\right] u\mathrm{d}u.
\end{aligned}$$

From this and **Lip-**2, one gets

$$\begin{aligned}
&\left|F(\boldsymbol{\theta}_2) - F(\boldsymbol{\theta}_1) - (\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1)'\dot{F}(\boldsymbol{\theta}_1) - \frac{(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1)'\ddot{F}(\boldsymbol{\theta}_1)(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1)}{2}\right| \\
&\leq \left\|\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1\right\|_2^2 \int_0^1 \left\|\ddot{F}\left(\boldsymbol{\theta}_1 + (1-u)(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1)\right) - \ddot{F}(\boldsymbol{\theta}_1)\right\|_2 u\mathrm{d}u \\
&\leq L_2 \left\|\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1\right\|_2^3 \int_0^1 (1-u)\,u\mathrm{d}u = \frac{L_2}{6}\left\|\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1\right\|_2^3.
\end{aligned}$$

QED

Lemma 3. *Under **Lip**-1, if $\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right) > \delta_2^{(i)}$, we have*

$$\left\|\mathbf{r}^{(i)}\right\|_2 \leq \frac{\delta_1^{(i)}\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right) + \delta_2^{(i)} L_1 \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right) - \delta_2^{(i)}}$$

and

$$\left\|\mathbf{r}^{(i)}\right\|_2 \le \frac{\delta_1^{(i)}\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right) + \delta_2^{(i)}\left(\left\|\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right\|_2\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 + \frac{L_2}{2}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2^2\right)}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right) - \delta_2^{(i)}}.$$

*Under **Lip**-2, if* $\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) > K\,|F|_{2,1}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 + \delta_2^{(i)}$, *we have*

$$\begin{aligned}\left\|\mathbf{r}^{(i)}\right\|_2 \le \delta_1^{(i)}&\frac{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - L_2\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - L_2\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 - \delta_2^{(i)}}\\ &+ \delta_2^{(i)}\frac{L_1\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - L_2\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 - \delta_2^{(i)}}\end{aligned}$$

and

$$\begin{aligned}\left\|\mathbf{r}^{(i)}\right\|_2 \le \delta_1^{(i)}&\frac{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - L_2\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - L_2\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 - \delta_2^{(i)}}\\ &+ \delta_2^{(i)}\frac{\left\|\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right\|_2\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 + \frac{L_2}{2}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2^2}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - L_2\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 - \delta_2^{(i)}}.\end{aligned}$$

Proof. We write

$$\begin{aligned}\left\|\mathbf{r}^{(i)}\right\|_2 &= \left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\left[\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right]^{-1}\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2\\ &\le \delta_1^{(i)} + \left\|\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right) - \ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\left[\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right]^{-1}\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2\\ &\le \delta_1^{(i)} + \left\|\mathbf{I}_K - \ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\left[\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right]^{-1}\right\|_2\left\|\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2\\ &\le \delta_1^{(i)} + \delta_2^{(i)}\left\|\left[\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right]^{-1}\right\|_2\left\|\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2\\ &\le \delta_1^{(i)} + \frac{\delta_2^{(i)}\left\|\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2}{\lambda_{\min}\left[\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right]}\end{aligned}\tag{8.2}$$

where we have used the fact that $\left\|\left[\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right]^{-1}\right\|_2 = \lambda_{\max}\left(\left[\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right]^{-1}\right) = \lambda_{\min}^{-1}\left[\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right]$.

First, we majorize $\left\|\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2$ in (8.2) as

$$\left\|\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 \le \left\|\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right) - \dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 + \left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 = \delta_1^{(i)} + \left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2.\tag{8.3}$$

Therefore, from the second and fourth inequalities of Lemma 2, we have

$$\left\|\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 \le \delta_1^{(i)} + L_1\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2$$

or

$$\left\|\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 \leq \delta_1^{(i)} + \left\|\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right\|_2 \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 + \frac{L_2}{2}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2^2.$$

Second, we deal with $\lambda_{\min}\left[\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right]$ in (8.2). From Weyl's inequality $\lambda_{\min}(\mathbf{A}+\mathbf{B}) \leq \lambda_{\min}(\mathbf{A}) + \lambda_{\max}(\mathbf{B})$ for Hermitian $\mathbf{A}$ and $\mathbf{B}$, taking $\mathbf{A} = \ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)$ and $\mathbf{B} = \ddot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)$, we get

$$\begin{aligned}\lambda_{\min}\left(\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right) &\geq \lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right) - \lambda_{\max}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right)\\ &\geq \lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right) - \left\|\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 = \lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right) - \delta_2^{(i)}.\end{aligned}$$

Replacing these formulas into (8.2), we get the first bounds in the forms

$$\left\|\mathbf{r}^{(i)}\right\|_2 \leq \delta_1^{(i)} + \frac{\delta_2^{(i)}\left(\delta_1^{(i)} + L_1\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2\right)}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right) - \delta_2^{(i)}}$$

and

$$\left\|\mathbf{r}^{(i)}\right\|_2 \leq \delta_1^{(i)} + \frac{\delta_2^{(i)}\left(\delta_1^{(i)} + \left\|\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right\|_2\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 + \frac{L_2}{2}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2^2\right)}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right) - \delta_2^{(i)}}.$$

This may be sufficient for most applications.

However, in the following we remove the dependence on $\boldsymbol{\theta}^{(i)}$ from $\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right)$ in the denominator. From $\lambda_{\min}\left(\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right) \geq \lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right) - \delta_2^{(i)}$, using the fifth inequality of Lemma 2:

$$\lambda_{\min}\left(\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right) \geq \lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - L_2\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 - \delta_2^{(i)}.$$

At last we get the formula of the statement. QED

Lemma 4. *Let the sequence* $\{x_n\}_{n\geq 0}$ *be such that*

$$x_{n+1} \leq a_n + b_n x_n, \quad n \geq 0,$$

for two sequences $\{a_n\}_{n\geq 0}$ *and* $\{b_n\}_{n\geq 0}$. *Then*

$$x_{n+1} \leq \sum_{j=0}^{n} a_j \prod_{k=j+1}^{n} b_k + x_0 \prod_{k=0}^{n} b_k, \tag{8.4}$$

where, by convention, empty products are equal to 1.

Remark 21. Recurrences of the form $x_{n+1} = a_n + b_n x_n$ have been considered, with some modifications, by Vervaat (1979); Bougerol and Picard (1992); Rachev and Samorodnitsky (1995); Babillot et al. (1997) among many others.

Proof. We prove the recurrence by induction. First of all, for $n = 0$, we have

$$x_1 \leq a_0 + b_0 x_0.$$

Moreover, if (8.4) holds for $n = h - 1$, it also holds for $n = h$. Indeed,

$$\begin{aligned} x_{h+1} &\le a_h + b_h x_h \\ &\le a_h + b_h \left(\sum_{j=0}^{h-1} a_j \prod_{k=j+1}^{h-1} b_k + x_0 \prod_{k=0}^{h-1} b_k \right) \\ &= \sum_{j=0}^{h} a_j \prod_{k=j+1}^{h} b_k + x_0 \prod_{k=0}^{h} b_k. \end{aligned}$$

This proves the claim. QED

Lemma 5. *For $-1 < \alpha$ and $\alpha \neq 1$,*

$$-\frac{\alpha}{8} n^{-1-\alpha} \le \sum_{k=1}^{n} k^{-\alpha} - \zeta(\alpha) - \frac{1}{1-\alpha} n^{1-\alpha} - \frac{1}{2} n^{-\alpha} \le 0.$$

For $\alpha = 1$,

$$0 \le \sum_{k=1}^{n} k^{-1} - \gamma_{(0)} - \ln n \le \frac{1}{n},$$

where $\gamma_{(0)}$ denotes the Euler–Mascheroni constant.

Remark 22. The proof of the first inequality is inspired by that in robjohn (https://math.stackexchange.com/users/13854/rob

Proof. We first note (see, e.g., Seri, 2015, Eq. (1)) that

$$\sum_{k=1}^{n} k^{-\alpha} = \zeta(\alpha) + \frac{1}{1-\alpha} n^{1-\alpha} + \frac{1}{2} n^{-\alpha} + O\left(n^{-1-\alpha}\right).$$

Through integration by parts of a Riemann–Stieltjes integral,

$$\begin{aligned} \sum_{k=1}^{n} k^{-\alpha} &= \int_{1^-}^{n^+} x^{-\alpha} \mathrm{d} \lfloor x \rfloor \\ &= \int_{1}^{n} x^{-\alpha} \mathrm{d}x - \int_{1^-}^{n^+} x^{-\alpha} \mathrm{d} \left(\{x\} - \frac{1}{2} \right) \\ &= \frac{1}{1-\alpha} \left(n^{1-\alpha} - 1 \right) + \frac{1}{2} n^{-\alpha} + \frac{1}{2} - \alpha \int_{1}^{n} x^{-1-\alpha} \left(\{x\} - \frac{1}{2} \right) \mathrm{d}x. \end{aligned} \tag{8.5}$$

We equate the two expressions and take the limit for $n \to \infty$:

$$\zeta(\alpha) = -\frac{1}{1-\alpha} + \frac{1}{2} - \alpha \int_{1}^{\infty} x^{-1-\alpha} \left(\{x\} - \frac{1}{2} \right) \mathrm{d}x.$$

Using this formula in (8.5), we get

$$\begin{aligned} \sum_{k=1}^{n} k^{-\alpha} &= \frac{1}{1-\alpha} \left(n^{1-\alpha} - 1 \right) + \frac{1}{2} n^{-\alpha} + \frac{1}{2} - \alpha \int_{1}^{n} x^{-1-\alpha} \left(\{x\} - \frac{1}{2} \right) \mathrm{d}x \\ &= \frac{1}{1-\alpha} \left(n^{1-\alpha} - 1 \right) + \frac{1}{2} n^{-\alpha} + \frac{1}{2} - \alpha \int_{1}^{n} x^{-1-\alpha} \left(\{x\} - \frac{1}{2} \right) \mathrm{d}x \end{aligned}$$

$$+\zeta(\alpha)+\frac{1}{1-\alpha}-\frac{1}{2}+\alpha\int_1^{\infty}x^{-1-\alpha}\left(\{x\}-\frac{1}{2}\right)\mathrm{d}x$$
$$=\zeta(\alpha)+\frac{1}{1-\alpha}n^{1-\alpha}+\frac{1}{2}n^{-\alpha}+\alpha\int_n^{\infty}x^{-1-\alpha}\left(\{x\}-\frac{1}{2}\right)\mathrm{d}x.$$

The last integral can be written as a telescoping series of integrals:

$$\int_n^{\infty}x^{-1-\alpha}\left(\{x\}-\frac{1}{2}\right)\mathrm{d}x=\sum_{k=n}^{\infty}\int_k^{k+1}x^{-1-\alpha}\left(\{x\}-\frac{1}{2}\right)\mathrm{d}x.$$

We note that

$$\left|\int_k^{k+1}x^{-1-\alpha}\left(\{x\}-\frac{1}{2}\right)\mathrm{d}x\right|$$
$$=\left|\int_k^{k+1}\left(x^{-1-\alpha}-\frac{(k+1)^{-1-\alpha}+k^{-1-\alpha}}{2}\right)\left(\{x\}-\frac{1}{2}\right)\mathrm{d}x\right|$$
$$\leq\max_{x\in[k,k+1]}\left|x^{-1-\alpha}-\frac{(k+1)^{-1-\alpha}+k^{-1-\alpha}}{2}\right|\int_k^{k+1}\left|\{x\}-\frac{1}{2}\right|\mathrm{d}x$$
$$=\frac{k^{-1-\alpha}-(k+1)^{-1-\alpha}}{8}$$

from which

$$\left|\int_n^{\infty}x^{-1-\alpha}\left(\{x\}-\frac{1}{2}\right)\mathrm{d}x\right|\leq\sum_{k=n}^{\infty}\frac{k^{-1-\alpha}-(k+1)^{-1-\alpha}}{8}=\frac{n^{-1-\alpha}}{8}.$$

This implies that

$$\left|\sum_{k=1}^{n}k^{-\alpha}-\zeta(\alpha)-\frac{1}{1-\alpha}n^{1-\alpha}-\frac{1}{2}n^{-\alpha}\right|\leq\frac{\alpha n^{-1-\alpha}}{8}.$$

However, we can improve the bound in one direction. Indeed,

$$\int_k^{k+1}x^{-1-\alpha}\left(\{x\}-\frac{1}{2}\right)\mathrm{d}x$$
$$=\int_0^{1}(x+k)^{-1-\alpha}\left(x-\frac{1}{2}\right)\mathrm{d}x$$
$$=\int_0^{\frac{1}{2}}\left[(k+1-x)^{-1-\alpha}-(x+k)^{-1-\alpha}\right]\left(\frac{1}{2}-x\right)\mathrm{d}x.$$

Now, for $x\in\left[0,\frac{1}{2}\right]$, $x+k<k+1-x$ and $(x+k)^{-1-\alpha}>(k+1-x)^{-1-\alpha}$. Therefore, the integral is negative. At last,

$$-\frac{n^{-1-\alpha}}{8}\leq\int_n^{\infty}x^{-1-\alpha}\left(\{x\}-\frac{1}{2}\right)\mathrm{d}x\leq 0.$$

From this, the first result follows. The inequalities for $\alpha=1$ are mentioned, e.g., in Jameson (2015, p. 75), where even more accurate bounds are proposed. QED

Lemma 6. *Let* $\left|\gamma_{i+1}-c_1(i+1)^{-\gamma}\right|\leq c_2(i+1)^{-\xi}$ for $\xi>\gamma>0$. If $\gamma\neq 1$ and $\xi\neq 1$,

$$(c_1\zeta(\gamma)-c_2\zeta(\xi))+\frac{c_1}{1-\gamma}i^{1-\gamma}+\frac{c_1}{2}i^{-\gamma}-\frac{c_1\gamma}{8}i^{-1-\gamma}-\frac{c_2}{1-\xi}i^{1-\xi}-\frac{c_2}{2}i^{-\xi}$$

$$\leq \sum_{k=0}^{i-1} \gamma_{k+1} \leq \left(c_1\zeta\left(\gamma\right) + c_2\zeta\left(\xi\right)\right) + \frac{c_1}{1-\gamma} i^{1-\gamma} + \frac{c_1}{2} i^{-\gamma} + \frac{c_2}{1-\xi} i^{1-\xi} + \frac{c_2}{2} i^{-\xi}.$$

If $\gamma = 1$,

$$\begin{aligned}
&\left(c_1\gamma_{(0)} - c_2\zeta\left(\xi\right)\right) + c_1 \ln i - \frac{c_2}{1-\xi} i^{1-\xi} - \frac{c_2}{2} i^{-\xi} \\
&\leq \sum_{k=0}^{i-1} \gamma_{k+1} \leq \left(c_1\gamma_{(0)} + c_2\zeta\left(\xi\right)\right) + c_1 \ln i + c_1 i^{-1} + \frac{c_2}{1-\xi} i^{1-\xi} + \frac{c_2}{2} i^{-\xi}.
\end{aligned}$$

Proof. From $\left|\gamma_{i+1} - c_1\left(i+1\right)^{-\gamma}\right| \leq c_2\left(i+1\right)^{-\xi}$, we have

$$\begin{aligned}
c_1\left(k+1\right)^{-\gamma} - c_2\left(k+1\right)^{-\xi} \leq \gamma_{k+1} \leq c_1\left(k+1\right)^{-\gamma} + c_2\left(k+1\right)^{-\xi}, \\
c_1\sum_{k=0}^{i-1}\left(k+1\right)^{-\gamma} - c_2\sum_{k=0}^{i-1}\left(k+1\right)^{-\xi} \leq \sum_{k=0}^{i-1}\gamma_{k+1} \leq c_1\sum_{k=0}^{i-1}\left(k+1\right)^{-\gamma} + c_2\sum_{k=0}^{i-1}\left(k+1\right)^{-\xi}, \\
c_1\sum_{k=1}^{i}k^{-\gamma} - c_2\sum_{k=1}^{i}k^{-\xi} \leq \sum_{k=0}^{i-1}\gamma_{k+1} \leq c_1\sum_{k=1}^{i}k^{-\gamma} + c_2\sum_{k=1}^{i}k^{-\xi}.
\end{aligned}$$

Through Lemma 5, the final results follow. QED

Lemma 7. *Let* $\left|\gamma_{i+1} - c_1\left(i+1\right)^{-\gamma}\right| \leq c_2\left(i+1\right)^{-\xi}$ for $\xi > \gamma > 0$. If $\gamma \neq 1$ and $\xi \neq 1$,

$$\begin{aligned}
\sum_{k=j+1}^{i}\gamma_{k+1} \leq\ & 2c_2\zeta\left(\xi\right) + \frac{c_1}{1-\gamma}\left[\left(i+1\right)^{1-\gamma} - \left(j+1\right)^{1-\gamma}\right] + \frac{c_1}{2}\left[\left(i+1\right)^{-\gamma} - \left(j+1\right)^{-\gamma}\right] \\
&+ \frac{c_2}{1-\xi}\left[\left(i+1\right)^{1-\xi} + \left(j+1\right)^{1-\xi}\right] + \frac{c_2}{2}\left[\left(i+1\right)^{-\xi} + \left(j+1\right)^{-\xi}\right] + \frac{c_1\gamma}{8}\left(j+1\right)^{-1-\gamma}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=j+1}^{i}\gamma_{k+1} \geq\ & -2c_2\zeta\left(\xi\right) + \frac{c_1}{1-\gamma}\left[\left(i+1\right)^{1-\gamma} - \left(j+1\right)^{1-\gamma}\right] + \frac{c_1}{2}\left[\left(i+1\right)^{-\gamma} - \left(j+1\right)^{-\gamma}\right] \\
&- \frac{c_2}{1-\xi}\left[\left(i+1\right)^{1-\xi} + \left(j+1\right)^{1-\xi}\right] - \frac{c_2}{2}\left[\left(i+1\right)^{-\xi} + \left(j+1\right)^{-\xi}\right] - \frac{c_1\gamma}{8}\left(i+1\right)^{-1-\gamma}.
\end{aligned}$$

If $\gamma = 1$,

$$\begin{aligned}
\sum_{k=j+1}^{i}\gamma_{k+1} \leq\ & 2c_2\zeta\left(\xi\right) + c_1\ln\left(\frac{i+1}{j+1}\right) + \frac{c_2}{1-\xi}\left[\left(i+1\right)^{1-\xi} + \left(j+1\right)^{1-\xi}\right] \\
&+ \frac{c_2}{2}\left[\left(i+1\right)^{-\xi} + \left(j+1\right)^{-\xi}\right] + c_1\left(i+1\right)^{-1}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=j+1}^{i}\gamma_{k+1} \geq\ & -2c_2\zeta\left(\xi\right) + c_1\ln\left(\frac{i+1}{j+1}\right) - \frac{c_2}{1-\xi}\left[\left(i+1\right)^{1-\xi} + \left(j+1\right)^{1-\xi}\right] \\
&- \frac{c_2}{2}\left[\left(i+1\right)^{-\xi} + \left(j+1\right)^{-\xi}\right] - c_1\left(j+1\right)^{-1}.
\end{aligned}$$

Proof. The proof is trivial from $\sum_{k=j}^{i-1}\gamma_{k+1}=\sum_{k=0}^{i-1}\gamma_{k+1}-\sum_{k=0}^{j-1}\gamma_{k+1}$ and Lemma 6. QED

Lemma 8. *Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be respectively a decreasing and an increasing function. Then,*

$$\sum_{i=a}^{b} f(i)\,g(i)\leq\int_{a-1}^{b} f(x)\,g(x+1)\,\mathrm{d}x.$$

Proof. The proof is trivial from

$$\int_{a-1}^{b} f(x)\,g(x+1)\,\mathrm{d}x=\sum_{i=a}^{b}\int_{i-1}^{i} f(x)\,g(x+1)\,\mathrm{d}x\geq\sum_{i=a}^{b} f(i)\,g(i)\,.$$

QED

Lemma 9. *Define*

$$S:=\sum_{j=0}^{i}(j+1)^{-A}\exp\left\{k_1(j+1)^{B_1}+\sum_{j=2}^{J}k_j(j+1)^{B_j}\right\}$$

where $k_1>0$, $1>B_1>0$, $B_j<B_1$ for $j=2,\ldots,J$. Then, as $i\to\infty$,

$$S\simeq\frac{(i+1)^{1-A-B_1}\exp\left(\sum_{j=1}^{J}k_j(i+1)^{B_j}\right)}{k_1B_1}.$$

Proof. We apply Euler–Maclaurin formula to S to get

$$\begin{aligned}S\simeq&\int_0^i(x+1)^{-A}\exp\left\{\sum_{j=1}^{J}k_j(x+1)^{B_j}\right\}\mathrm{d}x\\&+\frac{1}{2}\left\{\exp\left(\sum_{j=1}^{J}k_j\right)+(i+1)^{-A}\exp\left\{\sum_{j=1}^{J}k_j(i+1)^{B_j}\right\}\right\}.\end{aligned}\tag{8.6}$$

Using the substitution $x=z(i+1)-1$, the integral can be written as

$$\begin{aligned}&\int_0^i(x+1)^{-A}\exp\left\{\sum_{j=1}^{J}k_j(x+1)^{B_j}\right\}\mathrm{d}x\\&=(i+1)^{1-A}\int_{\frac{1}{i+1}}^{1}z^{-A}\exp\left\{\sum_{j=1}^{J}k_jz^{B_j}(i+1)^{B_j}\right\}\mathrm{d}z\\&=(i+1)^{1-A}\int_{\frac{1}{2}}^{1}z^{-A}\exp\left\{\sum_{j=1}^{J}k_jz^{B_j}(i+1)^{B_j}\right\}\mathrm{d}z\\&\quad+(i+1)^{1-A}\int_{\frac{1}{i+1}}^{\frac{1}{2}}z^{-A}\exp\left\{\sum_{j=1}^{J}k_jz^{B_j}(i+1)^{B_j}\right\}\mathrm{d}z.\end{aligned}\tag{8.7}$$

We start from the first integral in (8.7) that, through the substitution $t = 2(1-z)$, becomes

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1} z^{-A} \exp\left\{\sum_{j=1}^{J} k_j z^{B_j} (i+1)^{B_j}\right\} \mathrm{d}z \\
=&\frac{1}{2}\int_{0}^{1} \left(1-\frac{t}{2}\right)^{-A} \exp\left\{\sum_{j=1}^{J} k_j \left(1-\frac{t}{2}\right)^{B_j} (i+1)^{B_j}\right\} \mathrm{d}t \\
=&\frac{\exp\left(\sum_{j=2}^{J} k_j (i+1)^{B_j}\right)}{2}\int_{0}^{1} \left(1-\frac{t}{2}\right)^{-A} \\
&\cdot \exp\left\{k_1 \left(1-\frac{t}{2}\right)^{B_1} (i+1)^{B_1} + \sum_{j=2}^{J} k_j \left[\left(1-\frac{t}{2}\right)^{B_j} - 1\right] (i+1)^{B_j}\right\} \mathrm{d}t.
\end{aligned}$$

Now we apply Theorem 2.1 in Olver (1997, p. 326), stated for an integral $\int_0^k e^{-xp(t)+r(x,t)} q(x,t)\,\mathrm{d}t$. In the notation of Olver, $x = (i+1)^{B_1}$. As to condition (i) in his theorem, we have

$$\begin{aligned}
p(t) &= -k_1\left(1-\frac{t}{2}\right)^{B_1} = -k_1 + \frac{k_1 B_1 t}{2} + O\left(t^2\right), \\
p'(t) &= \frac{k_1 B_1}{2}\left(1-\frac{t}{2}\right)^{B_1-1} = \frac{k_1 B_1}{2} + O(t)
\end{aligned}$$

from which it is clear that $p(0) = -k_1$, $P = \frac{k_1 B_1}{2}$, $\mu = 1$ and $\mu_1 = 2$. Now, we pass to condition (ii). Then,

$$\begin{aligned}
r(x,t) &= \sum_{j=2}^{J} k_j \left[\left(1-\frac{t}{2}\right)^{B_j} - 1\right] x^{\frac{B_j}{B_1}}, \\
|r(x,t)| &\le \sum_{j=2}^{J} |k_j| \left|\left(1-\frac{t}{2}\right)^{B_j} - 1\right| x^{\frac{B_j}{B_1}} \\
&\le \left(\sum_{j=2}^{J} |k_j|\right) \max_j \left|\left(1-\frac{t}{2}\right)^{B_j} - 1\right| x^{\max_j \frac{B_j}{B_1}}.
\end{aligned}$$

We need to majorize this through $Rx^{\alpha}t^{\nu}$. We can take $\alpha = \max_j \frac{B_j}{B_1}$, $0 < \nu < 1$ and R large enough. At last, for $q(x,t) = \left(1-\frac{t}{2}\right)^{-A}$, we need

$$\left|q(x,t) - Qt^{\lambda-1}\right| \le Q_1 x^{\beta} t^{\lambda_1 - 1}.$$

It is clear that $\lambda = 1$ and $Q = 1$, so that $\left|q(x,t) - Qt^{\lambda-1}\right| = \left|1 - \left(1-\frac{t}{2}\right)^{-A}\right|$. This must be majorized by $Q_1 x^{\beta} t^{\lambda_1 - 1}$ where $\beta = 0$. We can take $1 < \lambda_1 < 2$ and Q_1 large enough. Then, $p(0) = -k_1$, $P = \frac{k_1 B_1}{2}$, $\mu = 1$, $\mu_1 = 2$, $\alpha = \max_j \frac{B_j}{B_1}$, $0 < \nu < 1$, $\lambda = 1$, $Q = 1$, $\beta = 0$, $\lambda_1 < 2$. The conditions $\nu \ge 0$, $\lambda > 0$, $\lambda_1 > 0$, $\alpha < 1 \wedge \frac{\nu}{\mu}$, $\beta < \frac{\lambda_1 - \lambda}{\mu}$ in the statement are verified by taking $\nu = 1 - \varepsilon$ and $\lambda_1 = 2 - \varepsilon$ for suitably small ε. At last, the integral behaves like

$$\frac{Q}{\mu}\Gamma\left(\frac{\lambda}{\mu}\right)\frac{e^{-xp(0)}}{(Px)^{\frac{\lambda}{\mu}}} = \frac{2e^{k_1(i+1)^{B_1}}}{k_1 B_1 (i+1)^{B_1}}$$

and the first term in (8.7) as

$$\frac{(i+1)^{1-A-B_1} \exp\left(\sum_{j=1}^{J} k_j (i+1)^{B_j}\right)}{k_1 B_1}.$$

The second integral in (8.7) can be majorized as

$$\begin{aligned}
&(i+1)^{1-A} \int_{\frac{1}{i+1}}^{\frac{1}{2}} z^{-A} \exp\left\{\sum_{j=1}^{J} k_j z^{B_j} (i+1)^{B_j}\right\} \mathrm{d}z \\
&\leq (i+1)^{1-A} \max_{z\in\left[\frac{1}{2},\frac{1}{i+1}\right]} \exp\left\{\sum_{j=1}^{J} k_j z^{B_j} (i+1)^{B_j}\right\} \int_{\frac{1}{i+1}}^{\frac{1}{2}} z^{-A}\mathrm{d}z \\
&= \exp\left\{\sum_{j=1}^{J} \left(k_j 2^{-B_j} (i+1)^{B_j}\right) \vee k_j\right\} \frac{2^{A-1} (i+1)^{1-A} - 1}{1-A}.
\end{aligned}$$

It is clear that this is of a lower order than the previous one.

At last, from (8.6), we get the final result. QED

Lemma 10. *Let* $\mathbf{X}$ *(*$\mathbf{X}_0$*) be the matrix associated with the points in* $\mathcal{P}(\boldsymbol{\theta}_0)$ *(*$\mathcal{P}_0(\boldsymbol{\theta}_0)$*). Then,*

$$\lambda_{\min}(\mathbf{X}'\mathbf{X}) \geq h^{2D} \lambda_{\min}(\mathbf{X}_0'\mathbf{X}_0).$$

Proof. Let us fix the location of the points $\boldsymbol{\theta}_{0,j} - \boldsymbol{\theta}_0$ for any j. Let us see what happens when we multiply each one of these vectors by a constant h. This should represent what happens when all points in $\mathcal{P}_0(\boldsymbol{\theta}_0)$ shrink towards $\boldsymbol{\theta}_0$. When passing from $\boldsymbol{\theta}_{0,j} - \boldsymbol{\theta}_0$ to $\boldsymbol{\theta}_j - \boldsymbol{\theta}_0 = h(\boldsymbol{\theta}_{0,j} - \boldsymbol{\theta}_0)$, the generic vector $\mathbf{x}_D(\boldsymbol{\theta}_{0,j})$ is multiplied by a vector containing powers of h from degree 0 to degree D, according to the degree of the respective element of $\mathbf{x}_D(\boldsymbol{\theta}_{0,j})$. Calling $\mathbf{h}$ the vector containing the previously described powers of h, we pass from $\mathbf{x}_D(\boldsymbol{\theta}_{0,j})$ to $\mathbf{x}_D(\boldsymbol{\theta}_j) = \mathbf{x}_D(\boldsymbol{\theta}_{0,j}) \odot \mathbf{h}$. When building the design matrix, we have $\mathbf{X} = \mathbf{X}_0 \odot (\boldsymbol{\iota}_N \mathbf{h}')$. From Styan (1973, p. 221, (2.11)),

$$\mathbf{X}_0 \odot (\boldsymbol{\iota}_N \mathbf{h}') = \mathrm{dg}(\boldsymbol{\iota}_N)\, \mathbf{X}_0 \mathrm{dg}(\mathbf{h}) = \mathbf{X}_0 \mathrm{dg}(\mathbf{h}),$$

from which $\mathbf{X}'\mathbf{X} = \mathrm{dg}(\mathbf{h})\, \mathbf{X}_0'\mathbf{X}_0 \mathrm{dg}(\mathbf{h})$. Now, from the variational property of eigenvalues,

$$\lambda_{\min}(\mathbf{X}'\mathbf{X}) = \lambda_{\min}(\mathrm{dg}(\mathbf{h})\, \mathbf{X}_0'\mathbf{X}_0 \mathrm{dg}(\mathbf{h})) \geq \lambda_{\min}\left(\mathrm{dg}(\mathbf{h})^2\right) \lambda_{\min}(\mathbf{X}_0'\mathbf{X}_0) = h^{2D} \lambda_{\min}(\mathbf{X}_0'\mathbf{X}_0).$$

QED

Lemma 11. *Let* $\mathbf{x}_D(\boldsymbol{\theta})$ *be the vector containing all monomials of elements of* $\boldsymbol{\theta} \in \mathbb{R}^K$ *up to degree* D*. Then,*

$$\max_{\boldsymbol{\theta}\in\rho\mathsf{B}} \|\mathbf{x}_D(\boldsymbol{\theta})\|_2 \leq \sqrt{\frac{\rho^{2(D+1)} - 1}{\rho^2 - 1}}.$$

Let D^k *denote the element-wise derivative with respect to the multi-index* k*. Then,*

$$\max_{|k|=S} \max_{\boldsymbol{\theta}\in\rho\mathsf{B}} \left\|D^k \mathbf{x}_D(\boldsymbol{\theta})\right\|_2 \leq S! \sqrt{\sum_{d=S}^{D} \binom{d}{S}^2 \rho^{2(d-S)}}.$$

Proof. In the proof we use the notation $\mathbf{x}^d(\boldsymbol{\theta})$ for the vector containing all monomials of degree d of elements of $\boldsymbol{\theta}$. It is clear that $\mathbf{x}^0(\boldsymbol{\theta}) \equiv 1$ and $\mathbf{x}^1(\boldsymbol{\theta}) \equiv \boldsymbol{\theta}$. Therefore, $\mathbf{x}_D(\boldsymbol{\theta}) = \left[\left(\mathbf{x}^0(\boldsymbol{\theta})\right)', \left(\mathbf{x}^1(\boldsymbol{\theta})\right)', \ldots, \left(\mathbf{x}^D(\boldsymbol{\theta})\right)'\right]'$.

From this,

$$\|\mathbf{x}_D(\boldsymbol{\theta})\|_2^2 = \sum_{d=0}^{D} \left\|\mathbf{x}^d(\boldsymbol{\theta})\right\|_2^2.$$

It is clear that $\mathbf{x}^d(\boldsymbol{\theta})$ is a subvector of the Kronecker product of d copies of $\boldsymbol{\theta}$, so that $\left\|\mathbf{x}^d(\boldsymbol{\theta})\right\|_2 \leq \left\|\boldsymbol{\theta}^{\otimes d}\right\|_2 = \|\boldsymbol{\theta}\|_2^d$. Therefore, $\left\|\mathbf{x}^0(\boldsymbol{\theta})\right\|_2 = 1$, $\left\|\mathbf{x}^1(\boldsymbol{\theta})\right\|_2 = \|\boldsymbol{\theta}\|_2 \leq \rho$ and $\left\|\mathbf{x}^d(\boldsymbol{\theta})\right\|_2 \leq \|\boldsymbol{\theta}\|_2^d \leq \rho^d$, from which

$$\|\mathbf{x}_D(\boldsymbol{\theta})\|_2^2 \leq \sum_{d=0}^{D} \rho^{2d}.$$

Now we turn to the derivatives. It is not difficult to see that the derivative of $\mathbf{x}_D(\boldsymbol{\theta})$ with respect to one element of $\boldsymbol{\theta}$, say θ_1, is composed of zeros and of the elements of $\mathbf{x}_{D-1}(\boldsymbol{\theta})$, each one multiplied by a different constant. The zeros do not contribute to the norm, therefore we remove them. By induction, $D^k\mathbf{x}_D(\boldsymbol{\theta})$ is composed of zeros and of the elements of $\mathbf{x}_{D-|k|}(\boldsymbol{\theta}) = \mathbf{x}_{D-S}(\boldsymbol{\theta})$, each one with a different multiplicative constant. Consider a multi-index ℓ and the monomial $\boldsymbol{\theta}^\ell = \prod_{j=1}^K \theta_j^{\ell_j}$. Then, $D^k\boldsymbol{\theta}^\ell = \prod_{j=1}^K \frac{\ell_j!}{(\ell_j-k_j)!}\theta_j^{\ell_j-k_j}$ provided $\ell_j \geq k_j$ for any j, and $D^k\boldsymbol{\theta}^\ell = 0$ if $\ell_j < k_j$ for some j. For fixed $|\ell|$ and $|k|$, the largest leading constant is obtained when ℓ and k contain only one element different from zero and the indexes of the two elements coincide. The leading constant is then $\frac{|\ell|!}{(|\ell|-|k|)!}$. We can thus majorize the constants multiplying the monomials of order $|\ell|-|k|$ in $D^k\mathbf{x}_D(\boldsymbol{\theta})$ times $\frac{|\ell|!}{(|\ell|-|k|)!}$. Therefore, the norm of $D^k\mathbf{x}_D(\boldsymbol{\theta})$ can be majorized by the norm of

$$\left[\frac{|k|!}{(|k|-|k|)!}\left(\mathbf{x}^0(\boldsymbol{\theta})\right)', \frac{(|k|+1)!}{(|k|+1-|k|)!}\left(\mathbf{x}^1(\boldsymbol{\theta})\right)', \ldots, \frac{D!}{(D-|k|)!}\left(\mathbf{x}^{D-|k|}(\boldsymbol{\theta})\right)'\right]'.$$

As a result,

$$\left\|D^k\mathbf{x}_D(\boldsymbol{\theta})\right\|_2^2 \leq \sum_{d=S}^{D} \frac{d!}{(d-S)!}\left\|\mathbf{x}^{d-S}(\boldsymbol{\theta})\right\|_2^2.$$

The norm of $\left\|\mathbf{x}^d(\boldsymbol{\theta})\right\|_2$ can be majorized by ρ^d as above. At last, we get $\sqrt{\sum_{d=S}^{D}\left(\frac{d!}{(d-S)!}\right)^2\rho^{2(d-S)}}$ from which the result follows. QED

8.2 Proofs of Optimization Results

8.2.1 Inexact Newton Methods

Proof of Theorem 1. We note that (3.1) can be written as in (3.2):

$$\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\left(\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{(i)}\right) = -\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) + \mathbf{r}^{(i)}$$

with

$$\mathbf{r}^{(i)} = \dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\left[\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)\right]^{-1}\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right).$$

Using the fact that $\dot{F}(\boldsymbol{\theta}^\star) = \mathbf{0}$, this can be written as

$$\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star = \left[\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right]^{-1} \left\{\mathbf{r}^{(i)} + \left[\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \ddot{F}(\boldsymbol{\theta}^\star)\right]\left(\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right) - \left[\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \dot{F}(\boldsymbol{\theta}^\star) - \ddot{F}(\boldsymbol{\theta}^\star)\left(\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right)\right]\right\}.$$

Taking norms,

$$\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star\right\|_2 \leq \left\|\left[\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right]^{-1}\right\|_2 \left\{\left\|\mathbf{r}^{(i)}\right\|_2 + \left\|\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \ddot{F}(\boldsymbol{\theta}^\star)\right\|_2 \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 + \left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \dot{F}(\boldsymbol{\theta}^\star) - \ddot{F}(\boldsymbol{\theta}^\star)\left(\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right)\right\|_2\right\}. \tag{8.8}$$

From the third inequality in Lemma 2 and from **Lip-**2, we respectively get

$$\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \dot{F}(\boldsymbol{\theta}^\star) - \ddot{F}(\boldsymbol{\theta}^\star)\left(\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right)\right\|_2 \leq \frac{L_2}{2}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2^2$$

and

$$\left\|\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \ddot{F}(\boldsymbol{\theta}^\star)\right\|_2 \leq L_2\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2.$$

Replacing these inequalities into (8.8) and using the equality $\left\|\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right)^{-1}\right\|_2 = \lambda_{\max}\left(\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right)^{-1}\right) = \lambda_{\min}^{-1}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right)$ and the second formula of Lemma 3, we get the final result. QED

Proof of Corollary 1. The first formula in the statement can be obtained rewriting the inequality in Remark 3 (i) as

$$\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star\right\|_2 \leq \frac{1}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right)}\left\{\delta_1^{(i)}\left(1 + \frac{\delta_2^{(i)}}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right) - \delta_2^{(i)}}\right) + \delta_2^{(i)}\frac{L_1}{\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right) - \delta_2^{(i)}}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 + \frac{3L_2}{2}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2^2\right\}$$

and remarking that, from the fifth inequality of Lemma 2,

$$\lambda_{\min}\left(\ddot{F}(\boldsymbol{\theta}^\star)\right) - L_2\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \leq \lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right).$$

The second formula can be obtained using the fact that, under **Hess**, $\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right) \geq m$ (see Bertsekas et al., 2003, p. 72). QED

Proof of Theorem 2. From Theorem 1 and Remark 3 (i), under **Lip-**2 and **Hess**, we can write

$$\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star\right\|_2 \leq \frac{1}{m - \delta_2^{(i)}}\left\{\delta_1^{(i)} + \frac{\delta_2^{(i)}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2}{m}M + \frac{3L_2}{2}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2^2\right\}, \tag{8.9}$$

where $M = \min\left\{\left\|\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right\|_2, L_1\right\}$. Define

$$a^{(i)} := \frac{\delta_1^{(i)}}{m - \delta_2^{(i)}},$$
$$b^{(i)} := \frac{\delta_2^{(i)} M}{m\left(m - \delta_2^{(i)}\right)} + \frac{3L_2}{2\left(m - \delta_2^{(i)}\right)}\Delta$$

for $i \geq 0$. If $\left\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^{\star}\right\|_2 \leq \Delta$,

$$\begin{aligned}\left\|\boldsymbol{\theta}^{(1)} - \boldsymbol{\theta}^{\star}\right\|_2 &\leq \frac{\delta_1^{(0)}}{m - \delta_2^{(0)}} + \left\{\frac{\delta_2^{(0)} M}{m\left(m - \delta_2^{(0)}\right)} + \frac{3L_2}{2\left(m - \delta_2^{(0)}\right)}\Delta\right\}\left\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^{\star}\right\|_2 \\ &\leq a^{(0)} + b^{(0)}\left\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^{\star}\right\|_2 \leq a^{(0)} + b^{(0)}\Delta.\end{aligned}$$

If $\left\|\boldsymbol{\theta}^{(1)} - \boldsymbol{\theta}^{\star}\right\|_2$ is smaller than Δ, we can reiterate the reasoning to get

$$\begin{aligned}\left\|\boldsymbol{\theta}^{(2)} - \boldsymbol{\theta}^{\star}\right\|_2 &\leq \frac{\delta_1^{(1)}}{m - \delta_2^{(1)}} + \left\{\frac{\delta_2^{(1)} M}{m\left(m - \delta_2^{(1)}\right)} + \frac{3L_2}{2\left(m - \delta_2^{(1)}\right)}\left\|\boldsymbol{\theta}^{(1)} - \boldsymbol{\theta}^{\star}\right\|_2\right\}\left\|\boldsymbol{\theta}^{(1)} - \boldsymbol{\theta}^{\star}\right\|_2 \\ &\leq \frac{\delta_1^{(1)}}{m - \delta_2^{(1)}} + \left\{\frac{\delta_2^{(1)} M}{m\left(m - \delta_2^{(1)}\right)} + \frac{3L_2}{2\left(m - \delta_2^{(1)}\right)}\Delta\right\}\left\|\boldsymbol{\theta}^{(1)} - \boldsymbol{\theta}^{\star}\right\|_2 \\ &= a^{(1)} + b^{(1)}\left\|\boldsymbol{\theta}^{(1)} - \boldsymbol{\theta}^{\star}\right\|_2.\end{aligned}$$

This happens if $\left\|\boldsymbol{\theta}^{(1)} - \boldsymbol{\theta}^{\star}\right\|_2 \leq a^{(0)} + b^{(0)}\Delta \leq \Delta$, i.e. if $a^{(0)} \leq \left(1 - b^{(0)}\right)\Delta$. Provided $a^{(i)} \leq \left(1 - b^{(i)}\right)\Delta$ for any i, the reasoning leads, through Lemma 4, to

$$\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2 \leq \sum_{j=0}^{i} a^{(j)} \prod_{k=j+1}^{i} b^{(k)} + \Delta \prod_{k=0}^{i} b^{(k)}.$$

Note that the requirement $a^{(i)} \leq \left(1 - b^{(i)}\right)\Delta$ is necessary to replace with Δ the occurrence of $\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2$ in braces in (8.9). If

$$b^{(i)} \leq c_1\left(1 + c_2\left(i+1\right)^{-\xi}\right),$$

from Lemma 5, we can write

$$\begin{aligned}\prod_{k=0}^{i} b^{(k)} &\leq c_1^{i+1} \exp\left\{\sum_{k=0}^{i} \ln\left(1 + c_2\left(k+1\right)^{-\xi}\right)\right\} \\ &\leq c_1^{i+1} \exp\left\{c_2 \sum_{k=0}^{i}\left(k+1\right)^{-\xi}\right\} = c_1^{i+1} \exp\left\{c_2 \sum_{k=1}^{i+1} k^{-\xi}\right\} \\ &= \begin{cases} c_1^{i+1} \exp\left\{c_2\left(\zeta\left(\xi\right) + \frac{1}{1-\xi}\left(i+1\right)^{1-\xi} + \frac{1}{2}\left(i+1\right)^{-\xi}\right)\right\}, & \xi > 0, \xi \neq 1, \\ c_1^{i+1} \exp\left\{c_2\left(\gamma_{(0)} + \ln\left(i+1\right) + \frac{1}{i+1}\right)\right\}, & \xi = 1.\end{cases}\end{aligned}$$

$$\simeq \begin{cases} c_1^{i+1} \exp\left\{c_2\zeta(\xi) + \frac{c_2}{1-\xi} i^{1-\xi}\right\}, & \xi > 0, \xi \neq 1, \\ c_1^{i+1} i^{c_2} \exp\left\{c_2 \gamma_{(0)}\right\}, & \xi = 1. \end{cases} \tag{8.10}$$

Let us suppose that $\xi \neq 1$. In the same way, from Lemma 5,

$$\begin{aligned} \prod_{k=j+1}^{i} b^{(k)} \leq & c_1^{i-j} \exp\left\{\sum_{k=j+1}^{i} \ln\left(1 + c_2 (k+1)^{-\xi}\right)\right\} \leq c_1^{i-j} \exp\left\{c_2 \sum_{k=j+1}^{i} (k+1)^{-\xi}\right\} \\ = & c_1^{i-j} \exp\left\{c_2 \left[\sum_{k=1}^{i+1} k^{-\xi} - \sum_{k=1}^{j+1} k^{-\xi}\right]\right\} \\ \leq & c_1^{i-j} \exp\left\{\frac{c_2}{1-\xi} (i+1)^{1-\xi} + \frac{c_2}{2} (i+1)^{-\xi} - \frac{c_2}{1-\xi} (j+1)^{1-\xi} \right. \\ & \left. - \frac{c_2}{2} (j+1)^{-\xi} + \frac{c_2 \xi}{8} (j+1)^{-1-\xi}\right\}. \end{aligned}$$

From $a^{(j)} = \frac{\delta_1^{(j)}}{m - \delta_2^{(j)}} \leq c_3 (j+1)^{-\delta} (1 + o(1))$,

$$\begin{aligned} \sum_{j=0}^{i} a^{(j)} \prod_{k=j+1}^{i} b^{(k)} \lesssim & c_3 c_1^i \exp\left\{\frac{c_2}{1-\xi} (i+1)^{1-\xi} + \frac{c_2}{2} (i+1)^{-\xi}\right\} \\ & \cdot \sum_{j=0}^{i} (j+1)^{-\delta} \exp\left\{j |\ln c_1| - \frac{c_2}{1-\xi} (j+1)^{1-\xi} - \frac{c_2}{2} (j+1)^{-\xi} \right. \\ & \left. + \frac{c_2 \xi}{8} (j+1)^{-1-\xi}\right\}. \end{aligned}$$

Using Lemma 8, we can majorize the sum through an integral:

$$\begin{aligned} & \sum_{j=0}^{i} (j+1)^{-\delta} \exp\left\{j |\ln c_1| - \frac{c_2}{1-\xi} (j+1)^{1-\xi} - \frac{c_2}{2} (j+1)^{-\xi} + \frac{c_2 \xi}{8} (j+1)^{-1-\xi}\right\} \\ \leq & \sum_{j=1}^{i+1} j^{-\delta} \exp\left\{(j-1) |\ln c_1| - \frac{c_2}{1-\xi} j^{1-\xi} + \frac{c_2 \xi}{8} j^{-1-\xi}\right\} \\ = & \sum_{j=2}^{i+1} j^{-\delta} \exp\left\{(j-1) |\ln c_1| - \frac{c_2}{1-\xi} j^{1-\xi} + \frac{c_2 \xi}{8} j^{-1-\xi}\right\} + \exp\left\{-\frac{c_2}{1-\xi} + \frac{c_2 \xi}{8}\right\} \\ \leq & \int_1^{i+1} x^{-\delta} \exp\left\{x |\ln c_1| - \frac{c_2}{1-\xi} x^{1-\xi} + \frac{c_2 \xi}{8} x^{-1-\xi}\right\} \mathrm{d}x + \exp\left\{-\frac{c_2}{1-\xi} + \frac{c_2 \xi}{8}\right\}. \end{aligned}$$

Through the change of variable $x = (i+1) z$, we can follow the proof of Lemma 9 to get

$$\begin{aligned} & \int_1^{i+1} x^{-\delta} \exp\left\{x |\ln c_1| - \frac{c_2}{1-\xi} x^{1-\xi} + \frac{c_2 \xi}{8} x^{-1-\xi}\right\} \mathrm{d}x \\ \simeq & (i+1)^{1-\delta} \frac{\exp\left\{(i+1) |\ln c_1| - \frac{c_2}{1-\xi} (i+1)^{1-\xi} + \frac{c_2 \xi}{8} (i+1)^{-1-\xi}\right\}}{(i+1) |\ln c_1|} \\ \simeq & \frac{c_1^{-i-1} (i+1)^{-\delta} \exp\left\{-\frac{c_2}{1-\xi} (i+1)^{1-\xi}\right\}}{|\ln c_1|} \end{aligned}$$

and

$$\sum_{j=0}^{i} a^{(j)} \prod_{k=j+1}^{i} b^{(k)} \lesssim c_3 c_1^i \exp\left\{\frac{c_2}{1-\xi}(i+1)^{1-\xi} + \frac{c_2}{2}(i+1)^{-\xi}\right\}$$

$$\cdot \left\{\frac{c_1^{-i-1}(i+1)^{-\delta}\exp\left\{-\frac{c_2}{1-\xi}(i+1)^{1-\xi}\right\}}{|\ln c_1|} + \exp\left\{-\frac{c_2}{1-\xi} + \frac{c_2\xi}{8}\right\}\right\}$$

$$\simeq \frac{c_3 i^{-\delta}}{c_1|\ln c_1|} + O\left(c_1^{-i} i^{-\delta}\exp\left\{-\frac{c_2}{1-\xi}(i+1)^{1-\xi}\right\}\right).$$

The other term is of a lower order. If $\xi = 1$, from Lemma 5,

$$\prod_{k=j+1}^{i} b^{(k)} \le c_1^{i-j}\exp\left\{\sum_{k=j+1}^{i}\ln\left(1+c_2(k+1)^{-\xi}\right)\right\} \le c_1^{i-j}\exp\left\{c_2\sum_{k=j+1}^{i}(k+1)^{-\xi}\right\}$$

$$= c_1^{i-j}\exp\left\{c_2\left[\sum_{k=1}^{i+1}k^{-\xi} - \sum_{k=1}^{j+1}k^{-\xi}\right]\right\}$$

$$\le c_1^{i-j}\left(\frac{i+1}{j+1}\right)^{c_2}\exp\left\{\frac{c_2}{i+1}\right\}$$

and, using the same method seen above,

$$\sum_{j=0}^{i} a^{(j)} \prod_{k=j+1}^{i} b^{(k)} \lesssim c_3 c_1^i (i+1)^{c_2}\exp\left\{\frac{c_2}{i+1}\right\}\sum_{j=0}^{i} c_1^{-j}(j+1)^{-\delta-c_2}$$

$$= c_3 c_1^{i+1}(i+1)^{c_2}\exp\left\{\frac{c_2}{i+1}\right\}\sum_{j=1}^{i+1} c_1^{-j} j^{-\delta-c_2}$$

$$\le c_3 c_1^{i+1}(i+1)^{c_2}\exp\left\{\frac{c_2}{i+1}\right\}\left\{\sum_{j=2}^{i+1} j^{-\delta-c_2}\exp\{j|\ln c_1|\} + c_1^{-1}\right\}$$

$$\le c_3 c_1^{i+1}(i+1)^{c_2}\exp\left\{\frac{c_2}{i+1}\right\}\left\{\int_1^{i+1} x^{-\delta-c_2}\exp\{(x+1)|\ln c_1|\}\,\mathrm{d}x + c_1^{-1}\right\}$$

$$= c_3 c_1^i (i+1)^{c_2}\exp\left\{\frac{c_2}{i+1}\right\}$$

$$\cdot\left\{(i+1)^{1-\delta-c_2}\int_{\frac{1}{i+1}}^{1} z^{-\delta-c_2}\exp\{(i+1)z|\ln c_1|\}\,\mathrm{d}z + 1\right\}$$

$$\simeq c_3 c_1^i (i+1)^{c_2}\exp\left\{\frac{c_2}{i+1}\right\}\left\{(i+1)^{1-\delta-c_2}\frac{\exp\{(i+1)|\ln c_1|\}}{(i+1)|\ln c_1|} + 1\right\}$$

$$\simeq \frac{c_3 i^{-\delta}}{c_1|\ln c_1|} + O\left(c_1^i i^{c_2}\right).$$

At last, if $\delta_1^{(i)} \equiv 0$, we have $a^{(i)} \equiv 0$ and $\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2 \le \Delta\prod_{k=0}^{i} b^{(k)}$. From (8.10), we have

$$\prod_{k=0}^{i} b^{(k)} \lesssim \begin{cases} e^{c_2\zeta(\xi)} c_1^{i+1}\exp\left\{\frac{c_2}{1-\xi} i^{1-\xi}\right\}, & \xi > 0, \xi \ne 1, \\ e^{c_2\gamma_{(0)}} c_1^{i+1} i^{c_2}, & \xi = 1, \end{cases}$$

and the final result follows. QED

Proof of Theorem 3. We can write

$$
\begin{aligned}
&\mathbb{P}\left\{\max_{1\le i\le n+1}\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^\star\right\|_2>\Delta\right\}\\
&=\mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(1)}-\boldsymbol{\theta}^\star\right\|_2>\Delta,\left\|\boldsymbol{\theta}^{(0)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta\right\}\\
&\quad+\mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(2)}-\boldsymbol{\theta}^\star\right\|_2>\Delta,\left\|\boldsymbol{\theta}^{(1)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta,\left\|\boldsymbol{\theta}^{(0)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta\right\}\\
&\quad+\mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(3)}-\boldsymbol{\theta}^\star\right\|_2>\Delta,\left\|\boldsymbol{\theta}^{(2)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta,\left\|\boldsymbol{\theta}^{(1)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta,\left\|\boldsymbol{\theta}^{(0)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta\right\}\\
&\quad+\cdots+\mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(n+1)}-\boldsymbol{\theta}^\star\right\|_2>\Delta,\left\|\boldsymbol{\theta}^{(n)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta,\ldots,\left\|\boldsymbol{\theta}^{(0)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta\right\}\\
&=\sum_{i=0}^{n}\mathbb{P}\left\{\left\{\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^\star\right\|_2>\Delta\right\},\bigcap_{j=0}^{i}\left\{\left\|\boldsymbol{\theta}^{(j)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta\right\}\right\}\\
&=\sum_{i=0}^{n}\mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^\star\right\|_2>\Delta\,\middle|\,\bigcap_{j=0}^{i}\left\{\left\|\boldsymbol{\theta}^{(j)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta\right\}\right\}\mathbb{P}\left\{\bigcap_{j=0}^{i}\left\{\left\|\boldsymbol{\theta}^{(j)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta\right\}\right\}\\
&=\sum_{i=0}^{n}\mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^\star\right\|_2>\Delta\,\middle|\,\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta\right\}\mathbb{P}\left\{\bigcap_{j=0}^{i}\left\{\left\|\boldsymbol{\theta}^{(j)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta\right\}\right\}
\end{aligned}
$$

where the last step derives from the fact that the sequence $\left\{\boldsymbol{\theta}^{(i)}\right\}$ is a Markov process. From (8.9) valid under Assumptions **Lip-**2 and **Hess**, using $m-\delta_2^{(i)}\le m$, we note that

$$
\begin{aligned}
&\left\{\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^\star\right\|_2>\Delta\right\}\\
&\subseteq\left\{\delta_1^{(i)}m+\delta_2^{(i)}\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^\star\right\|_2M+\frac{3L_2m}{2}\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^\star\right\|_2^2>\Delta m\left(m-\delta_2^{(i)}\right)\right\}.
\end{aligned}
\tag{8.11}
$$

Therefore, for a constant $\mu>0$,

$$
\begin{aligned}
&\mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^\star\right\|_2>\Delta\,\middle|\,\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta\right\}\\
&=\mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^\star\right\|_2>\Delta\,\middle|\,m-\mu>\delta_2^{(i)},\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta\right\}\\
&\qquad\cdot\mathbb{P}\left\{m-\mu>\delta_2^{(i)}\,\middle|\,\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta\right\}\\
&\quad+\mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^\star\right\|_2>\Delta\,\middle|\,m-\mu\le\delta_2^{(i)},\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta\right\}\\
&\qquad\cdot\mathbb{P}\left\{m-\mu\le\delta_2^{(i)}\,\middle|\,\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta\right\}\\
&\le\mathbb{P}\left\{\delta_1^{(i)}m+\delta_2^{(i)}\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^\star\right\|_2M+\frac{3L_2m}{2}\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^\star\right\|_2^2\right.\\
&\qquad\left.>\Delta m\left(m-\delta_2^{(i)}\right)\,\middle|\,m-\mu>\delta_2^{(i)},\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta\right\}\\
&\qquad\cdot\mathbb{P}\left\{m-\mu>\delta_2^{(i)}\,\middle|\,\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta\right\}\\
&\quad+\mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^\star\right\|_2>\Delta\,\middle|\,m-\mu\le\delta_2^{(i)},\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta\right\}\\
&\qquad\cdot\mathbb{P}\left\{m-\mu\le\delta_2^{(i)}\,\middle|\,\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^\star\right\|_2\le\Delta\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\le \mathbb{P}\left\{\delta_1^{(i)} m + \delta_2^{(i)} \Delta M + \frac{3L_2 m}{2}\Delta^2 > \Delta m \left(m - \delta_2^{(i)}\right)\right. \\
&\qquad \left| m - \mu > \delta_2^{(i)}, \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta \right\} \\
&\qquad \cdot \mathbb{P}\left\{m - \mu > \delta_2^{(i)} \,\middle|\, \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta \right\} \\
&\quad + \mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star\right\|_2 > \Delta \,\middle|\, m - \mu \le \delta_2^{(i)}, \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta \right\} \\
&\qquad \cdot \mathbb{P}\left\{m - \mu \le \delta_2^{(i)} \,\middle|\, \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta \right\} \\
&\le \mathbb{P}\left\{\delta_1^{(i)} m + \delta_2^{(i)} M\Delta + \frac{3L_2 m}{2}\Delta^2 > \Delta m \left(m - \delta_2^{(i)}\right),\right. \\
&\qquad \left. m - \mu > \delta_2^{(i)} \,\middle|\, \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta \right\} \\
&\quad + \mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star\right\|_2 > \Delta, m - \mu \le \delta_2^{(i)} \,\middle|\, \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta \right\} \\
&\le \mathbb{P}\left\{\delta_1^{(i)} m + \delta_2^{(i)} M\Delta + \frac{3L_2 m}{2} > \Delta m\mu,\right. \\
&\qquad \left. m - \mu > \delta_2^{(i)} \,\middle|\, \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta \right\} \\
&\quad + \mathbb{P}\left\{m - \mu \le \delta_2^{(i)} \,\middle|\, \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta \right\}
\end{aligned}
$$

where the second step derives from (8.11), the third step from the majorization $\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta$, the fifth step from the minorization $m - \delta_2^{(i)} > \mu$.

From this inequality and $\mathbb{P}\left\{\bigcap_{j=1}^{i} \left\{\left\|\boldsymbol{\theta}^{(j)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta\right\}\right\} \le \mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta\right\}$, using Markov's inequality, we get

$$
\begin{aligned}
&\mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star\right\|_2 > \Delta \,\middle|\, \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta \right\} \mathbb{P}\left\{\bigcap_{j=1}^{i} \left\{\left\|\boldsymbol{\theta}^{(j)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta\right\}\right\} \\
&\le \mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star\right\|_2 > \Delta \,\middle|\, \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta \right\} \mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta\right\} \\
&\le \mathbb{P}\left\{\delta_1^{(i)} m + \delta_2^{(i)} M\Delta > \Delta m\mu - \frac{3L_2 m}{2}\Delta^2,\right. \\
&\qquad \left. m - \mu > \delta_2^{(i)} \,\middle|\, \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta \right\} \mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta\right\} \\
&\quad + \mathbb{P}\left\{m - \mu \le \delta_2^{(i)} \,\middle|\, \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta \right\} \mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta\right\} \\
&\le \mathbb{P}\left\{\delta_1^{(i)} m + \delta_2^{(i)} M\Delta > \Delta m\mu - \frac{3L_2 m}{2}\Delta^2,\right. \\
&\qquad \left. m - \mu > \delta_2^{(i)}, \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta \right\} \\
&\quad + \mathbb{P}\left\{m - \mu \le \delta_2^{(i)}, \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \le \Delta \right\} \\
&\le \mathbb{P}\left\{\delta_1^{(i)} m + \delta_2^{(i)} M\Delta > \Delta m\mu - \frac{3L_2 m}{2}\Delta^2\right\} \\
&\quad + \mathbb{P}\left\{m - \mu \le \delta_2^{(i)}\right\} \\
&\le \frac{\mathbb{E}\left(m\delta_1^{(i)} + M\Delta\delta_2^{(i)}\right)^2}{m^2\Delta^2\left(\mu - \frac{3L_2}{2}\Delta\right)^2} + \frac{\mathbb{E}\left(\delta_2^{(i)}\right)^2}{(m-\mu)^2},
\end{aligned}
$$

provided $m > \mu > \frac{3L_2}{2}\Delta$.

If we take $\mu = (1-\varepsilon)\, m$, we get

$$\begin{aligned}
&\mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2 > \Delta \,\middle|\, \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 \leq \Delta\right\} \mathbb{P}\left\{\bigcap_{j=1}^{i}\left\{\left\|\boldsymbol{\theta}^{(j)} - \boldsymbol{\theta}^{\star}\right\|_2 \leq \Delta\right\}\right\} \\
&\leq \frac{\mathbb{E}\left(m\delta_1^{(i)} + M\Delta\delta_2^{(i)}\right)^2}{m^2\Delta^2\left((1-\varepsilon)\, m - \frac{3L_2}{2}\Delta\right)^2} + \frac{\mathbb{E}\left(\delta_2^{(i)}\right)^2}{\varepsilon^2 m^2}.
\end{aligned}$$

We use the inequality $(a+b)^2 \leq 2\left(a^2+b^2\right)$ to get

$$\begin{aligned}
&\mathbb{P}\left\{\max_{1\leq i\leq n+1}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 > \Delta\right\} \\
&= \sum_{i=0}^{n}\mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2 > \Delta \,\middle|\, \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 \leq \Delta\right\} \mathbb{P}\left\{\bigcap_{j=1}^{i}\left\{\left\|\boldsymbol{\theta}^{(j)} - \boldsymbol{\theta}^{\star}\right\|_2 \leq \Delta\right\}\right\} \\
&\leq \frac{2\sum_{i=0}^{n}\mathbb{E}\left(\delta_1^{(i)}\right)^2}{\Delta^2\left((1-\varepsilon)\, m - \frac{3L_2}{2}\Delta\right)^2} + \left[\frac{2\varepsilon^2 M^2 + \left((1-\varepsilon)\, m - \frac{3L_2}{2}\Delta\right)^2}{\varepsilon^2 m^2\left((1-\varepsilon)\, m - \frac{3L_2}{2}\Delta\right)^2}\right]\sum_{i=0}^{n}\mathbb{E}\left(\delta_2^{(i)}\right)^2.
\end{aligned}$$

As for the conditions, $m > \mu$ is automatically verified while $\mu > \frac{3L_2}{2}\Delta$ becomes $\Delta < \frac{2(1-\varepsilon)m}{3L_2}$.

Now, let us denote $\delta := (1-\varepsilon)\, m - \frac{3L_2}{2}\Delta$ with $\delta > 0$. As a result,

$$\begin{aligned}
\mathbb{P}\left\{\max_{1\leq i\leq n+1}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 > \Delta\right\} &\leq \frac{2}{\Delta^2\delta^2}\sum_{i=0}^{n}\mathbb{E}\left(\delta_1^{(i)}\right)^2 + \left(\frac{2M^2}{m^2\delta^2} + \frac{1}{\varepsilon^2 m^2}\right)\sum_{i=0}^{n}\mathbb{E}\left(\delta_2^{(i)}\right)^2 \\
&\leq K_1\left(\frac{1}{\Delta^2}\sum_{i=0}^{n}\mathbb{E}\left(\delta_1^{(i)}\right)^2 + \sum_{i=0}^{n}\mathbb{E}\left(\delta_2^{(i)}\right)^2\right)
\end{aligned}$$

for a constant K_1 depending on K, m, M, ε and δ. (In particular, one could take $K_1 \geq \max\left\{\frac{2}{\delta^2}, \frac{2M^2}{m^2\delta^2} + \frac{1}{\varepsilon^2 m^2}\right\}$.) Taking $\Delta = \left(\frac{\sum_{i=0}^{n}\mathbb{E}\left(\delta_1^{(i)}\right)^2}{\frac{K_2}{K_1} - \sum_{i=0}^{n}\mathbb{E}\left(\delta_2^{(i)}\right)^2}\right)^{\frac{1}{2}}$, we can write this as

$$\mathbb{P}\left\{\max_{1\leq i\leq n+1}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 > \left(\frac{\sum_{i=0}^{n}\mathbb{E}\left(\delta_1^{(i)}\right)^2}{\frac{K_2}{K_1} - \sum_{i=0}^{n}\mathbb{E}\left(\delta_2^{(i)}\right)^2}\right)^{\frac{1}{2}}\right\} \leq K_2.$$

Now, $\delta > 0$ can be written as

$$\begin{gathered}
(1-\varepsilon)\, m - \frac{3L_2}{2}\Delta > 0, \\
(1-\varepsilon)^2 m^2\left[\frac{K_2}{K_1} - \sum_{i=0}^{n}\mathbb{E}\left(\delta_2^{(i)}\right)^2\right] > \frac{9L_2^2}{4}\sum_{i=0}^{n}\mathbb{E}\left(\delta_1^{(i)}\right)^2
\end{gathered}$$

and this is verified under the conditions of the theorem. QED

8.2.2 Stochastic Approximation Schemes

Proof of Theorem 4. From $\boldsymbol{\theta}^{(i+1)} = \boldsymbol{\theta}^{(i)} - \gamma_{i+1}\dot{\hat{F}}\left(\boldsymbol{\theta}^{(i)}\right)$,

$$
\begin{aligned}
\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star} &= \boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star} - \gamma_{i+1}\left\{\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) + \dot{\hat{F}}\left(\boldsymbol{\theta}^{(i)}\right) - \dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\} \\
&= \boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star} - \gamma_{i+1}\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \gamma_{i+1}\boldsymbol{\eta}^{(i)} \\
&= \left(\mathbf{I}_K - \gamma_{i+1}\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right)\left(\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right) \\
&\quad - \gamma_{i+1}\left(\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \dot{F}\left(\boldsymbol{\theta}^{\star}\right) - \ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\left(\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right)\right) - \gamma_{i+1}\boldsymbol{\eta}^{(i)}
\end{aligned}
$$

where the third step comes from the rewriting

$$
\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) = \ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\left(\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right) + \left(\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \dot{F}\left(\boldsymbol{\theta}^{\star}\right) - \ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\left(\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right)\right)
$$

and from the fact that $\dot{F}\left(\boldsymbol{\theta}^{\star}\right) = 0$. Now, taking norms,

$$
\begin{aligned}
\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2 &\leq \left\|\mathbf{I}_K - \gamma_{i+1}\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right\|_2 \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 \\
&\quad + \gamma_{i+1}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \dot{F}\left(\boldsymbol{\theta}^{\star}\right) - \ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\left(\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right)\right\|_2 + \gamma_{i+1}\delta_1^{(i)} \\
&\leq \left\|\mathbf{I}_K - \gamma_{i+1}\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right\|_2 \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 + \gamma_{i+1}\frac{L_2}{2}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2^2 + \gamma_{i+1}\delta_1^{(i)}
\end{aligned}
$$

where the first step comes from $\delta_1^{(i)} = \left\|\boldsymbol{\eta}^{(i)}\right\|_2$ and the second step comes from the third inequality of Lemma 2. Provided $\gamma_{i+1}\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) < 1$,

$$
\left\|\mathbf{I}_K - \gamma_{i+1}\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right\|_2 = 1 - \gamma_{i+1}\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right).
$$

Therefore,

$$
\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2 \leq \left(1 - \gamma_{i+1}\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right)\right)\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 + \gamma_{i+1}\frac{L_2}{2}\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2^2 + \gamma_{i+1}\delta_1^{(i)}.
$$

QED

Proof of Theorem 5. In this case too, as in Theorem 2, we consider what happens when $\left\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^{\star}\right\|_2 \leq \Delta$. We define

$$
\begin{aligned}
a^{(i)} &:= \gamma_{i+1}\delta_1^{(i)}, \\
b^{(i)} &:= 1 - \gamma_{i+1}\left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right).
\end{aligned}
$$

Then, from the result in Theorem 4 and $\left\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^{\star}\right\|_2 \leq \Delta$,

$$
\begin{aligned}
\left\|\boldsymbol{\theta}^{(1)} - \boldsymbol{\theta}^{\star}\right\|_2 &\leq \left[1 - \gamma_1\left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right)\right]\left\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^{\star}\right\|_2 + \gamma_1\delta_1^{(0)} \\
&= a^{(0)} + b^{(0)}\left\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^{\star}\right\|_2.
\end{aligned}
$$

We now reason as in the proof of Corollary 2. If $a^{(i)} \leq \left(1 - b^{(i)}\right)\Delta$ for any i, we get, from Lemma 4,

$$\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2 \leq \sum_{j=0}^{i} a^{(j)} \prod_{k=j+1}^{i} b^{(k)} + \Delta \prod_{k=0}^{i} b^{(k)}.$$

We note that

$$\begin{aligned}\prod_{k=j+1}^{i} b^{(k)} &= \exp\left\{\sum_{k=j+1}^{i} \ln\left[1 - \gamma_{k+1}\left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right)\right]\right\} \\ &\leq \exp\left\{-\left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta\right)\sum_{k=j+1}^{i} \gamma_{k+1}\right\}.\end{aligned}$$

To simplify the computations, we define $K_3 := \lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right) - \frac{L_2}{2}\Delta$. Then,

$$\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2 \leq \sum_{j=0}^{i} a^{(j)} \exp\left\{-K_3 \sum_{k=j+1}^{i} \gamma_{k+1}\right\} + \Delta \exp\left\{-K_3 \sum_{k=0}^{i} \gamma_{k+1}\right\}. \tag{8.12}$$

We note that, in order to obtain convergence to 0 of the second term in the right-hand side of this inequality, we need $\sum_{k=0}^{i} \gamma_{k+1} \to \infty$ or $\gamma \leq 1$. We consider the two cases, $\gamma < 1$ and $\gamma = 1$, separately.

Let us start from the case $\gamma < 1$. We first consider the term $\exp\left\{-K_3 \sum_{k=0}^{i} \gamma_{k+1}\right\}$ in (8.12). In order to simplify the computations, we write this as

$$\exp\left\{-K_3 \sum_{k=0}^{i} \gamma_{k+1}\right\} = \exp\left\{-K_3 \sum_{k=0}^{i-1} \gamma_{k+1} - K_3 \gamma_{i+1}\right\}.$$

From the inequality $\left|\gamma_{k+1} - c_1 (k+1)^{-\gamma}\right| \leq c_2 (k+1)^{-\xi}$, Lemma 6 yields

$$\left(c_1 \zeta(\gamma) - c_2 \zeta(\xi)\right) + \frac{c_1}{1-\gamma} i^{1-\gamma} + \frac{c_1}{2} i^{-\gamma} - \frac{c_1 \gamma}{8} i^{-1-\gamma} - \frac{c_2}{1-\xi} i^{1-\xi} - \frac{c_2}{2} i^{-\xi} \leq \sum_{k=0}^{i-1} \gamma_{k+1}.$$

Therefore,

$$\begin{aligned}&\exp\left\{-K_3 \sum_{k=0}^{i} \gamma_{k+1}\right\} = \exp\left\{-K_3 \sum_{k=0}^{i-1} \gamma_{k+1} - K_3 \gamma_{i+1}\right\} \\ &\leq \exp\left\{-K_3 c_1 \zeta(\gamma) + K_3 c_2 \zeta(\xi) - \frac{K_3 c_1}{1-\gamma} i^{1-\gamma} - \frac{K_3 c_1}{2} i^{-\gamma} + \frac{K_3 c_1 \gamma}{8} i^{-1-\gamma}\right. \\ &\qquad \left. + \frac{K_3 c_2}{1-\xi} i^{1-\xi} + \frac{K_3 c_2}{2} i^{-\xi} - K_3 c_1 (i+1)^{-\gamma} + K_3 c_2 (i+1)^{-\xi}\right\} \\ &= O\left(\exp\left\{-\frac{K_3 c_1}{1-\gamma} i^{1-\gamma} + \frac{K_3 c_2}{1-\xi} i^{1-\xi}\right\}\right)\end{aligned} \tag{8.13}$$

where we have used the fact that, from $\left|\gamma_{i+1} - c_1 (i+1)^{-\gamma}\right| \leq c_2 (i+1)^{-\xi}$, $c_1 (i+1)^{-\gamma} - c_2 (i+1)^{-\xi} \leq \gamma_{i+1}$. All the summands $\frac{K_3 c_1}{2} i^{-\gamma}$, $\frac{K_3 c_1 \gamma}{8} i^{-1-\gamma}$, $\frac{K_3 c_2}{2} i^{-\xi}$, $K_3 c_1 (i+1)^{-\gamma}$ and $K_3 c_2 (i+1)^{-\xi}$ vanish asymptotically, the summand $\frac{K_3 c_1}{1-\gamma} i^{1-\gamma}$ diverges, while the behavior of $\frac{K_3 c_2}{1-\xi} i^{1-\xi}$ depends on the value of ξ. If $\xi \geq 1$,

$O\left(\exp\left\{-\frac{K_3c_1}{1-\gamma}i^{1-\gamma}+\frac{K_3c_2}{1-\xi}i^{1-\xi}\right\}\right)=O\left(\exp\left\{-\frac{K_3c_1}{1-\gamma}i^{1-\gamma}\right\}\right)$ while, if $\xi<1$, the term $\frac{K_3c_2}{1-\xi}i^{1-\xi}$ cannot be neglected.

Now we consider the first term in (8.12). From Lemma 7,

$$\begin{aligned}
&\exp\left\{-K_3\sum_{k=j+1}^{i}\gamma_{k+1}\right\}\\
&\leq\exp\left\{2K_3c_2\zeta(\xi)-\frac{K_3c_1}{1-\gamma}(i+1)^{1-\gamma}-\frac{K_3c_1}{2}(i+1)^{-\gamma}\right.\\
&\qquad\left.+\frac{K_3c_1\gamma}{8}(i+1)^{-1-\gamma}+\frac{K_3c_2}{1-\xi}(i+1)^{1-\xi}+\frac{K_3c_2}{2}(i+1)^{-\xi}\right\}\\
&\quad\cdot\exp\left\{\frac{K_3c_1}{1-\gamma}(j+1)^{1-\gamma}+\frac{K_3c_1}{2}(j+1)^{-\gamma}+\frac{K_3c_2}{1-\xi}(j+1)^{1-\xi}+\frac{K_3c_2}{2}(j+1)^{-\xi}\right\}.
\end{aligned}\tag{8.14}$$

The first term in the right-hand side of (8.12) becomes

$$\begin{aligned}
&\sum_{j=0}^{i}a^{(j)}\exp\left\{-K_3\sum_{k=j+1}^{i}\gamma_{k+1}\right\}\\
&\lesssim\exp\left\{2K_3c_2\zeta(\xi)-\frac{K_3c_1}{1-\gamma}(i+1)^{1-\gamma}-\frac{K_3c_1}{2}(i+1)^{-\gamma}\right.\\
&\qquad\left.+\frac{K_3c_1\gamma}{8}(i+1)^{-1-\gamma}+\frac{K_3c_2}{1-\xi}(i+1)^{1-\xi}+\frac{K_3c_2}{2}(i+1)^{-\xi}\right\} \qquad (8.15)\\
&\quad\cdot\sum_{j=0}^{i}a^{(j)}\exp\left\{\frac{K_3c_1}{1-\gamma}(j+1)^{1-\gamma}+\frac{K_3c_1}{2}(j+1)^{-\gamma}+\frac{K_3c_2}{1-\xi}(j+1)^{1-\xi}+\frac{K_3c_2}{2}(j+1)^{-\xi}\right\}\\
&\leq\exp\left\{2K_3c_2\zeta(\xi)-\frac{K_3c_1}{1-\gamma}(i+1)^{1-\gamma}+\frac{K_3c_1\gamma}{8}(i+1)^{-1-\gamma}+\frac{K_3c_2}{1-\xi}(i+1)^{1-\xi}+\frac{K_3c_2}{2}(i+1)^{-\xi}\right\}\\
&\quad\cdot\sum_{j=0}^{i}a^{(j)}\exp\left\{\frac{K_3c_1}{1-\gamma}(j+1)^{1-\gamma}+\frac{K_3c_1}{2}(j+1)^{-\gamma}+\frac{K_3c_2}{1-\xi}(j+1)^{1-\xi}+\frac{K_3c_2}{2}(j+1)^{-\xi}\right\} \qquad (8.16)
\end{aligned}$$

where we have used the majorization $-\frac{K_3c_1}{2}(i+1)^{-\gamma}\leq 0$. From the inequalities for γ_{j+1} and $\delta_1^{(j)}$,

$$a^{(j)}=\gamma_{j+1}\delta_1^{(j)}\lesssim c_1c_3(j+1)^{-\gamma-\delta}+c_2c_3(j+1)^{-\xi-\delta}.$$

We express the sum replacing either $(j+1)^{-\gamma-\delta}$ or $(j+1)^{-\xi-\delta}$ with a generic $(j+1)^{-\nu}$. We apply Lemma 9 to get

$$\begin{aligned}
&\sum_{j=0}^{i}a^{(j)}\exp\left\{-K_3\sum_{k=j+1}^{i}\gamma_{k+1}\right\}\\
&\leq\exp\left\{2K_3c_2\zeta(\xi)-\frac{K_3c_1}{1-\gamma}(i+1)^{1-\gamma}+\frac{K_3c_1\gamma}{8}(i+1)^{-1-\gamma}+\frac{K_3c_2}{1-\xi}(i+1)^{1-\xi}+\frac{K_3c_2}{2}(i+1)^{-\xi}\right\}\\
&\quad\cdot\sum_{j=0}^{i}a^{(j)}\exp\left\{\frac{K_3c_1}{1-\gamma}(j+1)^{1-\gamma}+\frac{K_3c_1}{2}(j+1)^{-\gamma}+\frac{K_3c_2}{1-\xi}(j+1)^{1-\xi}+\frac{K_3c_2}{2}(j+1)^{-\xi}\right\}\\
&\lesssim\exp\left\{2K_3c_2\zeta(\xi)-\frac{K_3c_1}{1-\gamma}(i+1)^{1-\gamma}+\frac{K_3c_1\gamma}{8}(i+1)^{-1-\gamma}+\frac{K_3c_2}{1-\xi}(i+1)^{1-\xi}+\frac{K_3c_2}{2}(i+1)^{-\xi}\right\}
\end{aligned}$$

$$
\begin{aligned}
&\cdot c_1 c_3 \frac{(i+1)^{1-\gamma-\delta-1+\gamma} \exp\left\{\frac{K_3 c_1}{1-\gamma}(i+1)^{1-\gamma} + \frac{K_3 c_1}{2}(i+1)^{-\gamma} + \frac{K_3 c_2}{1-\xi}(i+1)^{1-\xi} + \frac{K_3 c_2}{2}(i+1)^{-\xi}\right\}}{K_3 c_1} \\
&\simeq \frac{c_3}{K_3}(i+1)^{-\delta} \exp\left\{2K_3 c_2 \zeta(\xi) + \frac{K_3 c_1}{2}(i+1)^{-\gamma} + \frac{K_3 c_1 \gamma}{8}(i+1)^{-1-\gamma}\right. \\
&\qquad\qquad\qquad \left. + \frac{2K_3 c_2}{1-\xi}(i+1)^{1-\xi} + K_3 c_2 (i+1)^{-\xi}\right\} \\
&\simeq \frac{c_3 i^{-\delta}}{K_3} \exp\left\{2K_3 c_2 \zeta(\xi) + \frac{2K_3 c_2}{1-\xi}(i+1)^{1-\xi}\right\}
\end{aligned}
$$

where we have used the fact that $\frac{K_3 c_1 \gamma}{8}(i+1)^{-1-\gamma} + \frac{K_3 c_2}{2}(i+1)^{-\xi} \to 0$. If $\xi > 1$, $\sum_{j=0}^{i} a^{(j)} \exp\left\{-K_3 \sum_{k=j+1}^{i} \gamma_{k+1}\right\} \lesssim \frac{c_3 e^{2K_3 c_2 \zeta(\xi)} i^{-\delta}}{K_3}$. From (8.12),

$$
\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2 \lesssim \frac{c_3 e^{2K_3 c_2 \zeta(\xi)} i^{-\delta}}{K_3} + O\left(\exp\left\{-\frac{K_3 c_1}{1-\gamma} i^{1-\gamma} + \frac{K_3 c_2}{1-\xi} i^{1-\xi}\right\}\right) \simeq \frac{c_3 e^{2K_3 c_2 \zeta(\xi)} i^{-\delta}}{K_3}.
$$

Now we consider the case $\gamma = 1$. We start from the second summand in the right-hand side of (8.12). We recall, from Lemma 6 with $\gamma = 1$, that

$$
\sum_{k=0}^{i} \gamma_{k+1} \geq c_1 \gamma_{(0)} - c_2 \zeta(\xi) + c_1 \ln(i+1) - \frac{c_2}{1-\xi}(i+1)^{1-\xi} - \frac{c_2}{2}(i+1)^{-\xi}.
$$

As a result,

$$
\begin{aligned}
&\exp\left\{-K_3 \sum_{k=0}^{i} \gamma_{k+1}\right\} \\
&\leq \exp\left\{-K_3 c_1 \gamma_{(0)} + K_3 c_2 \zeta(\xi) - K_3 c_1 \ln(i+1) + \frac{K_3 c_2}{1-\xi}(i+1)^{1-\xi} + \frac{K_3 c_2}{2}(i+1)^{-\xi}\right\} \\
&\lesssim (i+1)^{-K_3 c_1} \exp\left\{-K_3 c_1 \gamma_{(0)} + K_3 c_2 \zeta(\xi)\right\} \qquad (8.17)
\end{aligned}
$$

where we have used the fact that $1 - \xi < 0$ and both $\frac{K_3 c_2}{2}(i+1)^{-\xi}$ and $-\frac{K_3 c_2}{\xi - 1} i^{1-\xi}$ vanish asymptotically. The second summand in the right-hand side of (8.12) becomes

$$
\Delta \exp\left\{-K_3 \sum_{k=0}^{i} \gamma_{k+1}\right\} \lesssim \Delta i^{-K_3 c_1} \exp\left\{-K_3 c_1 \gamma_{(0)} + K_3 c_2 \zeta(\xi)\right\}.
$$

Now we pass to the first summand in the right-hand side of (8.12), i.e. $\sum_{j=0}^{i} a^{(j)} \exp\left\{-K_3 \sum_{k=j+1}^{i} \gamma_{k+1}\right\}$. From Lemma 7 with $\gamma = 1$,

$$
\begin{aligned}
\sum_{k=j+1}^{i} \gamma_{k+1} \geq -2c_2 \zeta(\xi) + c_1 \ln\left(\frac{i+1}{j+1}\right) - \frac{c_2}{1-\xi}\left[(i+1)^{1-\xi} + (j+1)^{1-\xi}\right] \\
- \frac{c_2}{2}\left[(i+1)^{-\xi} + (j+1)^{-\xi}\right] - c_1 (j+1)^{-1}.
\end{aligned}
$$

and

$$
\begin{aligned}
&\exp\left\{-K_3 \sum_{k=j+1}^{i} \gamma_{k+1}\right\} \\
&\leq \left(\frac{j+1}{i+1}\right)^{K_3 c_1} \exp\left\{2K_3 c_2 \zeta(\xi) + \frac{K_3 c_2}{1-\xi}(i+1)^{1-\xi} + \frac{K_3 c_2}{2}(i+1)^{-\xi}\right\} \\
&\cdot \exp\left\{K_3 c_1 (j+1)^{-1} + \frac{K_3 c_2}{1-\xi}(j+1)^{1-\xi} + \frac{K_3 c_2}{2}(j+1)^{-\xi}\right\}. \qquad (8.18)
\end{aligned}
$$

The first term in the right-hand side of (8.12) becomes

$$
\begin{aligned}
\sum_{j=0}^{i} a^{(j)} \exp\left\{-K_3 \sum_{k=j+1}^{i} \gamma_{k+1}\right\} \lesssim\ & c_1 c_3 \sum_{j=0}^{i} (j+1)^{-1-\delta} \exp\left\{-K_3 \sum_{k=j+1}^{i} \gamma_{k+1}\right\} \\
&+ c_2 c_3 \sum_{j=0}^{i} (j+1)^{-\xi-\delta} \exp\left\{-K_3 \sum_{k=j+1}^{i} \gamma_{k+1}\right\}
\end{aligned}
$$

where

$$
\begin{aligned}
& c_1 c_3 \sum_{j=0}^{i} (j+1)^{-1-\delta} \exp\left\{-K_3 \sum_{k=j+1}^{i} \gamma_{k+1}\right\} \\
\lesssim & \frac{c_1 c_3}{(i+1)^{K_3 c_1}} \exp\left\{2K_3 c_2 \zeta(\xi) + \frac{K_3 c_2}{1-\xi}(i+1)^{1-\xi} + \frac{K_3 c_2}{2}(i+1)^{-\xi}\right\} \\
& \cdot \sum_{j=0}^{i} (j+1)^{K_3 c_1 - 1 - \delta} \exp\left\{K_3 c_1 (j+1)^{-1} + \frac{K_3 c_2}{1-\xi}(j+1)^{1-\xi} + \frac{K_3 c_2}{2}(j+1)^{-\xi}\right\} \\
\lesssim & \frac{c_1 c_3 e^{2K_3 c_2 \zeta(\xi)}}{(i+1)^{K_3 c_1}} \sum_{j=0}^{i} (j+1)^{K_3 c_1 - 1 - \delta} \\
& \cdot \exp\left\{K_3 c_1 (j+1)^{-1} + \frac{K_3 c_2}{1-\xi}(j+1)^{1-\xi} + \frac{K_3 c_2}{2}(j+1)^{-\xi}\right\}
\end{aligned}
$$

and we have used the fact that $\frac{K_3 c_2}{1-\xi}(i+1)^{1-\xi}$ and $\frac{K_3 c_2}{2}(i+1)^{-\xi}$ vanish asymptotically. Using the fact that, for small x, $e^x \simeq 1 + O(x)$, we have

$$
\begin{aligned}
& \sum_{j=0}^{i} (j+1)^{K_3 c_1 - 1 - \delta} \exp\left\{K_3 c_1 (j+1)^{-1} + \frac{K_3 c_2}{1-\xi}(j+1)^{1-\xi} + \frac{K_3 c_2}{2}(j+1)^{-\xi}\right\} \\
= & \sum_{j=0}^{i} (j+1)^{K_3 c_1 - 1 - \delta} + O\left(\sum_{j=0}^{i} (j+1)^{K_3 c_1 - \delta - \xi \wedge 2}\right).
\end{aligned}
$$

Reasoning in the same way, the remaining term behaves like

$$
c_2 c_3 \sum_{j=0}^{i} (j+1)^{-\xi-\delta} \exp\left\{-K_3 \sum_{k=j+1}^{i} \gamma_{k+1}\right\}
$$

$$\lesssim \frac{c_2 c_3}{(i+1)^{K_3 c_1}} \exp\left\{2K_3 c_2 \zeta(\xi) + \frac{K_3 c_2}{1-\xi}(i+1)^{1-\xi} + \frac{K_3 c_2}{2}(i+1)^{-\xi}\right\}$$
$$\cdot \sum_{j=0}^{i} (j+1)^{K_3 c_1 - \xi - \delta} \exp\left\{K_3 c_1 (j+1)^{-1} + \frac{K_3 c_2}{1-\xi}(j+1)^{1-\xi} + \frac{K_3 c_2}{2}(j+1)^{-\xi}\right\}$$
$$\lesssim \frac{c_2 c_3}{(i+1)^{K_3 c_1}} \exp\left\{2K_3 c_2 \zeta(\xi)\right\} \sum_{j=0}^{i} (j+1)^{K_3 c_1 - \xi - \delta}.$$

We have three different cases.

If $K_3 c_1 > \delta$, the Euler–Maclaurin formula yields

$$\sum_{j=0}^{i} (j+1)^{K_3 c_1 - 1 - \delta} = \sum_{j=1}^{i+1} j^{K_3 c_1 - 1 - \delta}$$
$$= \int_1^{i+1} x^{K_3 c_1 - 1 - \delta} \mathrm{d}x + O\left(i^{K_3 c_1 - 1 - \delta} + 1\right)$$
$$= \frac{(i+1)^{K_3 c_1 - \delta}}{K_3 c_1 - \delta} + O\left(i^{K_3 c_1 - 1 - \delta} + 1\right)$$

and

$$\sum_{j=0}^{i} a^{(j)} \exp\left\{-K_3 \sum_{k=j+1}^{i} \gamma_{k+1}\right\} \lesssim \frac{c_1 c_3 e^{2K_3 c_2 \zeta(\xi)} (i+1)^{-\delta}}{K_3 c_1 - \delta}$$
$$+ O\left(i^{-1-\delta} + i^{-K_3 c_1} + i^{-K_3 c_1} \sum_{j=0}^{i} (j+1)^{K_3 c_1 - \delta - \xi \wedge 2}\right).$$

Now, $i^{-K_3 c_1} = o\left(i^{-\delta}\right)$ and $i^{-1-\delta} = o\left(i^{-\delta}\right)$. The last summand, $i^{-K_3 c_1} \sum_{j=0}^{i} (j+1)^{K_3 c_1 - \delta - \xi \wedge 2}$, behaves like $i^{-\delta - \xi \wedge 2 + 1} = o\left(i^{-\delta}\right)$ if $K_3 c_1 + 1 > \xi \wedge 2 + \delta$, like $i^{-K_3 c_1} \ln i = o\left(i^{-\delta}\right)$ if $K_3 c_1 + 1 = \xi \wedge 2 + \delta$, and like $i^{-K_3 c_1} = o\left(i^{-\delta}\right)$ if $K_3 c_1 + 1 < \xi \wedge 2 + \delta$. Therefore, $\sum_{j=0}^{i} a^{(j)} \exp\left\{-K_3 \sum_{k=j+1}^{i} \gamma_{k+1}\right\} \lesssim \frac{c_1 c_3 e^{2K_3 c_2 \zeta(\xi)}}{K_3 c_1 - \delta} i^{-\delta}$ and

$$\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star\right\|_2 \lesssim \frac{c_1 c_3 e^{2K_3 c_2 \zeta(\xi)}}{K_3 c_1 - \delta} i^{-\delta} + \Delta i^{-K_3 c_1} \exp\left\{-K_3 c_1 \gamma_{(0)} + K_3 c_2 \zeta(\xi)\right\} \simeq \frac{c_1 c_3 e^{2K_3 c_2 \zeta(\xi)}}{K_3 c_1 - \delta} i^{-\delta}.$$

If $K_3 c_1 = \delta$, $\sum_{j=0}^{i} (j+1)^{-1} = \sum_{j=1}^{i+1} j^{-1} = \ln(i+1) + O(1)$ (see Lemma 5),

$$\sum_{j=0}^{i} a^{(j)} \exp\left\{-K_3 \sum_{k=j+1}^{i} \gamma_{k+1}\right\} \lesssim c_1 c_3 e^{2K_3 c_2 \zeta(\xi)} i^{-K_3 c_1} \ln i + O\left(i^{-K_3 c_1}\right)$$

and

$$\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star\right\|_2 \lesssim c_1 c_3 e^{2K_3 c_2 \zeta(\xi)} i^{-K_3 c_1} \ln i + O\left(i^{-K_3 c_1}\right).$$

If $K_3 c_1 < \delta$, it is easy to see, using the limit comparison test, that

$$\sum_{j=0}^{i} (j+1)^{K_3 c_1 - 1 - \delta} \exp\left\{K_3 c_1 (j+1)^{-1} + \frac{K_3 c_2}{1-\xi}(j+1)^{1-\xi} + \frac{K_3 c_2}{2}(j+1)^{-\xi}\right\} = O(1).$$

This implies that

$$\sum_{j=0}^{i} a^{(j)} \exp\left\{-K_3 \sum_{k=j+1}^{i} \gamma_{k+1}\right\} = O\left(i^{-K_3 c_1}\right)$$

and that

$$\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2 = O\left(i^{-K_3 c_1}\right).$$

A leading constant can be obtained through the (rough) majorization:

$$\begin{aligned}
&\sum_{j=0}^{i} (j+1)^{K_3 c_1 - 1 - \delta} \exp\left\{K_3 c_1 (j+1)^{-1} + \frac{K_3 c_2}{1-\xi} (j+1)^{1-\xi} + \frac{K_3 c_2}{2} (j+1)^{-\xi}\right\} \\
&\leq \sum_{j=1}^{i+1} j^{K_3 c_1 - 1 - \delta} \exp\left\{K_3 c_1 j^{-1} + \frac{K_3 c_2}{2} j^{-\xi}\right\} \leq \sum_{j=1}^{i+1} j^{K_3 c_1 - 1 - \delta} \exp\left\{K_3 \left(\frac{2c_1 + c_2}{2}\right) j^{-1}\right\} \\
&= \sum_{j=1}^{i+1} j^{K_3 c_1 - 1 - \delta} \sum_{k=0}^{\infty} \frac{K_3^k \left(\frac{2c_1+c_2}{2}\right)^k j^{-k}}{k!} = \sum_{k=0}^{\infty} \frac{K_3^k \left(\frac{2c_1+c_2}{2}\right)^k}{k!} \sum_{j=1}^{i+1} j^{K_3 c_1 - 1 - \delta - k} \\
&\leq \sum_{k=0}^{\infty} \frac{K_3^k \left(\frac{2c_1+c_2}{2}\right)^k}{k!} \zeta\left(1 + \delta + k - K_3 c_1\right).
\end{aligned}$$

Then,

$$\sum_{j=0}^{i} a^{(j)} \exp\left\{-K_3 \sum_{k=j+1}^{i} \gamma_{k+1}\right\} \lesssim i^{-K_3 c_1} c_1 c_3 e^{2 K_3 c_2 \zeta(\xi)} \sum_{k=0}^{\infty} \frac{K_3^k \left(\frac{2c_1+c_2}{2}\right)^k}{k!} \zeta\left(1 + \delta + k - K_3 c_1\right)$$

and

$$\begin{aligned}
\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2 &\lesssim i^{-K_3 c_1} c_1 c_3 e^{2 K_3 c_2 \zeta(\xi)} \sum_{k=0}^{\infty} \frac{K_3^k \left(\frac{2c_1+c_2}{2}\right)^k}{k!} \zeta\left(1 + \delta + k - K_3 c_1\right) \\
&\quad + \Delta i^{-K_3 c_1} \exp\left\{-K_3 c_1 \gamma_{(0)} + K_3 c_2 \zeta\left(\xi\right)\right\}.
\end{aligned}$$

If $\delta_1^{(i)} \equiv 0$, from (8.12),

$$\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2 \leq \Delta \exp\left\{-K_3 \sum_{k=0}^{i} \gamma_{k+1}\right\}.$$

For $\gamma < 1$,

$$\Delta \exp\left\{-K_3 \sum_{k=0}^{i} \gamma_{k+1}\right\} \lesssim \exp\left\{-K_3 c_1 \zeta\left(\gamma\right) + K_3 c_2 \zeta\left(\xi\right) - \frac{K_3 c_1}{1-\gamma} i^{1-\gamma}\right\},$$

and for $\gamma = 1$,

$$\Delta \exp\left\{-K_3 \sum_{k=0}^{i} \gamma_{k+1}\right\} \lesssim \Delta \exp\left\{-K_3 c_1 \gamma_{(0)} + K_3 c_2 \zeta\left(\xi\right)\right\} i^{-K_3 c_1}.$$

QED

Proof of Corollary 2. Reasoning as in the proof of Theorem 3 and using Theorem 4, we have

$$\mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2 > \Delta \,\middle|\, \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{\star}\right\|_2 \leq \Delta\right\} \mathbb{P}\left\{\bigcap_{j=1}^{i} \left\{\left\|\boldsymbol{\theta}^{(j)} - \boldsymbol{\theta}^{\star}\right\|_2 \leq \Delta\right\}\right\}$$

$$
\begin{aligned}
&\le \mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^{\star}\right\|_2>\Delta \,\middle|\, \left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2\le\Delta\right\}\mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2\le\Delta\right\}\\
&\le \mathbb{P}\left\{\left(1-\gamma_{i+1}\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right)\right)\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2+\gamma_{i+1}\frac{L_2}{2}\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2^2\right.\\
&\quad\left.+\gamma_{i+1}\delta_1^{(i)}>\Delta \,\middle|\, \left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2\le\Delta\right\}\mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2\le\Delta\right\}\\
&\le \mathbb{P}\left\{\left(1-\gamma_{i+1}\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right)\right)\Delta+\gamma_{i+1}\frac{L_2}{2}\Delta^2+\gamma_{i+1}\delta_1^{(i)}>\Delta\right.\\
&\quad\left.\middle|\, \left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2\le\Delta\right\}\mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2\le\Delta\right\}\\
&\le \mathbb{P}\left\{\delta_1^{(i)}>\left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right)-\frac{L_2}{2}\Delta\right)\Delta \,\middle|\, \left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2\le\Delta\right\}\\
&\quad\cdot\mathbb{P}\left\{\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2\le\Delta\right\}\\
&\le \mathbb{P}\left\{\delta_1^{(i)}>\left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right)-\frac{L_2}{2}\Delta\right)\Delta, \left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{\star}\right\|_2\le\Delta\right\}\\
&\le \mathbb{P}\left\{\delta_1^{(i)}>\left(\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right)-\frac{L_2}{2}\Delta\right)\Delta\right\}\\
&\le \frac{\mathbb{E}\left(\delta_1^{(i)}\right)^2}{\Delta^2\left[\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right)-\frac{L_2}{2}\Delta\right]^2},
\end{aligned}
$$

provided $\lambda_{\min}\left(\ddot{F}\left(\boldsymbol{\theta}^{\star}\right)\right)>\frac{L_2}{2}\Delta$.

The last result is proved as in Theorem 3. QED

Proof of Theorem 6. (i) We start decomposing (3.3) as

$$
\boldsymbol{\theta}^{(i+1)}=\boldsymbol{\theta}^{(i)}-\gamma_{i+1}\dot{\hat{F}}\left(\boldsymbol{\theta}^{(i)}\right)=\boldsymbol{\theta}^{(i)}-\gamma_{i+1}\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)-\gamma_{i+1}\boldsymbol{\eta}^{(i)}. \tag{8.19}
$$

Note that there is no guarantee that the expected value of $\boldsymbol{\eta}^{(i)}$ is $\mathbf{0}$.

From the first inequality in Lemma 2, replacing $\boldsymbol{\theta}_1$ with $\boldsymbol{\theta}^{(i)}$ and $\boldsymbol{\theta}_2$ with $\boldsymbol{\theta}^{(i+1)}$, we get

$$
F\left(\boldsymbol{\theta}^{(i+1)}\right)-F\left(\boldsymbol{\theta}^{(i)}\right)-\left(\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^{(i)}\right)'\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\le\frac{L_1}{2}\left\|\boldsymbol{\theta}^{(i)}-\boldsymbol{\theta}^{(i+1)}\right\|_2^2.
$$

We set $h_i:=F\left(\boldsymbol{\theta}^{(i)}\right)$. Using (8.19) and the inequality $(a+b)^2\le 2a^2+2b^2$,

$$
\begin{aligned}
h_{i+1}-h_i\le&-\gamma_{i+1}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2-\gamma_{i+1}\left[\boldsymbol{\eta}^{(i)}\right]'\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\\
&+\gamma_{i+1}^2\frac{L_1}{2}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)+\boldsymbol{\eta}^{(i)}\right\|_2^2\\
\le&-\gamma_{i+1}\left[\boldsymbol{\eta}^{(i)}\right]'\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)-\gamma_{i+1}\left(1-\gamma_{i+1}L_1\right)\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2\\
&+\gamma_{i+1}^2L_1\left\|\boldsymbol{\eta}^{(i)}\right\|_2^2,\\
\gamma_{i+1}\left(1-\gamma_{i+1}L_1\right)\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2\le&\; h_i-h_{i+1}-\gamma_{i+1}\left[\boldsymbol{\eta}^{(i)}\right]'\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)+\gamma_{i+1}^2L_1\left\|\boldsymbol{\eta}^{(i)}\right\|_2^2.
\end{aligned}
$$

We take expectations:

$$\begin{aligned}\gamma_{i+1}\left(1-\gamma_{i+1}L_1\right)\mathbb{E}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2 \leq & \mathbb{E}\left(h_i-h_{i+1}\right)-\gamma_{i+1}\mathbb{E}\left[\boldsymbol{\eta}^{(i)}\right]'\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)+\gamma_{i+1}^2L_1\mathbb{E}\left\|\boldsymbol{\eta}^{(i)}\right\|_2^2\\ \leq & \mathbb{E}\left(h_i-h_{i+1}\right)+\gamma_{i+1}\left|\mathbb{E}\left[\boldsymbol{\eta}^{(i)}\right]'\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right|+\gamma_{i+1}^2L_1\mathbb{E}\left\|\boldsymbol{\eta}^{(i)}\right\|_2^2.\end{aligned}$$

From Assumption **MaV** and $\max_{\boldsymbol{\theta}\in\Theta}\left\|\dot{F}\left(\boldsymbol{\theta}\right)\right\|_2 \leq c_1 < \infty$ we have

$$\begin{aligned}\left|\mathbb{E}\left[\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right]'\boldsymbol{\eta}^{(i)}\right| &= \left|\mathbb{E}\mathbb{E}\left[\left[\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right]'\boldsymbol{\eta}^{(i)}\left|\mathcal{F}_i\right.\right]\right|\\ &= \left|\mathbb{E}\left[\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right]'\mathbb{E}\left[\boldsymbol{\eta}^{(i)}\left|\mathcal{F}_i\right.\right]\right|\\ &\leq \mathbb{E}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2\left\|\mathbb{E}\left[\boldsymbol{\eta}^{(i)}\left|\mathcal{F}_i\right.\right]\right\|_2\\ &\leq \mathbb{E}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2\left\|\mathbb{E}\left[\boldsymbol{\eta}^{(i)}\left|\mathcal{F}_i\right.\right]\right\|_2\\ &\leq b_i\mathbb{E}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 \leq c_1b_i,\\ \mathbb{E}\left\|\boldsymbol{\eta}^{(i)}\right\|_2^2 &\leq \mathbb{E}\mathbb{E}\left[\left\|\boldsymbol{\eta}^{(i)}\right\|_2^2\left|\mathcal{F}_i\right.\right] \leq \sigma_i.\end{aligned}$$

Summing up,

$$\gamma_{i+1}\left(1-\gamma_{i+1}L_1\right)\mathbb{E}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2 \leq \mathbb{E}\left(h_i-h_{i+1}\right)+\gamma_{i+1}c_1b_i+\gamma_{i+1}^2L_1\sigma_i.$$

Using the fact that $1-\gamma_{i+1}L_1 \geq c_2 > 0$ and summing from $i=0$ to $i=n$, we get

$$\begin{aligned}c_2\gamma_{i+1}\mathbb{E}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2 &\leq \mathbb{E}\left(h_i-h_{i+1}\right)+\gamma_{i+1}c_1b_i+\gamma_{i+1}^2L_1\sigma_i,\\ c_2\mathbb{E}\sum_{i=0}^{n}\gamma_{i+1}\mathbb{E}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2 &\leq \sum_{i=0}^{n}\mathbb{E}\left(h_i-h_{i+1}\right)+c_1\sum_{i=0}^{n}\gamma_{i+1}b_i+L_1\sum_{i=0}^{n}\gamma_{i+1}^2\sigma_i, \qquad (8.20)\\ \mathbb{E}\min_{0\leq i\leq n}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2 &\leq \frac{c_1\sum_{i=0}^{n}\gamma_{i+1}b_i+\mathbb{E}\left(h_0-h_{n+1}\right)+L_1\sum_{i=0}^{n}\gamma_{i+1}^2\sigma_i}{c_2\sum_{i=0}^{n}\gamma_{i+1}}.\end{aligned}$$

From this, at last,

$$\min_{0\leq i\leq n}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 = O_{\mathbb{P}}\left(\left(\frac{\sum_{i=0}^{n}\gamma_{i+1}b_i+\mathbb{E}\left(h_0-h_{n+1}\right)+\sum_{i=0}^{n}\gamma_{i+1}^2\sigma_i}{\sum_{i=0}^{n}\gamma_{i+1}}\right)^{\frac{1}{2}}\right).$$

If $\boldsymbol{\theta}^{(0)}$ is fixed and we minorize $h_{n+1}=F\left(\boldsymbol{\theta}^{(n+1)}\right)$ with $F\left(\boldsymbol{\theta}^{\star}\right)$, we have

$$\mathbb{E}\left(h_0-h_{n+1}\right) \leq \mathbb{E}\left(F\left(\boldsymbol{\theta}^{(0)}\right)-F\left(\boldsymbol{\theta}^{\star}\right)\right) = F\left(\boldsymbol{\theta}^{(0)}\right)-F\left(\boldsymbol{\theta}^{\star}\right)$$

and

$$\min_{0\leq i\leq n}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 = O_{\mathbb{P}}\left(\left(\frac{1+\sum_{i=0}^{n}\gamma_{i+1}b_i+\sum_{i=0}^{n}\gamma_{i+1}^2\sigma_i}{\sum_{i=0}^{n}\gamma_{i+1}}\right)^{\frac{1}{2}}\right).$$

(ii) We use Cauchy–Schwarz inequality to get

$$\begin{aligned}\left(\sum_{i=0}^{n}\mathbb{E}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2\right)^2 &\leq \left(\sum_{i=0}^{n}\left(\mathbb{E}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2\right)^{\frac{1}{2}}\right)^2\\ &= \left(\sum_{i=0}^{n}\left(\gamma_{i+1}\mathbb{E}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2\right)^{\frac{1}{2}}\gamma_{i+1}^{-\frac{1}{2}}\right)^2\\ &\leq \sum_{i=0}^{n}\gamma_{i+1}\mathbb{E}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2\sum_{i=0}^{n}\gamma_{i+1}^{-1},\end{aligned}$$

or

$$\frac{1}{n}\sum_{i=0}^{n}\mathbb{E}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 \leq \frac{1}{n}\left(\sum_{i=0}^{n}\gamma_{i+1}\mathbb{E}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2\sum_{i=0}^{n}\gamma_{i+1}^{-1}\right)^{\frac{1}{2}}.$$

From (8.20), we get

$$\frac{1}{n}\sum_{i=0}^{n}\mathbb{E}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 \leq \frac{1}{n}\left(\sum_{i=0}^{n}\gamma_{i+1}^{-1}\right)^{\frac{1}{2}}\left(\frac{c_1}{c_2}\sum_{i=0}^{n}\gamma_{i+1}b_i + \frac{\sum_{i=0}^{n}\mathbb{E}\left(h_i - h_{i+1}\right)}{c_2} + \frac{L_1}{c_2}\sum_{i=0}^{n}\gamma_{i+1}^2\sigma_i\right)^{\frac{1}{2}}.$$

(iii) Under **Hess**, the function F is strongly convex with parameter $m > 0$, see Bertsekas et al. (2003, p. 72). In that case, from Eq. (1.16) in Bertsekas et al. (2003, p. 72), we have

$$\left(\dot{F}\left(\boldsymbol{\theta}_1\right) - \dot{F}\left(\boldsymbol{\theta}_2\right)\right)'\left(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\right) \geq m\left\|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\right\|_2^2$$

for any $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathbb{R}^K$. Taking $\boldsymbol{\theta}_1 = \boldsymbol{\theta}^{(i)}$ and $\boldsymbol{\theta}_2 = \boldsymbol{\theta}^\star$, and using Cauchy–Schwarz inequality, we get

$$\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 \leq \frac{1}{m}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2$$

from which the result follows. QED

Proof of Theorem 7. We write $\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star\right\|_2^2$ as

$$\begin{aligned}&\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star\right\|_2^2\\ &= \left\|\boldsymbol{\theta}^{(i)} - \gamma_{i+1}\boldsymbol{\eta}^{(i)} - \gamma_{i+1}\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \boldsymbol{\theta}^\star\right\|_2^2\\ &= \left\|\boldsymbol{\theta}^{(i)} - \gamma_{i+1}\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \boldsymbol{\theta}^\star\right\|_2^2 + \gamma_{i+1}^2\left\|\boldsymbol{\eta}^{(i)}\right\|_2^2\\ &\quad - 2\gamma_{i+1}\left(\boldsymbol{\theta}^{(i)} - \gamma_{i+1}\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \boldsymbol{\theta}^\star\right)'\boldsymbol{\eta}^{(i)}\\ &= \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2^2 + \gamma_{i+1}^2\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2 - 2\gamma_{i+1}\left(\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right)'\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\\ &\quad + \gamma_{i+1}^2\left\|\boldsymbol{\eta}^{(i)}\right\|_2^2 - 2\gamma_{i+1}\left(\boldsymbol{\theta}^{(i)} - \gamma_{i+1}\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \boldsymbol{\theta}^\star\right)'\boldsymbol{\eta}^{(i)}.\end{aligned} \tag{8.21}$$

Under **Hess**, Eq. (1.16) in Bertsekas et al. (2003, p. 72) yields

$$\left(\dot{F}\left(\boldsymbol{\theta}_1\right) - \dot{F}\left(\boldsymbol{\theta}_2\right)\right)'\left(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\right) \geq m\left\|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\right\|_2^2.$$

Taking $\boldsymbol{\theta}_1 = \boldsymbol{\theta}^{(i)}$ and $\boldsymbol{\theta}_2 = \boldsymbol{\theta}^\star$ and using $\dot{F}(\boldsymbol{\theta}^\star) \equiv \mathbf{0}$, we get

$$\left(\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right)' \left(\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right) \geq m \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2^2.$$

Moreover, under **Lip-**1, the second inequality in Lemma 2 yields

$$\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 \leq L_1 \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2.$$

These can be replaced in (8.21) to get

$$\begin{aligned}
&\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star\right\|_2^2 \\
&\leq \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2^2 + \gamma_{i+1}^2 L_1^2 \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2^2 - 2\gamma_{i+1} m \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2^2 \\
&\quad + \gamma_{i+1}^2 \left\|\boldsymbol{\eta}^{(i)}\right\|_2^2 - 2\gamma_{i+1}\left(\boldsymbol{\theta}^{(i)} - \gamma_{i+1}\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \boldsymbol{\theta}^\star\right)' \boldsymbol{\eta}^{(i)} \\
&\leq \left(1 - 2\gamma_{i+1} m + \gamma_{i+1}^2 L_1^2\right) \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2^2 \\
&\quad + \gamma_{i+1}^2 \left\|\boldsymbol{\eta}^{(i)}\right\|_2^2 - 2\gamma_{i+1}\left(\boldsymbol{\theta}^{(i)} - \gamma_{i+1}\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \boldsymbol{\theta}^\star\right)' \boldsymbol{\eta}^{(i)}.
\end{aligned}$$

Taking expectations conditionally on $\mathcal{F}_i$, we get

$$\begin{aligned}
&\mathbb{E}\left\{\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star\right\|_2^2 | \mathcal{F}_i\right\} \\
&\leq \left(1 - 2\gamma_{i+1} m + \gamma_{i+1}^2 L_1^2\right) \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2^2 + \gamma_{i+1}^2 \mathbb{E}\left\{\left\|\boldsymbol{\eta}^{(i)}\right\|_2^2 | \mathcal{F}_i\right\} \\
&\quad - 2\gamma_{i+1} \mathbb{E}\left\{\left(\boldsymbol{\theta}^{(i)} - \gamma_{i+1}\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \boldsymbol{\theta}^\star\right)' \boldsymbol{\eta}^{(i)} | \mathcal{F}_i\right\}.
\end{aligned} \tag{8.22}$$

From **MaV**, we have

$$\begin{aligned}
&\left|\mathbb{E}\left\{\left(\boldsymbol{\theta}^{(i)} - \gamma_{i+1}\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \boldsymbol{\theta}^\star\right)' \boldsymbol{\eta}^{(i)} | \mathcal{F}_i\right\}\right| \\
&= \left|\left(\boldsymbol{\theta}^{(i)} - \gamma_{i+1}\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \boldsymbol{\theta}^\star\right)' \mathbb{E}\left\{\boldsymbol{\eta}^{(i)} | \mathcal{F}_i\right\}\right| \\
&\leq \left\|\boldsymbol{\theta}^{(i)} - \gamma_{i+1}\dot{F}\left(\boldsymbol{\theta}^{(i)}\right) - \boldsymbol{\theta}^\star\right\|_2 \left\|\mathbb{E}\left\{\boldsymbol{\eta}^{(i)} | \mathcal{F}_i\right\}\right\|_2 \\
&\leq \left(\left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2 + \gamma_{i+1}\left\|\dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2\right) b_i \\
&\leq b_i \left(1 + \gamma_{i+1} L_1\right) \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2
\end{aligned}$$

where the last step uses the second inequality in Lemma 2. Therefore, replacing this into (8.22) and using **MaV**,

$$\begin{aligned}
&\mathbb{E}\left\{\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star\right\|_2^2 | \mathcal{F}_i\right\} \\
&\leq \left(1 - 2\gamma_{i+1} m + \gamma_{i+1}^2 L_1^2\right) \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2^2 + \gamma_{i+1}^2 \sigma_i \\
&\quad + 2\gamma_{i+1} b_i \left(1 + \gamma_{i+1} L_1\right) \left\|\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star\right\|_2.
\end{aligned} \tag{8.23}$$

We use the inequality $2ab \le a^2 + b^2$ to write

$$\begin{aligned}
&2\gamma_{i+1} b_i \left(1+\gamma_{i+1}L_1\right) \left\| \boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star \right\|_2 \\
&\le 2\left\{ \frac{\gamma_{i+1}\left(1+\gamma_{i+1}L_1\right) b_i}{c_i} \right\} \left\{ c_i \left\| \boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star \right\|_2 \right\} \\
&\le \frac{\gamma_{i+1}^2 \left(1+\gamma_{i+1}L_1\right)^2 b_i^2}{c_i^2} + c_i^2 \left\| \boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star \right\|_2^2
\end{aligned} \tag{8.24}$$

for a sequence c_i. We choose c_i in such a way to balance, in (8.24), the two terms $\frac{\gamma_{i+1}^2(1+\gamma_{i+1}L_1)^2 b_i^2}{c_i^2}$ and $c_i^2\Delta^2$, intended as a replacement for $c_i^2 \left\| \boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star \right\|_2^2$. As a result, $c_i^2 = \frac{\gamma_{i+1}(1+\gamma_{i+1}L_1) b_i}{\Delta}$. Therefore, (8.23) becomes

$$\begin{aligned}
&\mathbb{E}\left\{ \left\| \boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star \right\|_2^2 | \mathcal{F}_i \right\} \\
&\le \left(1 - 2\gamma_{i+1} m + \gamma_{i+1}^2 L_1^2 + \Delta^{-1}\gamma_{i+1}\left(1+\gamma_{i+1}L_1\right) b_i\right) \left\| \boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star \right\|_2^2 \\
&\quad + \gamma_{i+1}\left(\sigma_i \gamma_{i+1} + \Delta\left(1+\gamma_{i+1}L_1\right) b_i\right).
\end{aligned}$$

We set

$$\begin{aligned}
A^{(i)} &= \gamma_{i+1}\left(\sigma_i \gamma_{i+1} + \Delta\left(1+\gamma_{i+1}L_1\right) b_i\right), \\
B^{(i)} &= 1 - 2\gamma_{i+1} m + \gamma_{i+1}^2 L_1^2 + \Delta^{-1}\gamma_{i+1}\left(1+\gamma_{i+1}L_1\right) b_i.
\end{aligned}$$

From

$$\begin{aligned}
\mathbb{E}\left\| \boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star \right\|_2^2 &= \mathbb{E}\mathbb{E}\left\{ \left\| \boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star \right\|_2^2 | \mathcal{F}_i \right\} \le \mathbb{E}\left\{ A^{(i)} + B^{(i)} \left\| \boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star \right\|_2^2 \right\} \\
&= A^{(i)} + B^{(i)} \mathbb{E}\left\| \boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^\star \right\|_2^2,
\end{aligned}$$

Lemma 4 allows us to prove the recurrence

$$\mathbb{E}\left\| \boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star \right\|_2^2 \le \sum_{j=0}^{i} A^{(j)} \prod_{k=j+1}^{i} B^{(k)} + \left\| \boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^\star \right\|_2^2 \prod_{k=0}^{i} B^{(k)}$$

that, taking into account $\left\| \boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^\star \right\|_2^2 \le \Delta^2$, becomes

$$\mathbb{E}\left\| \boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^\star \right\|_2^2 \le \sum_{j=0}^{i} A^{(j)} \prod_{k=j+1}^{i} B^{(k)} + \Delta^2 \prod_{k=0}^{i} B^{(k)}. \tag{8.25}$$

From $\left| b_i - c_3 (i+1)^{-\beta} \right| \le c_4 (i+1)^{-\zeta}$, we can see that $b_i \le K_4$ for a constant $K_4 \ge 0$ and we can write

$$\begin{aligned}
\prod_{k=0}^{i} B^{(k)} &= \prod_{k=0}^{i} \left[1 - 2\gamma_{i+1} m + \gamma_{i+1}^2 L_1^2 + \Delta^{-1}\gamma_{i+1}\left(1+\gamma_{i+1}L_1\right) b_i\right] \\
&= \exp\left\{ \sum_{k=0}^{i} \ln\left[1 - 2\gamma_{i+1} m + \gamma_{i+1}^2 L_1^2 + \Delta^{-1}\gamma_{i+1}\left(1+\gamma_{i+1}L_1\right) b_i\right] \right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \exp\left\{-2m\sum_{k=0}^{i}\gamma_{i+1}+L_1^2\sum_{k=0}^{i}\gamma_{i+1}^2+\Delta^{-1}\sum_{k=0}^{i}\gamma_{i+1}b_i+\Delta^{-1}L_1\sum_{k=0}^{i}\gamma_{i+1}^2 b_i\right\}\\
&\leq \exp\left\{-2m\sum_{k=0}^{i}\gamma_{i+1}+\left(L_1^2+\Delta^{-1}K_4L_1\right)\sum_{k=0}^{i}\gamma_{i+1}^2+\Delta^{-1}\sum_{k=0}^{i}\gamma_{i+1}b_i\right\}\\
&\leq \exp\left\{-2m\sum_{k=0}^{i}\gamma_{i+1}+\left(\left(L_1^2+\Delta^{-1}K_4L_1\right)\vee\left(\Delta^{-1}\right)\right)\sum_{k=0}^{i}\left(\gamma_{i+1}^2+\gamma_{i+1}b_i\right)\right\}\\
&\leq \exp\left\{-2m\sum_{k=0}^{i}\gamma_{i+1}+K_5\sum_{k=0}^{i}\left(\gamma_{i+1}^2+\gamma_{i+1}b_i\right)\right\}
\end{aligned}
$$

where we have set $K_5 := \left(L_1^2+\Delta^{-1}K_4L_1\right)\vee\Delta^{-1}$. In the same way,

$$
\prod_{k=j+1}^{i}B^{(k)}\leq \exp\left\{-2m\sum_{k=j+1}^{i}\gamma_{i+1}+K_5\sum_{k=j+1}^{i}\left(\gamma_{i+1}^2+\gamma_{i+1}b_i\right)\right\}.
$$

As concerns $\sum_{k=0}^{i}\gamma_{i+1}$, we use Lemma 6 to get

$$
\left(c_1\zeta\left(\gamma\right)-c_2\zeta\left(\xi\right)\right)+\frac{c_1}{1-\gamma}i^{1-\gamma}+\frac{c_1}{2}i^{-\gamma}-\frac{c_1\gamma}{8}i^{-1-\gamma}-\frac{c_2}{1-\xi}i^{1-\xi}-\frac{c_2}{2}i^{-\xi}\leq\sum_{k=0}^{i-1}\gamma_{k+1}.
$$

Now we turn to $\sum_{k=0}^{i-1}\left(\gamma_{k+1}^2+\gamma_{k+1}b_k\right)$. From $\left|\gamma_{i+1}-c_1\left(i+1\right)^{-\gamma}\right|\leq c_2\left(i+1\right)^{-\xi}$ and $\left|b_i-c_3\left(i+1\right)^{-\beta}\right|\leq c_4\left(i+1\right)^{-\zeta}$, we have

$$
\begin{aligned}
\gamma_{i+1}^2 &\leq \left(c_1\left(i+1\right)^{-\gamma}+c_2\left(i+1\right)^{-\xi}\right)^2=c_1^2\left(i+1\right)^{-2\gamma}+c_2^2\left(i+1\right)^{-2\xi}+2c_1c_2\left(i+1\right)^{-\gamma-\xi},\\
\gamma_{i+1}b_i &\leq \left(c_1\left(i+1\right)^{-\gamma}+c_2\left(i+1\right)^{-\xi}\right)\left(c_3\left(i+1\right)^{-\beta}+c_4\left(i+1\right)^{-\zeta}\right)\\
&= c_1c_3\left(i+1\right)^{-\gamma-\beta}+c_2c_4\left(i+1\right)^{-\xi-\zeta}+c_2c_3\left(i+1\right)^{-\beta-\xi}+c_1c_4\left(i+1\right)^{-\gamma-\zeta},\\
\gamma_{i+1}^2+\gamma_{i+1}b_i &\leq c_1^2\left(i+1\right)^{-2\gamma}+c_2^2\left(i+1\right)^{-2\xi}+c_1c_3\left(i+1\right)^{-\gamma-\beta}+c_2c_4\left(i+1\right)^{-\xi-\zeta}\\
&\quad+2c_1c_2\left(i+1\right)^{-\gamma-\xi}+c_2c_3\left(i+1\right)^{-\beta-\xi}+c_1c_4\left(i+1\right)^{-\gamma-\zeta}.
\end{aligned}
$$

For the minorization, we use the obvious inequalities

$$
\begin{aligned}
\gamma_{i+1} &\geq c_1\left(i+1\right)^{-\gamma}-c_2\left(i+1\right)^{-\xi},\\
b_i &\geq c_3\left(i+1\right)^{-\beta}-c_4\left(i+1\right)^{-\zeta}.
\end{aligned}
$$

However, when transforming them into minorizations for γ_{i+1}^2 and $\gamma_{i+1}b_i$, one must pay attention to the fact that the right-hand sides of these inequalities could be negative. We then introduce the set of indexes

$$
\mathcal{I}:=\left\{i: c_1\left(i+1\right)^{-\gamma}\geq c_2\left(i+1\right)^{-\xi}, c_3\left(i+1\right)^{-\beta}\geq c_4\left(i+1\right)^{-\zeta}\right\}.
$$

We note that

$$
\mathcal{I}:=\left\{i: i\geq\max\left\{\left(\frac{c_2}{c_1}\right)^{\frac{1}{\xi-\gamma}},\left(\frac{c_4}{c_3}\right)^{\frac{1}{\zeta-\beta}}\right\}-1\right\},
$$

so that the set of i's not belonging to $\mathcal{I}$ is finite and has cardinality at most, say, I. We then have

$$\begin{aligned}\gamma_{i+1} &\geq \left(c_1 (i+1)^{-\gamma} - c_2 (i+1)^{-\xi}\right) 1_{\{i\in\mathcal{I}\}},\\ b_i &\geq \left(c_3 (i+1)^{-\beta} - c_4 (i+1)^{-\zeta}\right) 1_{\{i\in\mathcal{I}\}},\end{aligned}$$

and

$$\begin{aligned}\gamma_{i+1}^2 &\geq \left[c_1 (i+1)^{-\gamma} - c_2 (i+1)^{-\xi}\right]^2 1_{\{i\in\mathcal{I}\}}\\ &= \left[c_1^2 (i+1)^{-2\gamma} + c_2^2 (i+1)^{-2\xi} - 2c_1c_2 (i+1)^{-\gamma-\xi}\right] 1_{\{i\in\mathcal{I}\}},\\ \gamma_{i+1}b_i &\geq \left[c_1 (i+1)^{-\gamma} - c_2 (i+1)^{-\xi}\right] 1_{\{i\in\mathcal{I}\}} \left[c_3 (i+1)^{-\beta} - c_4 (i+1)^{-\zeta}\right] 1_{\{i\in\mathcal{I}\}}\\ &= \left[c_1c_3 (i+1)^{-\gamma-\beta} + c_2c_4 (i+1)^{-\xi-\zeta} - c_2c_3 (i+1)^{-\beta-\xi} - c_1c_4 (i+1)^{-\gamma-\zeta}\right] 1_{\{i\in\mathcal{I}\}},\\ \gamma_{i+1}^2 + \gamma_{i+1}b_i &\geq \Big[c_1^2 (i+1)^{-2\gamma} + c_2^2 (i+1)^{-2\xi} + c_1c_3 (i+1)^{-\gamma-\beta} + c_2c_4 (i+1)^{-\xi-\zeta}\\ &\quad -2c_1c_2 (i+1)^{-\gamma-\xi} - c_2c_3 (i+1)^{-\beta-\xi} - c_1c_4 (i+1)^{-\gamma-\zeta}\Big] 1_{\{i\in\mathcal{I}\}}.\end{aligned}$$

We note that, for a suitable choice of the constants $k_{j_1}, k_{j_2}, \alpha_{j_1}$ and α_{j_2}, the majorization and minorization of $\gamma_{i+1}^2 + \gamma_{i+1}b_i$ can be written as

$$\gamma_{i+1}^2 + \gamma_{i+1}b_i \leq \sum_{j_1} k_{j_1} (i+1)^{-\alpha_{j_1}} + \sum_{j_2} k_{j_2} (i+1)^{-\alpha_{j_2}}$$

and

$$\begin{aligned}\gamma_{i+1}^2 + \gamma_{i+1}b_i &\geq \sum_{j_1} k_{j_1} (i+1)^{-\alpha_{j_1}} 1_{\{i\in\mathcal{I}\}} - \sum_{j_2} k_{j_2} (i+1)^{-\alpha_{j_2}} 1_{\{i\in\mathcal{I}\}}\\ &\geq \sum_{j_1} k_{j_1} (i+1)^{-\alpha_{j_1}} 1_{\{i\in\mathcal{I}\}} - \sum_{j_2} k_{j_2} (i+1)^{-\alpha_{j_2}}.\end{aligned}$$

Note that the exponents α_{j_1}'s correspond to 2γ, 2ξ, $\gamma+\beta$ and $\xi+\zeta$ and the exponents α_{j_2}'s correspond to $\gamma+\xi$, $\beta+\xi$ and $\gamma+\zeta$. Using Lemma 5, the corresponding inequalities on $\sum_{k=0}^{i-1}\left(\gamma_{k+1}^2 + \gamma_{k+1}b_k\right)$ can be written as

$$\begin{aligned}\sum_{k=0}^{i-1}\left(\gamma_{k+1}^2 + \gamma_{k+1}b_k\right) &\leq \sum_{j_1} k_{j_1} \sum_{k=0}^{i-1} (k+1)^{-\alpha_{j_1}} + \sum_{j_2} k_{j_2} \sum_{k=0}^{i-1} (k+1)^{-\alpha_{j_2}}\\ &= \sum_{j_1} k_{j_1} \sum_{k=1}^{i} k^{-\alpha_{j_1}} + \sum_{j_2} k_{j_2} \sum_{k=1}^{i} k^{-\alpha_{j_2}}\\ &\leq \sum_{j_1} k_{j_1} \left(\zeta(\alpha_{j_1}) + \frac{1}{1-\alpha_{j_1}} i^{1-\alpha_{j_1}} + \frac{1}{2} i^{-\alpha_{j_1}}\right)\\ &\quad + \sum_{j_2} k_{j_2} \left(\zeta(\alpha_{j_2}) + \frac{1}{1-\alpha_{j_2}} i^{1-\alpha_{j_2}} + \frac{1}{2} i^{-\alpha_{j_2}}\right) \qquad (8.26)\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{i-1}\left(\gamma_{i+1}^2+\gamma_{i+1}b_i\right) &\geq \sum_{j_1} k_{j_1}\sum_{k=0}^{i-1}(k+1)^{-\alpha_{j_1}}1_{\{k\in\mathcal{I}\}}-\sum_{j_2}k_{j_2}\sum_{k=0}^{i-1}(k+1)^{-\alpha_{j_2}}\\
&=\sum_{j_1}k_{j_1}\sum_{k=1}^{i}k^{-\alpha_{j_1}}1_{\{k-1\in\mathcal{I}\}}-\sum_{j_2}k_{j_2}\sum_{k=1}^{i}k^{-\alpha_{j_2}}\\
&\geq\sum_{j_1}k_{j_1}\sum_{k=1}^{i}k^{-\alpha_{j_1}}-\sum_{j_2}k_{j_2}\sum_{k=1}^{i}k^{-\alpha_{j_2}}-K_6\\
&\geq\sum_{j_1}k_{j_1}\left(\zeta\left(\alpha_{j_1}\right)+\frac{1}{1-\alpha_{j_1}}i^{1-\alpha_{j_1}}+\frac{1}{2}i^{-\alpha_{j_1}}-\frac{\alpha_{j_1}}{8}i^{-1-\alpha_{j_1}}\right)\\
&\quad-\sum_{j_2}k_{j_2}\left(\zeta\left(\alpha_{j_2}\right)+\frac{1}{1-\alpha_{j_2}}i^{1-\alpha_{j_2}}+\frac{1}{2}i^{-\alpha_{j_2}}\right)-K_6 \qquad (8.27)\end{aligned}$$

for a suitable constant K_6, provided $2\gamma, 2\xi, \gamma+\beta, \xi+\zeta, \gamma+\xi, \beta+\xi$ and $\gamma+\zeta$ are larger than -1 and different from 1. If one of them is equal to 1, the modification to the proof is trivial, i.e. the leading constant $\zeta\left(\alpha_{j_k}\right)$ is replaced by $\gamma_{(0)}$, the term $\frac{1}{1-\alpha_{j_k}}i^{1-\alpha_{j_k}}$ is interpreted as a limit for $\alpha_{j_k}\to 1$, and the remaining terms are modified accordingly. The choice of the coefficients α_{j_1}'s and α_{j_2}'s implies that all terms of the form $i^{-\alpha_{j_1}}$, $i^{-1-\alpha_{j_1}}$ and $i^{-\alpha_{j_2}}$ vanish asymptotically for large i.

Now we consider the behavior of $\Delta^2\prod_{k=0}^{i}B^{(k)}$. We have

$$\begin{aligned}&\Delta^2\prod_{k=0}^{i}B^{(k)}\\
&\leq\Delta^2\exp\left\{-2m\sum_{k=0}^{i}\gamma_{k+1}+K_5\sum_{k=0}^{i}\left(\gamma_{k+1}^2+\gamma_{k+1}b_k\right)\right\}\\
&\leq\Delta^2\exp\left\{-2m\left[\left(c_1\zeta\left(\gamma\right)-c_2\zeta\left(\xi\right)\right)+\frac{c_1}{1-\gamma}i^{1-\gamma}+\frac{c_1}{2}i^{-\gamma}-\frac{c_1\gamma}{8}i^{-1-\gamma}-\frac{c_2}{1-\xi}i^{1-\xi}-\frac{c_2}{2}i^{-\xi}\right]\right.\\
&\qquad+K_5\sum_{j_1}k_{j_1}\left(\zeta\left(\alpha_{j_1}\right)+\frac{1}{1-\alpha_{j_1}}i^{1-\alpha_{j_1}}+\frac{1}{2}i^{-\alpha_{j_1}}\right)\\
&\qquad+K_5\sum_{j_2}k_{j_2}\left(\zeta\left(\alpha_{j_2}\right)+\frac{1}{1-\alpha_{j_2}}i^{1-\alpha_{j_2}}+\frac{1}{2}i^{-\alpha_{j_2}}\right)\\
&\qquad\left.-2m\gamma_{i+1}+K_5\left(\gamma_{i+1}^2+\gamma_{i+1}b_i\right)\right\}\\
&=O\left(\exp\left\{-\frac{2mc_1}{1-\gamma}i^{1-\gamma}+\frac{2mc_2}{1-\xi}i^{1-\xi}+K_5\sum_{j_1}\frac{k_{j_1}}{1-\alpha_{j_1}}i^{1-\alpha_{j_1}}+K_5\sum_{j_2}\frac{k_{j_2}}{1-\alpha_{j_2}}i^{1-\alpha_{j_2}}\right\}\right)\\
&=O\left(\exp\left\{-\frac{2mc_1}{1-\gamma}i^{1-\gamma}\left(1+o\left(1\right)\right)\right\}\right).\end{aligned}$$

Now we turn to the first term of (8.25). The minorization on $\sum_{k=j+1}^{i}\gamma_{i+1}$ comes from Lemma 7:

$$\begin{aligned}\sum_{k=j+1}^{i}\gamma_{k+1}\geq{}&-2c_2\zeta\left(\xi\right)+\frac{c_1}{1-\gamma}\left[\left(i+1\right)^{1-\gamma}-\left(j+1\right)^{1-\gamma}\right]+\frac{c_1}{2}\left[\left(i+1\right)^{-\gamma}-\left(j+1\right)^{-\gamma}\right]\\
&-\frac{c_2}{1-\xi}\left[\left(i+1\right)^{1-\xi}+\left(j+1\right)^{1-\xi}\right]-\frac{c_2}{2}\left[\left(i+1\right)^{-\xi}+\left(j+1\right)^{-\xi}\right]-\frac{c_1\gamma}{8}\left(i+1\right)^{-1-\gamma}.\end{aligned}$$

The majorization on $\sum_{k=j+1}^{i}\left(\gamma_{k+1}^2+\gamma_{k+1}b_k\right)$ is obtained from (8.26) and (8.27) as

$$
\begin{aligned}
&\sum_{k=j+1}^{i}\left(\gamma_{k+1}^2+\gamma_{k+1}b_k\right)\\
&=\sum_{k=0}^{i}\left(\gamma_{k+1}^2+\gamma_{k+1}b_k\right)-\sum_{k=0}^{j}\left(\gamma_{k+1}^2+\gamma_{k+1}b_k\right)\\
&\leq \sum_{j_1} k_{j_1}\left(\frac{1}{1-\alpha_{j_1}}(i+1)^{1-\alpha_{j_1}}+\frac{1}{2}(i+1)^{-\alpha_{j_1}}\right)\\
&\quad+\sum_{j_2} k_{j_2}\left(2\zeta\left(\alpha_{j_2}\right)+\frac{1}{1-\alpha_{j_2}}(i+1)^{1-\alpha_{j_2}}+\frac{1}{2}(i+1)^{-\alpha_{j_2}}\right)\\
&\quad-\sum_{j_1} k_{j_1}\left(\frac{1}{1-\alpha_{j_1}}(j+1)^{1-\alpha_{j_1}}+\frac{1}{2}(j+1)^{-\alpha_{j_1}}-\frac{\alpha_{j_1}}{8}(j+1)^{-1-\alpha_{j_1}}\right)\\
&\quad+\sum_{j_2} k_{j_2}\left(\frac{1}{1-\alpha_{j_2}}(j+1)^{1-\alpha_{j_2}}+\frac{1}{2}(j+1)^{-\alpha_{j_2}}\right)+K_6.
\end{aligned}
$$

As a result,

$$
\begin{aligned}
\prod_{k=j+1}^{i} B^{(k)} &\leq \exp\left\{-2m\sum_{k=j+1}^{i}\gamma_{k+1}+K_5\sum_{k=j+1}^{i}\left(\gamma_{k+1}^2+\gamma_{k+1}b_k\right)\right\}\\
&\leq \exp\Bigg\{4mc_2\zeta(\xi)-\frac{2mc_1}{1-\gamma}(i+1)^{1-\gamma}-mc_1(i+1)^{-\gamma}+\frac{mc_1\gamma}{4}(i+1)^{-1-\gamma}\\
&\quad+\frac{2mc_2}{1-\xi}(i+1)^{1-\xi}+mc_2(i+1)^{-\xi}\\
&\quad+K_5\sum_{j_1} k_{j_1}\left(\frac{1}{1-\alpha_{j_1}}(i+1)^{1-\alpha_{j_1}}+\frac{1}{2}(i+1)^{-\alpha_{j_1}}\right)\\
&\quad+K_5\sum_{j_2} k_{j_2}\left(2\zeta\left(\alpha_{j_2}\right)+\frac{1}{1-\alpha_{j_2}}(i+1)^{1-\alpha_{j_2}}+\frac{1}{2}(i+1)^{-\alpha_{j_2}}\right)\\
&\quad+\frac{2mc_1}{1-\gamma}(j+1)^{1-\gamma}+mc_1(j+1)^{-\gamma}+\frac{2mc_2}{1-\xi}(j+1)^{1-\xi}+mc_2(j+1)^{-\xi}\\
&\quad-K_5\sum_{j_1} k_{j_1}\left(\frac{1}{1-\alpha_{j_1}}(j+1)^{1-\alpha_{j_1}}+\frac{1}{2}(j+1)^{-\alpha_{j_1}}-\frac{\alpha_{j_1}}{8}(j+1)^{-1-\alpha_{j_1}}\right)\\
&\quad+K_5\sum_{j_2} k_{j_2}\left(\frac{1}{1-\alpha_{j_2}}(j+1)^{1-\alpha_{j_2}}+\frac{1}{2}(j+1)^{-\alpha_{j_2}}\right)+K_5K_6\Bigg\}.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
A^{(i)} &= \gamma_{i+1}^2\sigma_i+\Delta\gamma_{i+1}b_i+\Delta L_1\gamma_{i+1}^2 b_i\\
&\lesssim c_1^2c_5(i+1)^{-2\gamma-\sigma}+c_2^2c_5(i+1)^{-2\xi-\sigma}+2c_1c_2c_5(i+1)^{-\gamma-\xi-\sigma}\\
&\quad+\Delta c_1c_3(i+1)^{-\gamma-\beta}+\Delta c_2c_4(i+1)^{-\xi-\zeta}\\
&\quad+\Delta c_2c_3(i+1)^{-\beta-\xi}+\Delta c_1c_4(i+1)^{-\gamma-\zeta}\\
&\quad+\Delta L_1c_1^2c_3(i+1)^{-2\gamma-\beta}+\Delta L_1c_1^2c_4(i+1)^{-2\gamma-\zeta}
\end{aligned}
$$

$$
\begin{aligned}
&+ \Delta L_1 c_2^2 c_3 (i+1)^{-2\xi-\beta} + \Delta L_1 c_2^2 c_4 (i+1)^{-2\xi-\zeta} \\
&+ 2\Delta L_1 c_1 c_2 c_3 (i+1)^{-\gamma-\xi-\beta} + 2\Delta L_1 c_1 c_2 c_4 (i+1)^{-\gamma-\xi-\zeta} \\
&\lesssim K_7 (i+1)^{-\gamma-(\gamma+\sigma)\wedge\beta}
\end{aligned}
$$

for a suitable constant K_7. As a result,

$$
\begin{aligned}
&\sum_{j=0}^{i} A^{(j)} \prod_{k=j+1}^{i} B^{(k)} \\
&\lesssim K_7 \exp\Bigg\{ 4mc_2\zeta(\xi) - \frac{2mc_1}{1-\gamma}(i+1)^{1-\gamma} - mc_1 (i+1)^{-\gamma} + \frac{mc_1\gamma}{4}(i+1)^{-1-\gamma} \\
&+ \frac{2mc_2}{1-\xi}(i+1)^{1-\xi} + mc_2 (i+1)^{-\xi} \\
&+ K_5 \sum_{j_1} k_{j_1} \left(\frac{1}{1-\alpha_{j_1}} (i+1)^{1-\alpha_{j_1}} + \frac{1}{2}(i+1)^{-\alpha_{j_1}} \right) \\
&+ K_5 \sum_{j_2} k_{j_2} \left(2\zeta(\alpha_{j_2}) + \frac{1}{1-\alpha_{j_2}} (i+1)^{1-\alpha_{j_2}} + \frac{1}{2}(i+1)^{-\alpha_{j_2}} \right) + K_5 K_6 \Bigg\} \\
&\cdot \sum_{j=0}^{i} (j+1)^{-\gamma-(\gamma+\sigma)\wedge\beta} \exp\Bigg\{ \frac{2mc_1}{1-\gamma}(j+1)^{1-\gamma} + mc_1 (j+1)^{-\gamma} + \frac{2mc_2}{1-\xi}(j+1)^{1-\xi} + mc_2 (j+1)^{-\xi} \\
&- K_5 \sum_{j_1} k_{j_1} \left(\frac{1}{1-\alpha_{j_1}} (j+1)^{1-\alpha_{j_1}} + \frac{1}{2}(j+1)^{-\alpha_{j_1}} - \frac{\alpha_{j_1}}{8}(j+1)^{-1-\alpha_{j_1}} \right) \\
&+ K_5 \sum_{j_2} k_{j_2} \left(\frac{1}{1-\alpha_{j_2}} (j+1)^{1-\alpha_{j_2}} + \frac{1}{2}(j+1)^{-\alpha_{j_2}} \right) \Bigg\}.
\end{aligned}
$$

We are led to consider the second term in this equation. We apply Lemma 9 to get

$$
\begin{aligned}
&\sum_{j=0}^{i} (j+1)^{-\gamma-(\gamma+\sigma)\wedge\beta} \exp\Bigg\{ \frac{2mc_1}{1-\gamma}(j+1)^{1-\gamma} + mc_1 (j+1)^{-\gamma} + \frac{2mc_2}{1-\xi}(j+1)^{1-\xi} + mc_2 (j+1)^{-\xi} \\
&- K_5 \sum_{j_1} k_{j_1} \left(\frac{1}{1-\alpha_{j_1}} (j+1)^{1-\alpha_{j_1}} + \frac{1}{2}(j+1)^{-\alpha_{j_1}} - \frac{\alpha_{j_1}}{8}(j+1)^{-1-\alpha_{j_1}} \right) \\
&+ K_5 \sum_{j_2} k_{j_2} \left(\frac{1}{1-\alpha_{j_2}} (j+1)^{1-\alpha_{j_2}} + \frac{1}{2}(j+1)^{-\alpha_{j_2}} \right) \Bigg\} \\
&\simeq \frac{(i+1)^{-(\gamma+\sigma)\wedge\beta}}{2mc_1} \exp\Bigg\{ \frac{2mc_1}{1-\gamma}(i+1)^{1-\gamma} + mc_1 (i+1)^{-\gamma} + \frac{2mc_2}{1-\xi}(i+1)^{1-\xi} + mc_2 (i+1)^{-\xi} \\
&- K_5 \sum_{j_1} k_{j_1} \left(\frac{1}{1-\alpha_{j_1}} (i+1)^{1-\alpha_{j_1}} + \frac{1}{2}(i+1)^{-\alpha_{j_1}} - \frac{\alpha_{j_1}}{8}(i+1)^{-1-\alpha_{j_1}} \right) \\
&+ K_5 \sum_{j_2} k_{j_2} \left(\frac{1}{1-\alpha_{j_2}} (i+1)^{1-\alpha_{j_2}} + \frac{1}{2}(i+1)^{-\alpha_{j_2}} \right) \Bigg\} \\
&\simeq \frac{(i+1)^{-(\gamma+\sigma)\wedge\beta}}{2mc_1} \exp\Bigg\{ \frac{2mc_1}{1-\gamma}(i+1)^{1-\gamma} + \frac{2mc_2}{1-\xi}(i+1)^{1-\xi}
\end{aligned}
$$

$$-K_5 \sum_{j_1} k_{j_1} \frac{1}{1-\alpha_{j_1}} (i+1)^{1-\alpha_{j_1}} + K_5 \sum_{j_2} k_{j_2} \frac{1}{1-\alpha_{j_2}} (i+1)^{1-\alpha_{j_2}} \Bigg\} .$$

As a result,

$$\sum_{j=0}^{i} A^{(j)} \prod_{k=j+1}^{i} B^{(k)} = O\left((i+1)^{-(\gamma+\sigma)\wedge\beta} \exp\left\{ \frac{4mc_2}{1-\xi} (i+1)^{1-\xi} + 2K_5 \sum_{j_2} \frac{k_{j_2}}{1-\alpha_{j_2}} (i+1)^{1-\alpha_{j_2}} \right\} \right).$$

This is

$$\sum_{j=0}^{i} A^{(j)} \prod_{k=j+1}^{i} B^{(k)} = O\left(i^{-(\gamma+\sigma)\wedge\beta} \right)$$

provided $1-\xi < 0$ and $1-\alpha_{j_2} < 0$. Using the fact that α_{j_2} is $\gamma+\xi$, $\beta+\xi$ and $\gamma+\zeta$, these conditions become $1 < \xi$, $1 < \gamma+\xi$, $1 < \beta+\xi$ and $1 < \gamma+\zeta$, i.e. $1 < \xi$ and $1 < \gamma+\zeta$.

Now we consider the case $\gamma = 1$. Using (8.17), the second summand in (8.25) can be written as

$$\Delta^2 \prod_{k=0}^{i} B^{(k)} = O\left(\Delta^2 \exp\left\{ -2m \sum_{k=0}^{i} \gamma_{k+1} \right\} \right) = O\left(i^{-2mc_1} \right).$$

As to the first summand in (8.25), (8.18) yields

$$\begin{aligned} &\exp\left\{ -2m \sum_{k=j+1}^{i} \gamma_{k+1} \right\} \\ &\leq \left(\frac{j+1}{i+1} \right)^{2mc_1} \exp\left\{ 4mc_2 \zeta(\xi) + \frac{2mc_2}{1-\xi} (i+1)^{1-\xi} + mc_2 (i+1)^{-\xi} \right\} \\ &\cdot \exp\left\{ 2mc_1 (j+1)^{-1} + \frac{2mc_2}{1-\xi} (j+1)^{1-\xi} + mc_2 (j+1)^{-\xi} \right\}. \end{aligned}$$

Therefore, using $A^{(j)} = O\left((j+1)^{-2-\sigma\wedge(\beta-1)} \right)$, $\frac{2mc_2}{1-\xi} (i+1)^{1-\xi} = o(1)$ and $mc_2 (i+1)^{-\xi} = o(1)$, we have

$$\begin{aligned} &\sum_{j=0}^{i} A^{(j)} \exp\left\{ -2m \sum_{k=j+1}^{i} \gamma_{k+1} \right\} \\ &= O\Bigg((i+1)^{-2mc_1} \sum_{j=0}^{i} (j+1)^{2mc_1-2-\sigma\wedge(\beta-1)} \\ &\quad \cdot \exp\left\{ 2mc_1 (j+1)^{-1} + \frac{2mc_2}{1-\xi} (j+1)^{1-\xi} + mc_2 (j+1)^{-\xi} \right\} \Bigg). \end{aligned}$$

The proof is similar to the one of Theorem 5.

If $2mc_1 - 2 - \sigma\wedge(\beta-1) > -1$ or $2mc_1 > \sigma\wedge(\beta-1)+1$, the expansion $e^x \simeq 1+x$ for small x yields

$$\sum_{j=0}^{i} (j+1)^{2mc_1-2-\sigma\wedge(\beta-1)} \exp\left\{ 2mc_1 (j+1)^{-1} + \frac{2mc_2}{1-\xi} (j+1)^{1-\xi} + mc_2 (j+1)^{-\xi} \right\}$$

$$
\begin{aligned}
&= \sum_{j=0}^{i} (j+1)^{2mc_1-2-\sigma\wedge(\beta-1)} + O\left(\sum_{j=0}^{i} (j+1)^{2mc_1-2-\sigma\wedge(\beta-1)-1\wedge(\xi-1)}\right) \\
&= O\left(i^{2mc_1-1-\sigma\wedge(\beta-1)}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{j=0}^{i} A^{(j)} \exp\left\{-2m\sum_{k=j+1}^{i}\gamma_{k+1}\right\} &= O\left(i^{-1-\sigma\wedge(\beta-1)}\right), \\
\mathbb{E}\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^{\star}\right\|_2^2 &= O\left(i^{-1-\sigma\wedge(\beta-1)} + i^{-2mc_1}\right) \\
&= O\left(i^{-1-\sigma\wedge(\beta-1)}\right), \\
\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^{\star}\right\|_2 &= O_{\mathbb{P}}\left(i^{-\frac{1+\sigma\wedge(\beta-1)}{2}}\right).
\end{aligned}
$$

If $2mc_1 - 2 - \sigma\wedge(\beta-1) = -1$ or $2mc_1 = \sigma\wedge(\beta-1)+1$, the same method gives

$$
\begin{aligned}
&\sum_{j=0}^{i} (j+1)^{-1} \exp\left\{2mc_1 (j+1)^{-1} + \frac{2mc_2}{1-\xi}(j+1)^{1-\xi} + mc_2 (j+1)^{-\xi}\right\} \\
&= \sum_{j=0}^{i} (j+1)^{-1} + O\left(\sum_{j=0}^{i} (j+1)^{-1-(\xi-1)\wedge 1}\right) \\
&= O(\ln i)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{j=0}^{i} A^{(j)} \exp\left\{-2m\sum_{k=j+1}^{i}\gamma_{k+1}\right\} &= O\left(i^{-2mc_1}\ln i\right), \\
\mathbb{E}\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^{\star}\right\|_2^2 &= O\left(i^{-2mc_1}\ln i\right), \\
\left\|\boldsymbol{\theta}^{(i+1)}-\boldsymbol{\theta}^{\star}\right\|_2 &= O_{\mathbb{P}}\left(i^{-mc_1}\ln^{\frac{1}{2}} i\right).
\end{aligned}
$$

At last, when $2mc_1 - 2 - \sigma\wedge(\beta-1) < -1$ or $2mc_1 < \sigma\wedge(\beta-1)+1$, the limit comparison test shows that

$$
\sum_{j=0}^{i} (j+1)^{2mc_1-2-\sigma\wedge(\beta-1)} \exp\left\{2mc_1 (j+1)^{-1} + \frac{2mc_2}{1-\xi}(j+1)^{1-\xi} + mc_2 (j+1)^{-\xi}\right\}
$$

converges as far as $\sum_{j=0}^{i}(j+1)^{2mc_1-2-\sigma\wedge(\beta-1)}$ does. Therefore, we have

$$
\begin{aligned}
&\sum_{j=0}^{i} (j+1)^{2mc_1-2-\sigma\wedge(\beta-1)} \exp\left\{2mc_1 (j+1)^{-1} + \frac{2mc_2}{1-\xi}(j+1)^{1-\xi} + mc_2 (j+1)^{-\xi}\right\} \\
&\leq \sum_{j=0}^{i} (j+1)^{2mc_1-2-\sigma\wedge(\beta-1)} \exp\left\{2mc_1 + \frac{2mc_2}{1-\xi} + mc_2\right\} = O(1).
\end{aligned}
$$

This implies that

$$\sum_{j=0}^{i} A^{(j)} \exp\left\{-2m \sum_{k=j+1}^{i} \gamma_{k+1}\right\} = O\left(i^{-2mc_1}\right),$$

$$\mathbb{E}\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2^2 = O\left(i^{-2mc_1}\right),$$

$$\left\|\boldsymbol{\theta}^{(i+1)} - \boldsymbol{\theta}^{\star}\right\|_2 = O_{\mathbb{P}}\left(i^{-mc_1}\right).$$

QED

8.3 Proofs of Approximation Results

Proof of Theorem 8. We build a predicted value of $F(\boldsymbol{\theta}_0)$ as

$$\tilde{F}(\boldsymbol{\theta}_0) = \mathbf{x}_D'(\boldsymbol{\theta}_0)(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \mathbf{x}_D'(\boldsymbol{\theta}_0)(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\overline{\mathbf{y}} + \boldsymbol{\varepsilon}).$$

As explained in the text, we center the points in $\mathcal{P}(\boldsymbol{\theta}_0)$ in $\boldsymbol{\theta}_0$, so that $\boldsymbol{\theta}_0 \equiv \mathbf{0}$. The vector of regressors associated with the origin is $\mathbf{x}_D(\boldsymbol{\theta}_0) \equiv \mathbf{e}_1$. Moreover, $\overline{\mathbf{y}}$ and $\boldsymbol{\varepsilon}$ are the vectors whose elements are, respectively, $F(\boldsymbol{\theta}_j)$ and $\hat{F}(\boldsymbol{\theta}_j) - F(\boldsymbol{\theta}_j)$ for $\boldsymbol{\theta}_j \in \mathcal{P}(\boldsymbol{\theta}_0)$. It should therefore be clear that $\boldsymbol{\varepsilon}$ may be deterministic or stochastic but, in the second case, it does not have, in general, zero expectation. We have

$$\left|\tilde{F}(\boldsymbol{\theta}_0) - F(\boldsymbol{\theta}_0)\right| \leq \left|\mathbf{x}_D'(\boldsymbol{\theta}_0)(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{y}} - F(\boldsymbol{\theta}_0)\right| + \left|\mathbf{x}_D'(\boldsymbol{\theta}_0)(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\varepsilon}\right|. \tag{8.28}$$

Let us start from the last term. We have

$$\left|\mathbf{x}_D'(\boldsymbol{\theta}_0)(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\varepsilon}\right| \leq \left\|\mathbf{x}_D(\boldsymbol{\theta}_0)\right\|_2 \left\|(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\right\|_2 \left\|\boldsymbol{\varepsilon}\right\|_2 = \left\|\mathbf{x}_D(\boldsymbol{\theta}_0)\right\|_2 \left\|\boldsymbol{\varepsilon}\right\|_2 \sqrt{\lambda_{\max}\left((\mathbf{X}'\mathbf{X})^{-1}\right)} \leq \frac{\left\|\mathbf{x}_D(\boldsymbol{\theta}_0)\right\|_2 \left\|\boldsymbol{\varepsilon}\right\|_2}{\sqrt{\lambda_{\min}(\mathbf{X}'\mathbf{X})}}.$$

Now, $\left\|\mathbf{x}_D(\boldsymbol{\theta}_0)\right\|_2 = \left\|\mathbf{e}_1\right\|_2 = 1$. Therefore, using Lemma 10,

$$\left|\mathbf{x}_D'(\boldsymbol{\theta}_0)(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\varepsilon}\right| \leq h^{-D}\frac{\left\|\boldsymbol{\varepsilon}\right\|_2}{\sqrt{P\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}}.$$

We are left with the other term in (8.28), i.e. $\left|\mathbf{x}_D'(\boldsymbol{\theta}_0)(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{y}} - F(\boldsymbol{\theta}_0)\right|$. The first term, $\mathbf{x}_D'(\boldsymbol{\theta}_0)(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{y}}$, is $p^{\star}(\boldsymbol{\theta}_0)$ where $p^{\star} \in \mathbb{P}_D$ is the polynomial minimizing

$$\sum_{j:\boldsymbol{\theta}_j \in \mathcal{P}(\boldsymbol{\theta}_0)} \left|F(\boldsymbol{\theta}_j) - p^{\star}(\boldsymbol{\theta}_j)\right|^2 = \inf_{p \in \mathbb{P}_d} \sum_{j:\boldsymbol{\theta}_j \in \mathcal{P}(\boldsymbol{\theta}_0)} \left|F(\boldsymbol{\theta}_j) - p(\boldsymbol{\theta}_j)\right|^2.$$

If $\boldsymbol{\theta}_0 \in \mathcal{P}(\boldsymbol{\theta}_0)$,

$$\left|F(\boldsymbol{\theta}_0) - p^{\star}(\boldsymbol{\theta}_0)\right|^2 \leq \sum_{j:\boldsymbol{\theta}_j \in \mathcal{P}(\boldsymbol{\theta}_0)} \left|F(\boldsymbol{\theta}_j) - p^{\star}(\boldsymbol{\theta}_j)\right|^2.$$

Let $p^{\star\star} \in \mathbb{P}_D$ be a polynomial such that

$$\left\|F\left(\boldsymbol{\theta}\right)-p^{\star\star}\left(\boldsymbol{\theta}\right)\right\|_{\rho\mathsf{B}}=\inf_{p\in\mathbb{P}_d}\left\|F\left(\boldsymbol{\theta}\right)-p\left(\boldsymbol{\theta}\right)\right\|_{\rho\mathsf{B}}.$$

Then,

$$\begin{aligned}\left|F\left(\boldsymbol{\theta}_0\right)-p^{\star}\left(\boldsymbol{\theta}_0\right)\right|^2 &\leq \sum_{j:\boldsymbol{\theta}_j\in\mathcal{P}(\boldsymbol{\theta}_0)}\left|F\left(\boldsymbol{\theta}_j\right)-p^{\star}\left(\boldsymbol{\theta}_j\right)\right|^2\\ &\leq \sum_{j:\boldsymbol{\theta}_j\in\mathcal{P}(\boldsymbol{\theta}_0)}\left|F\left(\boldsymbol{\theta}_j\right)-p^{\star\star}\left(\boldsymbol{\theta}_j\right)\right|^2\\ &\leq P\left\|F\left(\boldsymbol{\theta}\right)-p^{\star\star}\left(\boldsymbol{\theta}\right)\right\|_{\rho\mathsf{B}}^2.\end{aligned}$$

Therefore,

$$\left|\tilde{F}\left(\boldsymbol{\theta}_0\right)-F\left(\boldsymbol{\theta}_0\right)\right|\leq P^{\frac{1}{2}}\left\|F\left(\boldsymbol{\theta}\right)-p^{\star\star}\left(\boldsymbol{\theta}\right)\right\|_{\rho\mathsf{B}}+h^{-D}\frac{\left\|\boldsymbol{\varepsilon}\right\|_2}{\sqrt{P\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}}.$$

If $\boldsymbol{\theta}_0\notin\mathcal{P}\left(\boldsymbol{\theta}_0\right)$, we can write, for any $\boldsymbol{\theta}_j\in\mathcal{P}\left(\boldsymbol{\theta}_0\right)$,

$$\left|F\left(\boldsymbol{\theta}_0\right)-p^{\star}\left(\boldsymbol{\theta}_0\right)\right|\leq\left|F\left(\boldsymbol{\theta}_j\right)-p^{\star}\left(\boldsymbol{\theta}_j\right)\right|+\left|F\left(\boldsymbol{\theta}_j\right)-F\left(\boldsymbol{\theta}_0\right)\right|+\left|p^{\star}\left(\boldsymbol{\theta}_j\right)-p^{\star}\left(\boldsymbol{\theta}_0\right)\right|.$$

Here, $\left|F\left(\boldsymbol{\theta}_j\right)-p^{\star}\left(\boldsymbol{\theta}_j\right)\right|$ can be majorized as we did above. For the other two terms, we have

$$\left|F\left(\boldsymbol{\theta}_j\right)-F\left(\boldsymbol{\theta}_0\right)\right|\leq|F|_{0,1}\left\|\boldsymbol{\theta}_j\right\|_2\leq|F|_{0,1}\,\widetilde{\rho}$$

and

$$\begin{aligned}\left|p^{\star}\left(\boldsymbol{\theta}_j\right)-p^{\star}\left(\boldsymbol{\theta}_0\right)\right| &= \left|\left[\mathbf{x}_D'\left(\boldsymbol{\theta}_j\right)-\mathbf{x}_D'\left(\boldsymbol{\theta}_0\right)\right]\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\overline{\mathbf{y}}\right|\\ &\leq\left\|\mathbf{x}_D\left(\boldsymbol{\theta}_j\right)-\mathbf{x}_D\left(\boldsymbol{\theta}_0\right)\right\|_2\left\|\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\right\|_2\left\|\overline{\mathbf{y}}\right\|_2\\ &\leq\frac{\left\|\mathbf{x}_D\left(\boldsymbol{\theta}_j\right)-\mathbf{x}_D\left(\boldsymbol{\theta}_0\right)\right\|_2\left\|\overline{\mathbf{y}}\right\|_2}{\sqrt{\lambda_{\min}\left(\mathbf{X}'\mathbf{X}\right)}}.\end{aligned}$$

From Lemma 11, we have

$$\begin{aligned}\left\|\mathbf{x}_D\left(\boldsymbol{\theta}_j\right)-\mathbf{x}_D\left(\boldsymbol{\theta}_0\right)\right\|_2^2=\left\|\mathbf{x}_D\left(\boldsymbol{\theta}_j\right)-\mathbf{e}_1\right\|_2^2 &= \left\|\mathbf{x}_D\left(\boldsymbol{\theta}_j\right)\right\|_2^2-1\\ &\leq\frac{1-\widetilde{\rho}^{2(D+1)}}{1-\widetilde{\rho}^2}-1=\widetilde{\rho}^2\left(\frac{1-\widetilde{\rho}^{2D}}{1-\widetilde{\rho}^2}\right).\end{aligned}$$

Moreover,

$$\left\|\overline{\mathbf{y}}\right\|_2^2\leq P\sup_{\boldsymbol{\theta}_j\in\mathcal{P}(\boldsymbol{\theta}_0)}\left|F\left(\boldsymbol{\theta}_j\right)\right|^2=P\left\|F\left(\boldsymbol{\theta}_j\right)\right\|_{\mathcal{P}(\boldsymbol{\theta}_0)}^2.$$

At last,

$$\left|p^{\star}\left(\boldsymbol{\theta}_j\right)-p^{\star}\left(\boldsymbol{\theta}_0\right)\right|^2\leq\widetilde{\rho}\left(\frac{1-\widetilde{\rho}^{2D}}{1-\widetilde{\rho}^2}\right)^{\frac{1}{2}}\frac{\|F\|_{\mathcal{P}(\boldsymbol{\theta}_0)}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}'\mathbf{X}\right)}}\leq h^{-D}\widetilde{\rho}\left(\frac{1-\widetilde{\rho}^{2D}}{1-\widetilde{\rho}^2}\right)^{\frac{1}{2}}\frac{\|F\|_{\mathcal{P}(\boldsymbol{\theta}_0)}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}}$$

and

$$\left|\tilde{F}(\boldsymbol{\theta}_0) - F(\boldsymbol{\theta}_0)\right| \leq P^{\frac{1}{2}} \left\|F(\boldsymbol{\theta}) - p^{\star\star}(\boldsymbol{\theta})\right\|_{\rho\mathsf{B}} + |F|_{0,1}\,\widetilde{\rho}$$
$$+ h^{-D}\widetilde{\rho}\left(\frac{1-\widetilde{\rho}^{2D}}{1-\widetilde{\rho}^{2}}\right)^{\frac{1}{2}} \frac{\|F\|_{\mathcal{P}(\boldsymbol{\theta}_0)}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}} + h^{-D}P^{-\frac{1}{2}}\frac{\|\boldsymbol{\varepsilon}\|_2}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}}.$$

Now, we want to majorize $\sup_{\boldsymbol{\theta}\in\rho\mathsf{B}}\left|F(\boldsymbol{\theta}) - p^{\star\star}(\boldsymbol{\theta})\right|$. Let $p^{\star\star\star}$ be the Taylor expansion of order d of $F(\boldsymbol{\theta})$ around $\boldsymbol{\theta}_0 \equiv \mathbf{0}$. Then,

$$\left\|F(\boldsymbol{\theta}) - p^{\star\star}(\boldsymbol{\theta})\right\|_{\rho\mathsf{B}} \leq \left\|F(\boldsymbol{\theta}) - p^{\star\star\star}(\boldsymbol{\theta})\right\|_{\rho\mathsf{B}}.$$

We use Lemma 1 with $\gamma = 1$:

$$|R_0(\mathbf{0};\boldsymbol{\theta})| \leq \frac{K^D}{(D-1)!}\gamma^D \|\boldsymbol{\theta}\|_2^{D+1} |F|_{D,1} \leq \frac{K^D}{(D-1)!}\rho^{D+1}|F|_{D,1}$$

and, through $\rho = h\rho_0$, we get the final result. QED

Proof of Theorem 9. We build a predicted value of $F(\boldsymbol{\theta})$ as

$$\tilde{F}(\boldsymbol{\theta}) = \mathbf{x}_D'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \mathbf{x}_D'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\overline{\mathbf{y}}+\boldsymbol{\varepsilon}).$$

The decomposition and the interpretation of the other quantities are similar to those in Theorem 8. We have

$$\begin{aligned}\left|D^k\tilde{F}(\boldsymbol{\theta}) - D^kF(\boldsymbol{\theta})\right| &\leq \left|D^k\mathbf{x}_D'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{y}} - D^kF(\boldsymbol{\theta})\right| \\ &\quad + \left|D^k\mathbf{x}_D'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\varepsilon}\right| \\ &\leq \left|D^k\mathbf{x}_D'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{y}} - D^k\mathbf{x}_D'(\boldsymbol{\theta})\boldsymbol{\beta}\right| \\ &\quad + \left|D^k\mathbf{x}_D'(\boldsymbol{\theta})\boldsymbol{\beta} - D^kF(\boldsymbol{\theta})\right| \\ &\quad + \left|D^k\mathbf{x}_D'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\varepsilon}\right|\end{aligned}$$

where $\boldsymbol{\beta}$ will be chosen below. From this, we get

$$\begin{aligned}&\max_{|k|=S}\left\|D^k\tilde{F}(\boldsymbol{\theta}) - D^kF(\boldsymbol{\theta})\right\|_{\rho\mathsf{B}} \\ &\leq \max_{|k|=S}\left\|D^k\mathbf{x}_D'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{y}} - D^k\mathbf{x}_D'(\boldsymbol{\theta})\boldsymbol{\beta}\right\|_{\rho\mathsf{B}} \\ &\quad + \max_{|k|=S}\left\|D^k\mathbf{x}_D'(\boldsymbol{\theta})\boldsymbol{\beta} - D^kF(\boldsymbol{\theta})\right\|_{\rho\mathsf{B}} \\ &\quad + \max_{|k|=S}\left\|D^k\mathbf{x}_D'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\varepsilon}\right\|_{\rho\mathsf{B}}.\end{aligned} \tag{8.29}$$

Let us start from the last term in (8.29). By Cauchy–Schwarz inequality, we have

$$\begin{aligned}\left|D^k\mathbf{x}_D'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\varepsilon}\right| &\leq \left\|D^k\mathbf{x}_D(\boldsymbol{\theta})\right\|_2\left\|(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\right\|_2\|\boldsymbol{\varepsilon}\|_2 \\ &\leq \left\|D^k\mathbf{x}_D(\boldsymbol{\theta})\right\|_2\|\boldsymbol{\varepsilon}\|_2\sqrt{\lambda_{\max}\left((\mathbf{X}'\mathbf{X})^{-1}\right)} \\ &\leq \frac{\left\|D^k\mathbf{x}_D(\boldsymbol{\theta})\right\|_2\|\boldsymbol{\varepsilon}\|_2}{\sqrt{\lambda_{\min}(\mathbf{X}'\mathbf{X})}}.\end{aligned}$$

Therefore, using Lemma 10,

$$\left|D^{k}\mathbf{x}_{D}'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\varepsilon}\right|\leq h^{-D}\frac{P^{-\frac{1}{2}}\left\|D^{k}\mathbf{x}_{D}(\boldsymbol{\theta})\right\|_{2}\|\boldsymbol{\varepsilon}\|_{2}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_{0}'\mathbf{X}_{0}\right)}}.$$

From Lemma 11,

$$\begin{aligned}\max_{|k|=S}\max_{\boldsymbol{\theta}\in\rho\mathsf{B}}\left\|D^{k}\mathbf{x}_{D}'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\varepsilon}\right\|_{2}&\leq h^{-D}\frac{P^{-\frac{1}{2}}\max_{\boldsymbol{\theta}\in\rho\mathsf{B}}\|\boldsymbol{\varepsilon}\|_{2}\max_{|k|=S}\max_{\boldsymbol{\theta}\in\rho\mathsf{B}}\left\|D^{k}\mathbf{x}_{D}(\boldsymbol{\theta})\right\|_{2}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_{0}'\mathbf{X}_{0}\right)}}\\&\leq h^{-D}\frac{P^{-\frac{1}{2}}\max_{\boldsymbol{\theta}\in\rho\mathsf{B}}\|\boldsymbol{\varepsilon}\|_{2}\,S!\sqrt{\sum_{d=S}^{D}\binom{d}{S}^{2}\rho^{2(d-S)}}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_{0}'\mathbf{X}_{0}\right)}}.\end{aligned}$$

Now we turn to the second term of (8.29). Let $\boldsymbol{\beta}$ be the vector of coefficients of the Taylor expansion of order D of $F(\boldsymbol{\theta})$ around $\mathbf{0}$. From Lemma 1,

$$|R_{k}(\mathbf{0};\boldsymbol{\theta})|\leq\frac{K^{D-|k|}}{(D-|k|-1)!}\gamma^{D-|k|}\|\boldsymbol{\theta}\|_{2}^{D-|k|+1}|F|_{D,1}\leq\frac{K^{D-|k|}}{(D-|k|-1)!}\rho^{D-|k|+1}|F|_{D,1}\tag{8.30}$$

and the second term in (8.29) becomes

$$\max_{|k|=S}\left\|D^{k}\mathbf{x}_{D}'(\boldsymbol{\theta})\boldsymbol{\beta}-D^{k}F(\boldsymbol{\theta})\right\|_{\rho\mathsf{B}}\leq\max_{|k|=S}|R_{k}(\mathbf{0};\boldsymbol{\theta})|\leq\frac{K^{D-S}}{(D-S-1)!}\rho^{D-S+1}|F|_{D,1}.$$

We are left with the first term in (8.29), i.e.

$$\begin{aligned}&\max_{|k|=S}\left\|D^{k}\mathbf{x}_{D}'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{y}}-D^{k}\mathbf{x}_{D}'(\boldsymbol{\theta})\boldsymbol{\beta}\right\|_{\rho\mathsf{B}}\\&=\max_{|k|=S}\left\|D^{k}\mathbf{x}_{D}'(\boldsymbol{\theta})\left((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{y}}-\boldsymbol{\beta}\right)\right\|_{\rho\mathsf{B}}.\end{aligned}$$

It is clear that $\mathbf{x}_{D}'(\boldsymbol{\theta})\left((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{y}}-\boldsymbol{\beta}\right)$ is a polynomial of order D in $\boldsymbol{\theta}$. In order to majorize

$$\left\|D^{k}\mathbf{x}_{D}'(\boldsymbol{\theta})\left((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{y}}-\boldsymbol{\beta}\right)\right\|_{\rho\mathsf{B}}$$

we use $|k|$ times a version of Markov brothers' inequality. We recall that, for a multidimensional polynomial p of order n in d variables, the generalization of Markov brothers' inequality in Kellogg (1928, Theorem VI) states that

$$\sup_{\mathbf{x}\in\mathsf{B}}\sqrt{\sum_{j=1}^{d}\left(\frac{\partial p(\mathbf{x})}{\partial x_{j}}\right)^{2}}\leq n^{2}\sup_{\mathbf{x}\in\mathsf{B}}|p(\mathbf{x})|.$$

Therefore,

$$\left\|D^{k}\mathbf{x}_{D}'(\boldsymbol{\theta})\left((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{y}}-\boldsymbol{\beta}\right)\right\|_{\rho\mathsf{B}}\leq\left(\frac{D!}{(D-|k|)!}\right)^{2}\frac{1}{\rho^{|k|}}\left\|\mathbf{x}_{D}'(\boldsymbol{\theta})\left((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{y}}-\boldsymbol{\beta}\right)\right\|_{\rho\mathsf{B}}.$$

We can write the term in the right-hand side of this equation as

$$\left|\mathbf{x}_D'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{y}} - \mathbf{x}_D'(\boldsymbol{\theta})\boldsymbol{\beta}\right| \leq \left|\mathbf{x}_D'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{y}} - F(\boldsymbol{\theta})\right| + \left|F(\boldsymbol{\theta}) - \mathbf{x}_D'(\boldsymbol{\theta})\boldsymbol{\beta}\right|.$$

The last term can be bounded as in (8.30):

$$\left\|\mathbf{x}_D'(\boldsymbol{\theta})\boldsymbol{\beta} - F(\boldsymbol{\theta})\right\|_{\rho\mathsf{B}} \leq |R_0(\mathbf{0};\boldsymbol{\theta})| \leq \frac{K^D}{(D-1)!}\rho^{D+1}|F|_{D,1}.$$

We only need to bound $\sup_{\boldsymbol{\theta}\in\rho\mathsf{B}}\left|\mathbf{x}_D'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{y}} - F(\boldsymbol{\theta})\right|$. Using Theorem 2 in Calvi and Levenberg (2008),

$$\begin{aligned}&\left\|\left|\mathbf{x}_D'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{y}} - F(\boldsymbol{\theta})\right|\right\|_{\rho\mathsf{B}}\\ &\leq \left(1 + C_D(\mathcal{P}(\boldsymbol{\theta}_0),\rho\mathsf{B})\left(1+P^{\frac{1}{2}}\right)\right)\inf_{p\in\mathbb{P}_D}\|F(\boldsymbol{\theta}) - p(\boldsymbol{\theta})\|_{\rho\mathsf{B}}.\end{aligned}$$

Here $C_D(\mathcal{P}(\boldsymbol{\theta}_0),\rho\mathsf{B})$ is the constant defined as

$$C_D(\mathcal{P}(\boldsymbol{\theta}_0),\rho\mathsf{B}) := \sup_{p\in\mathbb{P}_D}\frac{\|p(\boldsymbol{\theta})\|_{\rho\mathsf{B}}}{\|p(\boldsymbol{\theta})\|_{\mathcal{P}(\boldsymbol{\theta}_0)}}.$$

We show that this constant can be written in a different way. Indeed, we know that $\rho = h\rho_0$ and that $\mathcal{P}(\boldsymbol{\theta}_0) = h\mathcal{P}_0(\boldsymbol{\theta}_0)$. Then,

$$\begin{aligned}\sup_{p\in\mathbb{P}_D}\frac{\|p(\boldsymbol{\theta})\|_{\rho\mathsf{B}}}{\|p(\boldsymbol{\theta})\|_{\mathcal{P}(\boldsymbol{\theta}_0)}} &= \sup_{p\in\mathbb{P}_D}\frac{\sup_{h^{-1}\boldsymbol{\theta}\in\rho_0\mathsf{B}}|p(\boldsymbol{\theta})|}{\sup_{h^{-1}\boldsymbol{\theta}\in\mathcal{P}_0(\boldsymbol{\theta}_0)}|p(\boldsymbol{\theta})|}\\ &= \sup_{p\in\mathbb{P}_D}\frac{\sup_{\boldsymbol{\theta}\in\rho_0\mathsf{B}}|p(h\boldsymbol{\theta})|}{\sup_{\boldsymbol{\theta}\in\mathcal{P}_0(\boldsymbol{\theta}_0)}|p(h\boldsymbol{\theta})|}\\ &= \sup_{p\in\mathbb{P}_D}\frac{\|p(\boldsymbol{\theta})\|_{\rho_0\mathsf{B}}}{\|p(\boldsymbol{\theta})\|_{\mathcal{P}_0(\mathbf{0})}}.\end{aligned}$$

Using the definition in the text, we write $C_D(\mathcal{P}_0)$ instead of $C_D(\mathcal{P}(\boldsymbol{\theta}_0),\rho\mathsf{B})$. As to $\inf_{p\in\mathbb{P}_D}\sup_{\boldsymbol{\theta}\in\rho\mathsf{B}}|F(\boldsymbol{\theta}) - p(\boldsymbol{\theta})|$, let $p^\star$ be the polynomial for which

$$\inf_{p\in\mathbb{P}_D}\|F(\boldsymbol{\theta}) - p(\boldsymbol{\theta})\|_{\rho\mathsf{B}} = \|F(\boldsymbol{\theta}) - p^\star(\boldsymbol{\theta})\|_{\rho\mathsf{B}}.$$

If we replace $p^\star$ with another polynomial in $\mathbb{P}_D$, the result will be a majorization of this term. We can use the Taylor expansion of order D of $F(\boldsymbol{\theta})$ around $\mathbf{0}$:

$$\|F(\boldsymbol{\theta}) - p^\star(\boldsymbol{\theta})\|_{\rho\mathsf{B}} \leq |R_0(\mathbf{0};\boldsymbol{\theta})| \leq \frac{K^D}{(D-1)!}\rho^{D+1}|F|_{D,1}.$$

As a result:

$$\max_{|k|=S}\left\|D^k\tilde{F}(\boldsymbol{\theta}) - D^kF(\boldsymbol{\theta})\right\|_{\rho\mathsf{B}}$$

$$\leq h^{-D}\frac{P^{-\frac{1}{2}}\max_{\boldsymbol{\theta}\in\rho\mathsf{B}}\|\boldsymbol{\varepsilon}\|_2 S!\sqrt{\sum_{d=S}^{D}\binom{d}{S}^2\rho^{2(d-S)}}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}}$$
$$+\left[\frac{\left(2+C_D(\mathcal{P}_0)\left(1+P^{\frac{1}{2}}\right)\right)(D!)^2K^S}{((D-S)!)^2(D-1)!}+\frac{1}{(D-S-1)!}\right]K^{D-S}\rho^{D-S+1}|F|_{D,1}.$$

The result follows linking ρ and h, through $\rho=h\rho_0$. QED

Proof of Corollary 3. We want to majorize $\sup_{\boldsymbol{\theta}\in\rho\mathsf{B}}\left\|\dot{\tilde{F}}(\boldsymbol{\theta})-\dot{F}(\boldsymbol{\theta})\right\|_2$ for $D=1$ or $D=2$ and $\sup_{\boldsymbol{\theta}\in\rho\mathsf{B}}\left\|\ddot{\tilde{F}}(\boldsymbol{\theta})-\ddot{F}(\boldsymbol{\theta})\right\|_2$ for $D=2$. Using the triangle inequality, for $S=1$ we have

$$\begin{aligned}&\left\|\left[D^i\tilde{F}(\boldsymbol{\theta})-D^iF(\boldsymbol{\theta})\right]\right\|_2\\ &\leq\left\|\left[D^i\mathbf{x}_D'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{y}}-D^i\mathbf{x}_D'(\boldsymbol{\theta})\boldsymbol{\beta}\right]\right\|_2\\ &\quad+\left\|\left[D^i\mathbf{x}_D'(\boldsymbol{\theta})\boldsymbol{\beta}-D^iF(\boldsymbol{\theta})\right]\right\|_2\\ &\quad+\left\|\left[D^i\mathbf{x}_D'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\varepsilon}\right]\right\|_2\end{aligned}\tag{8.31}$$

where $\boldsymbol{\beta}$ will be chosen below. For $S=2$, the inequality is the same, with D^{ij} replacing D^i. In the following, we will use D^k instead of either D^i or D^{ij}.

For the last term in (8.31), we have

$$\begin{aligned}\left\|\left[D^k\mathbf{x}_D'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\varepsilon}\right]\right\|_2&\leq\left\|\left[D^k\mathbf{x}_D'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\varepsilon}\right]\right\|_F\\ &=\left(\sum_{|k|=S}\left|D^k\mathbf{x}_D'(\boldsymbol{\theta})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\varepsilon}\right|^2\right)^{\frac{1}{2}}\\ &\leq\left(\sum_{|k|=S}\left\|D^k\mathbf{x}_D(\boldsymbol{\theta})\right\|_2^2\left\|(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\right\|_2^2\|\boldsymbol{\varepsilon}\|_2^2\right)^{\frac{1}{2}}\\ &\leq K^{\frac{S}{2}}\max_{|k|=S}\left\|D^k\mathbf{x}_D(\boldsymbol{\theta})\right\|_2\left\|(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\right\|_2\|\boldsymbol{\varepsilon}\|_2\\ &\leq\frac{K^{\frac{S}{2}}S!\sqrt{\sum_{d=S}^{D}\binom{d}{S}^2\rho^{2(d-S)}}\|\boldsymbol{\varepsilon}\|_2}{\sqrt{\lambda_{\min}(\mathbf{X}'\mathbf{X})}}=\frac{K^{\frac{S}{2}}S!\sqrt{\sum_{d=S}^{D}\binom{d}{S}^2\rho^{2(d-S)}}\|\boldsymbol{\varepsilon}\|_2}{P^{\frac{1}{2}}h^D\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}}.\end{aligned}$$

In the second term of (8.31), let $\boldsymbol{\beta}$ be the vector of coefficients of the Taylor expansion of order D of $F(\boldsymbol{\theta})$ around $\mathbf{0}$. Then,

$$\left\|\left[D^k\mathbf{x}_D'(\boldsymbol{\theta})\boldsymbol{\beta}-D^kF(\boldsymbol{\theta})\right]\right\|_2=\begin{cases}\left\|\dot{F}(\boldsymbol{\theta})-\dot{F}(\mathbf{0})\right\|_2\leq L_1\|\boldsymbol{\theta}\|_2, & S=1,D=1,\\ \left\|\dot{F}(\boldsymbol{\theta})-\dot{F}(\mathbf{0})-\ddot{F}(\mathbf{0})\boldsymbol{\theta}\right\|_2\leq\frac{L_2}{2}\|\boldsymbol{\theta}\|_2^2, & S=1,D=2,\\ \left\|\ddot{F}(\boldsymbol{\theta})-\ddot{F}(\mathbf{0})\right\|_2\leq L_2\|\boldsymbol{\theta}\|_2, & S=2,D=2.\end{cases}$$

The first one comes from **Lip-**1, the second one from the third inequality in Lemma 2, the third one from **Lip-**2.

The first term in (8.31) is a polynomial. In the case $S=1$, it can be majorized through the generalization of Markov brothers' inequality in Kellogg (1928, Theorem VI) seen above:

$$\begin{aligned}
&\sup_{\boldsymbol{\theta}\in\rho\mathsf{B}}\left(\sum_{|k|=1}\left|D^{k}\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\overline{\mathbf{y}}-D^{k}\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\boldsymbol{\beta}\right|^{2}\right)^{\frac{1}{2}}\\
&\leq\frac{D^{2}}{\rho}\sup_{\boldsymbol{\theta}\in\rho\mathsf{B}}\left|\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\overline{\mathbf{y}}-\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\boldsymbol{\beta}\right|.
\end{aligned}$$

In the case $S=2$ and $D=2$, the same inequality yields

$$\begin{aligned}
&\sup_{\boldsymbol{\theta}\in\rho\mathsf{B}}\left(\sum_{i}\sum_{j}\left|D^{ij}\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\overline{\mathbf{y}}-D^{ij}\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\boldsymbol{\beta}\right|^{2}\right)^{\frac{1}{2}}\\
&\leq\left(\sum_{i}\left\{\sup_{\boldsymbol{\theta}\in\rho\mathsf{B}}\left[\sum_{j}\left|D^{ij}\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\overline{\mathbf{y}}-D^{ij}\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\boldsymbol{\beta}\right|^{2}\right]^{\frac{1}{2}}\right\}^{2}\right)^{\frac{1}{2}}\\
&\leq\left(\sum_{i}\left\{\frac{\left(D-1\right)^{2}}{\rho}\sup_{\boldsymbol{\theta}\in\rho\mathsf{B}}\left|D^{i}\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\overline{\mathbf{y}}-D^{i}\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\boldsymbol{\beta}\right|\right\}^{2}\right)^{\frac{1}{2}}\\
&\leq\frac{K^{\frac{1}{2}}\left(D-1\right)^{2}}{\rho^{2}}\sup_{i}\sup_{\boldsymbol{\theta}\in\rho\mathsf{B}}\left|D^{i}\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\overline{\mathbf{y}}-D^{i}\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\boldsymbol{\beta}\right|\\
&\leq\frac{K^{\frac{1}{2}}\left(D-1\right)^{2}D^{2}}{\rho^{2}}\sup_{\boldsymbol{\theta}\in\rho\mathsf{B}}\left|\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\overline{\mathbf{y}}-\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\boldsymbol{\beta}\right|.
\end{aligned}$$

We can write the term in the right-hand side of both equations as

$$\left|\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\overline{\mathbf{y}}-\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\boldsymbol{\beta}\right|\leq\left|\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\overline{\mathbf{y}}-F\left(\boldsymbol{\theta}\right)\right|+\left|F\left(\boldsymbol{\theta}\right)-\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\boldsymbol{\beta}\right|.$$

As above,

$$\begin{aligned}
&\left\|\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\overline{\mathbf{y}}-F\left(\boldsymbol{\theta}\right)\right\|_{\rho\mathsf{B}}\\
&\leq\left(1+C_{D}\left(\mathcal{P}_{0}\right)\left(1+P^{\frac{1}{2}}\right)\right)\inf_{p\in\mathbb{P}_{D}}\left\|F\left(\boldsymbol{\theta}\right)-p\left(\boldsymbol{\theta}\right)\right\|_{\rho\mathsf{B}}\\
&\leq\left(1+C_{D}\left(\mathcal{P}_{0}\right)\left(1+P^{\frac{1}{2}}\right)\right)\left\|F\left(\boldsymbol{\theta}\right)-p^{\star}\left(\boldsymbol{\theta}\right)\right\|_{\rho\mathsf{B}}
\end{aligned}$$

where $p^{\star}$ is the Taylor expansion of order D of $F\left(\boldsymbol{\theta}\right)$ around $\mathbf{0}$. Therefore,

$$\left|F\left(\boldsymbol{\theta}\right)-p^{\star}\left(\boldsymbol{\theta}\right)\right|=\begin{cases}\left|F\left(\boldsymbol{\theta}\right)-F\left(\mathbf{0}\right)-\boldsymbol{\theta}'\dot{F}\left(\mathbf{0}\right)\right|\leq\frac{L_{1}}{2}\left\|\boldsymbol{\theta}\right\|_{2}^{2}, & D=1,\\ \left|F\left(\boldsymbol{\theta}\right)-F\left(\mathbf{0}\right)-\boldsymbol{\theta}'\dot{F}\left(\mathbf{0}\right)-\frac{\boldsymbol{\theta}'\ddot{F}\left(\mathbf{0}\right)\boldsymbol{\theta}}{2}\right|\leq\frac{L_{2}}{6}\left\|\boldsymbol{\theta}\right\|_{2}^{3}, & D=2.\end{cases}$$

The statement follows by collecting the terms. QED

Proof of Theorem 10. The proof of the first result follows the one of Theorem 9. Indeed, we can write

$$\mathbb{E}D^{k}\tilde{F}\left(\boldsymbol{\theta}\right)-D^{k}F\left(\boldsymbol{\theta}\right)=D^{k}\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\overline{\mathbf{y}}-D^{k}F\left(\boldsymbol{\theta}\right)+D^{k}\mathbf{x}_{D}'\left(\boldsymbol{\theta}\right)\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbb{E}\boldsymbol{\varepsilon},$$

so that it is apparent that the result is the same as in Theorem 9, with $\max_{|k|=S}\left\|\mathbb{E}D^k\tilde{F}(\boldsymbol{\theta})-D^kF(\boldsymbol{\theta})\right\|_{\rho\mathsf{B}}$ replacing $\max_{|k|=S}\left\|D^k\tilde{F}(\boldsymbol{\theta})-D^kF(\boldsymbol{\theta})\right\|_{\rho\mathsf{B}}$ and $\mathbb{E}\boldsymbol{\varepsilon}$ replacing $\boldsymbol{\varepsilon}$.

Now we turn to the second result. As

$$D^k\tilde{F}(\boldsymbol{\theta})-\mathbb{E}D^k\tilde{F}(\boldsymbol{\theta})=D^k\mathbf{x}_D'(\boldsymbol{\theta})\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\left(\boldsymbol{\varepsilon}-\mathbb{E}\boldsymbol{\varepsilon}\right),$$

from Lemma 11, we have

$$\begin{aligned}
&\max_{|k|=S}\max_{\boldsymbol{\theta}\in\rho\mathsf{B}}\mathbb{E}\left|D^k\tilde{F}(\boldsymbol{\theta})-D^kF(\boldsymbol{\theta})-\mathbb{E}\left(D^k\tilde{F}(\boldsymbol{\theta})-D^kF(\boldsymbol{\theta})\right)\right|^2\\
&=\max_{|k|=S}\max_{\boldsymbol{\theta}\in\rho\mathsf{B}}\mathbb{E}\left|D^k\tilde{F}(\boldsymbol{\theta})-\mathbb{E}D^k\tilde{F}(\boldsymbol{\theta})\right|^2=\max_{|k|=S}\max_{\boldsymbol{\theta}\in\rho\mathsf{B}}\mathbb{E}\left|D^k\mathbf{x}_D'(\boldsymbol{\theta})\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\left(\boldsymbol{\varepsilon}-\mathbb{E}\boldsymbol{\varepsilon}\right)\right|^2\\
&\leq\max_{|k|=S}\max_{\boldsymbol{\theta}\in\rho\mathsf{B}}\left\|D^k\mathbf{x}_D(\boldsymbol{\theta})\right\|_2^2\left\|\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\right\|_2^2\mathbb{E}\left\|\boldsymbol{\varepsilon}-\mathbb{E}\boldsymbol{\varepsilon}\right\|_2^2\\
&\leq(S!)^2\left(\sum_{d=S}^D\binom{d}{S}^2\rho^{2(d-S)}\right)\left\|\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\right\|_2^2\max_{\boldsymbol{\theta}\in\rho\mathsf{B}}\mathbb{E}\left\|\boldsymbol{\varepsilon}-\mathbb{E}\boldsymbol{\varepsilon}\right\|_2^2\\
&\leq\frac{h^{-2D}(S!)^2}{\lambda_{\min}\left(\mathbf{X}_0'\mathbf{X}_0\right)}\left(\sum_{d=S}^D\binom{d}{S}^2\rho^{2(d-S)}\right)\max_{\boldsymbol{\theta}\in\rho\mathsf{B}}\mathbb{E}\left\|\boldsymbol{\varepsilon}-\mathbb{E}\boldsymbol{\varepsilon}\right\|_2^2.
\end{aligned}$$

QED

8.4 Proofs of Results Specific to the Approximating Algorithm

Proof of Corollary 4. Using **AUB**,

$$\|\boldsymbol{\varepsilon}\|_2=\sqrt{\sum_{j=1}^P\varepsilon_j^2}=\sqrt{\sum_{j=1}^P\left(\hat{F}(\boldsymbol{\theta}_j)-F(\boldsymbol{\theta}_j)\right)^2}\leq P^{\frac{1}{2}}a_N.$$

From Theorem 9, under **Fun-**D, one gets

$$\begin{aligned}
\left(\delta_1^{(i)}\right)^2&=\sum_{|k|=1}\left|D^k\tilde{F}\left(\boldsymbol{\theta}^{(i)}\right)-D^kF\left(\boldsymbol{\theta}^{(i)}\right)\right|^2\\
&\leq K\max_{|k|=1}\left|D^k\tilde{F}\left(\boldsymbol{\theta}^{(i)}\right)-D^kF\left(\boldsymbol{\theta}^{(i)}\right)\right|^2,\\
\delta_1^{(i)}&\leq K^{\frac{1}{2}}\max_{|k|=1}\left|D^k\tilde{F}\left(\boldsymbol{\theta}^{(i)}\right)-D^kF\left(\boldsymbol{\theta}^{(i)}\right)\right|\\
&\leq\rho^{-D}\frac{\rho_0^DK^{\frac{1}{2}}\sqrt{\sum_{d=1}^Dd^2\rho^{2(d-1)}}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}}a_N\\
&\quad+\left[\frac{\left(2+C_D(\mathcal{P}_0)\left(1+P^{\frac{1}{2}}\right)\right)D^2K}{D-1}+1\right]\frac{K^{D-\frac{1}{2}}\rho^D|F|_{D,1}}{(D-2)!}
\end{aligned}$$

and

$$\left(\delta_2^{(i)}\right)^2=\left\|\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)-\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2\leq\left\|\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)-\ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_F^2$$

$$
\begin{aligned}
&= \sum_{|k|=2} \left| D^k \tilde{F}\left(\boldsymbol{\theta}^{(i)}\right) - D^k F\left(\boldsymbol{\theta}^{(i)}\right) \right|^2 \\
&\le K^2 \max_{|k|=2} \left| D^k \tilde{F}\left(\boldsymbol{\theta}^{(i)}\right) - D^k F\left(\boldsymbol{\theta}^{(i)}\right) \right|^2, \\
\delta_2^{(i)} &\le K \max_{|k|=2} \left| D^k \tilde{F}\left(\boldsymbol{\theta}^{(i)}\right) - D^k F\left(\boldsymbol{\theta}^{(i)}\right) \right| \\
&\le \rho^{-D} \frac{2\rho_0^D K \sqrt{\sum_{d=2}^{D} \binom{d}{2}^2 \rho^{2(d-2)}}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}} a_N \\
&\quad + \left[\frac{\left(2 + C_D\left(\mathcal{P}_0\right)\left(1 + P^{\frac{1}{2}}\right)\right) D^2 (D-1) K^2}{D-2} + 1 \right] \frac{K^{D-1}\rho^{D-1} |F|_{D,1}}{(D-3)!}.
\end{aligned}
$$

Under **MaV2**, we have

$$
\begin{aligned}
\left\|\mathbb{E}\boldsymbol{\varepsilon}\right\|_2 &= \sqrt{\sum_{j=1}^{P} \left(\mathbb{E}\hat{F}\left(\boldsymbol{\theta}_j\right) - F\left(\boldsymbol{\theta}_j\right)\right)^2} \\
&\le \sqrt{\sum_{j=1}^{P} \left(\mathbb{E}\left|\mathbb{E}\left(\hat{F}\left(\boldsymbol{\theta}_j\right) | \mathcal{F}_i\right) - F\left(\boldsymbol{\theta}_j\right)\right|\right)^2} \\
&\le P^{\frac{1}{2}} B_i
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\left\|\boldsymbol{\varepsilon} - \mathbb{E}\boldsymbol{\varepsilon}\right\|_2^2 &= \sum_{j=1}^{P} \mathbb{E}\left(\varepsilon_j - \mathbb{E}\varepsilon_j\right)^2 \\
&\le \sum_{j=1}^{P} \mathbb{E}\varepsilon_j^2 \le \sum_{j=1}^{P} \mathbb{E}\mathbb{E}\left(\left(\hat{F}\left(\boldsymbol{\theta}_j\right) - F\left(\boldsymbol{\theta}_j\right)\right)^2 | \mathcal{F}_i\right) \le P\Sigma_i.
\end{aligned}
$$

Now we turn to $\mathbb{E}\left(\delta_1^{(i)}\right)^2$ and $\mathbb{E}\left(\delta_2^{(i)}\right)^2$. We note that, using the triangle inequality,

$$
\begin{aligned}
\mathbb{E}\left(\delta_1^{(i)}\right)^2 &= \mathbb{E}\left\|\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right) - \dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2 = \sum_{|k|=1} \mathbb{E}\left|D^k \tilde{F}\left(\boldsymbol{\theta}^{(i)}\right) - D^k F\left(\boldsymbol{\theta}^{(i)}\right)\right|^2 \\
&\le K \max_{|k|=1} \mathbb{E}\left|D^k \tilde{F}\left(\boldsymbol{\theta}^{(i)}\right) - D^k F\left(\boldsymbol{\theta}^{(i)}\right)\right|^2 \\
&\le K \left\{ \max_{|k|=1} \mathbb{E}\left|D^k \tilde{F}\left(\boldsymbol{\theta}^{(i)}\right) - D^k F\left(\boldsymbol{\theta}^{(i)}\right) - \mathbb{E}\left[D^k \tilde{F}\left(\boldsymbol{\theta}^{(i)}\right) - D^k F\left(\boldsymbol{\theta}^{(i)}\right) | \mathcal{F}_i\right]\right| \right. \\
&\quad \left. + \max_{|k|=1} \mathbb{E}\left|\mathbb{E}\left[D^k \tilde{F}\left(\boldsymbol{\theta}^{(i)}\right) | \mathcal{F}_i\right] - D^k F\left(\boldsymbol{\theta}^{(i)}\right)\right| \right\}^2 \\
&\le K \left\{ \left(\max_{|k|=1} \mathbb{E}\left|D^k \tilde{F}\left(\boldsymbol{\theta}^{(i)}\right) - D^k F\left(\boldsymbol{\theta}^{(i)}\right) - \mathbb{E}\left[D^k \tilde{F}\left(\boldsymbol{\theta}^{(i)}\right) - D^k F\left(\boldsymbol{\theta}^{(i)}\right) | \mathcal{F}_i\right]\right|^2 \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \max_{|k|=1} \mathbb{E}\left|\mathbb{E}\left[D^k \tilde{F}\left(\boldsymbol{\theta}^{(i)}\right) | \mathcal{F}_i\right] - D^k F\left(\boldsymbol{\theta}^{(i)}\right)\right| \right\}^2
\end{aligned}
$$

and

$$\begin{aligned}
\mathbb{E}\left(\delta_2^{(i)}\right)^2 &= \mathbb{E}\left\|\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right) - \ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2 \leq \mathbb{E}\left\|\ddot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right) - \ddot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_F^2 \\
&= \sum_{|k|=2} \mathbb{E}\left|D^k\tilde{F}\left(\boldsymbol{\theta}^{(i)}\right) - D^kF\left(\boldsymbol{\theta}^{(i)}\right)\right|^2 \\
&\leq K^2 \max_{|k|=2} \mathbb{E}\left|D^k\tilde{F}\left(\boldsymbol{\theta}^{(i)}\right) - D^kF\left(\boldsymbol{\theta}^{(i)}\right)\right|^2 \\
&\leq K^2 \left\{\max_{|k|=2} \mathbb{E}\left|D^k\tilde{F}\left(\boldsymbol{\theta}^{(i)}\right) - D^kF\left(\boldsymbol{\theta}^{(i)}\right) - \mathbb{E}\left[D^k\tilde{F}\left(\boldsymbol{\theta}^{(i)}\right) - D^kF\left(\boldsymbol{\theta}^{(i)}\right)|\mathcal{F}_i\right]\right|\right. \\
&\quad \left. + \max_{|k|=2} \mathbb{E}\left|\mathbb{E}\left[D^k\tilde{F}\left(\boldsymbol{\theta}^{(i)}\right)|\mathcal{F}_i\right] - D^kF\left(\boldsymbol{\theta}^{(i)}\right)\right|\right\}^2 \\
&\leq K^2 \left\{\left(\max_{|k|=2} \mathbb{E}\left|D^k\tilde{F}\left(\boldsymbol{\theta}^{(i)}\right) - D^kF\left(\boldsymbol{\theta}^{(i)}\right) - \mathbb{E}\left[D^k\tilde{F}\left(\boldsymbol{\theta}^{(i)}\right) - D^kF\left(\boldsymbol{\theta}^{(i)}\right)|\mathcal{F}_i\right]\right|^2\right)^{\frac{1}{2}}\right. \\
&\quad \left. + \max_{|k|=2} \mathbb{E}\left|\mathbb{E}\left[D^k\tilde{F}\left(\boldsymbol{\theta}^{(i)}\right)|\mathcal{F}_i\right] - D^kF\left(\boldsymbol{\theta}^{(i)}\right)\right|\right\}^2.
\end{aligned}$$

The final formulas are easily obtained through Theorem 10.

Now we consider b_i and σ_i as defined in Assumption **MaV**. From the inequality

$$\begin{aligned}
&\left\|\mathbb{E}\left[\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right)|\mathcal{F}_i\right] - \dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2 \\
&= \left\{\sum_{|k|=1} \left|\mathbb{E}\left[D^k\tilde{F}(\boldsymbol{\theta})|\mathcal{F}_i\right] - D^kF(\boldsymbol{\theta})\right|^2\right\}^{\frac{1}{2}} \\
&\leq K^{\frac{1}{2}} \max_{|k|=1} \left\|\mathbb{E}\left[D^k\tilde{F}(\boldsymbol{\theta})|\mathcal{F}_i\right] - D^kF(\boldsymbol{\theta})\right\|_{\boldsymbol{\theta}^{(i)}\oplus\rho\mathsf{B}},
\end{aligned}$$

we note that we can take, from Theorem 10,

$$\begin{aligned}
b_i &= \rho^{-D}\frac{\rho_0^D K^{\frac{1}{2}}\sqrt{\sum_{d=1}^D d^2\rho^{2(d-1)}}}{\sqrt{\lambda_{\min}\left(\frac{1}{P}\mathbf{X}_0'\mathbf{X}_0\right)}} B_i \\
&\quad + \left[\frac{\left(2 + C_D(\mathcal{P}_0)\left(1 + P^{\frac{1}{2}}\right)\right)D^2K}{(D-1)!} + \frac{1}{(D-2)!}\right] K^{D-\frac{1}{2}}\rho^D |F|_{D,1}.
\end{aligned}$$

We also have

$$\mathbb{E}\left(\delta_1^{(i)}\right)^2 = \mathbb{E}\left\|\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right) - \dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2 = \mathbb{E}\left\{\mathbb{E}\left(\left\|\dot{\tilde{F}}\left(\boldsymbol{\theta}^{(i)}\right) - \dot{F}\left(\boldsymbol{\theta}^{(i)}\right)\right\|_2^2 |\mathcal{F}_i\right)\right\} \leq \mathbb{E}\sigma_i = \sigma_i.$$

Therefore, we can take the upper bound on $\mathbb{E}\left(\delta_1^{(i)}\right)^2$ as σ_i. QED

Proof of Theorem 11. From Corollary 4, under **AUB** and **Fun-**D, $\delta_1^{(i)} = O\left(\rho_i^{-D}a_N + \rho_i^D\right) = O\left(i^{D\rho-\alpha} + i^{-D\rho}\right)$ and $\delta_2^{(i)} = O\left(\rho_i^{-D}a_N + \rho_i^{D-1}\right) = O\left(i^{D\rho-\alpha} + i^{-(D-1)\rho}\right)$ and one can indeed take two constants $K_8 > 0$ and $K_9 > 0$ such that $\delta_1^{(i)} \leq K_8(i+1)^{-(\alpha-D\rho)\wedge(D\rho)}(1+o(1))$ and $\delta_2^{(i)} \leq K_9(i+1)^{-(\alpha-D\rho)\wedge((D-1)\rho)}$. This implies that we can identify, in Theorem 2, $\xi = (\alpha - D\rho)\wedge((D-1)\rho)$ and $\delta = (\alpha - D\rho)\wedge(D\rho)$. Moreover, **Fun-**2 implies that **Lip-**2 holds with $L_2 = K|F|_{2,1}$.

Now we pass to the conditions in Theorem 2 (i). The fifth condition, $\frac{\delta_1^{(i)}}{m-\delta_2^{(i)}} \leq c_3 (i+1)^{-\delta} (1+o(1))$, is ensured by taking $\delta = (\alpha - D\rho) \wedge (D\rho)$ and c_3 large enough, i.e. such that $\frac{K_8}{m} \leq c_3$. Taking $c_1 = \frac{3K|F|_{2,1}}{2m}\Delta$, the fourth condition, $\frac{1}{m-\delta_2^{(i)}} \left(\frac{\delta_2^{(i)} M}{m} + \frac{3L_2}{2}\Delta \right) \leq c_1 \left(1 + c_2 (i+1)^{-\xi}\right)$, can be written as

$$\delta_2^{(i)} \leq \frac{3mK |F|_{2,1} \Delta c_2 (i+1)^{-\xi}}{2M + 3K |F|_{2,1} \Delta + 3K |F|_{2,1} \Delta c_2 (i+1)^{-\xi}}. \tag{8.32}$$

If we take $\xi = (\alpha - D\rho) \wedge ((D-1)\rho)$, it is possible to choose K_9 in such a way that

$$\delta_2^{(i)} \leq \frac{3mK |F|_{2,1} \Delta c_2 (i+1)^{-\xi}}{2M + 3K |F|_{2,1} \Delta + 3K |F|_{2,1} \Delta c_2 (i+1)^{-\xi}} \leq K_9 (i+1)^{-\xi}.$$

(As an example, $K_9 = mc_2$.) The third condition, $\delta_2^{(i)} < m$, using (8.32), is guaranteed if

$$\delta_2^{(i)} \leq \frac{3mK |F|_{2,1} \Delta c_2 (i+1)^{-\xi}}{2M + 3K |F|_{2,1} \Delta + 3K |F|_{2,1} \Delta c_2 (i+1)^{-\xi}} < m,$$

or $0 < 2mM + 3mK |F|_{2,1} \Delta$ and this is automatically true. The second condition, $\delta_1^{(i)} + \delta_2^{(i)} \left(1 + \frac{M}{m}\right) \Delta \leq \left(m - \frac{3L_2}{2}\Delta\right) \Delta$, is reproduced as it is in the statement. From $1 > c_1 > 0$, we have $\frac{2m}{3K|F|_{2,1}} > \Delta > 0$. From Theorem 2, the final result follows. QED

Proof of Theorem 12. We first identify c_1 and c_2 in Theorem 5 as C_1 and C_2 in this theorem. From the inequalities on ρ_i and a_N, from Corollary 4 and Remark 15, $\delta_1^{(i)} = O\left(\rho_i^{-D} a_N + \rho_i^D\right) = O\left(i^{D\rho-\alpha} + i^{-D\rho}\right)$, and one can take a constant $K_{10} > 0$ such that $\delta_1^{(i)} \leq K_{10} (i+1)^{-(\alpha - D\rho)\wedge(D\rho)} (1 + o(1))$. As a result, in Theorem 5, $\delta = (\alpha - D\rho) \wedge (D\rho)$ and the final result follows. QED

Proof of Theorem 13. The result follows replacing in Theorem 3 and Corollary 2 the formulas of Corollary 4. QED

Proof of Theorem 14. We apply Theorem 7 and Corollary 4. From the latter, we have

$$\begin{aligned} b_i &\asymp \rho_i^{-D} B_i + \rho_i^D \asymp i^{D\rho - \frac{\nu}{2}} + i^{-D\rho}, \\ \sigma_i &\asymp \left\{ \rho_i^{-D} \left(\Sigma_i^{\frac{1}{2}} + B_i \right) + \rho_i^D \right\}^2 \asymp i^{2D\rho - \nu} + i^{-2D\rho}. \end{aligned}$$

Therefore, $\beta = \left(\frac{\nu}{2} - D\rho\right) \wedge (D\rho)$, $\zeta = \left(\frac{\nu}{2} - D\rho\right) \vee (D\rho)$ and $\sigma = (\nu - 2D\rho) \wedge (2D\rho)$. Note that the constants c_3, c_4 and c_5 do not really matter for the final result. The condition $1 < \gamma + \zeta$ boils down to $\gamma + \frac{\nu}{2} > 1 + D\rho$ and $\gamma + D\rho > 1$. As a result,

$$(\sigma + 1) \wedge \beta = (\nu - 2D\rho + 1) \wedge (2D\rho + 1) \wedge \left(\frac{\nu}{2} - D\rho\right) \wedge (D\rho) = \left(\frac{\nu}{2} - D\rho\right) \wedge (D\rho)$$

and

$$(\gamma + \sigma) \wedge \beta = (\gamma + \nu - 2D\rho) \wedge (\gamma + 2D\rho) \wedge \left(\frac{\nu}{2} - D\rho\right) \wedge (D\rho) = \left(\frac{\nu}{2} - D\rho\right) \wedge (D\rho),$$

and the final result follows. QED

8.5 Proofs of Computation Results

Proof of Proposition 1. We start from the nearest matrix in the Frobenius norm and we follow the notation in Higham (1988). First, note that our matrix is automatically symmetric so that the skew-symmetric matrix $\mathbf{C}$ in Higham (1988, Theorem 2.1) is identically equal to $\mathbf{0}$. Then, the nearest psd matrix in the Frobenius norm is given in the proof of Theorem 2.1 in Higham (1988) (the formulation in the statement of the same result is less interesting for us). From the same source, it is apparent that the Frobenius distance between the two matrices is:

$$\left\| \ddot{\bar{F}}(\boldsymbol{\theta}) - \mathbf{U}\boldsymbol{\Lambda}_{+}\mathbf{U}' \right\|_F = \sqrt{\sum_{j:\lambda_j<0} \lambda_j^2}.$$

It can be shown that in our case, as the matrix $\ddot{\bar{F}}(\boldsymbol{\theta})$ is symmetric and normal, $\mathbf{U}\boldsymbol{\Lambda}_{+}\mathbf{U}'$ is also a nearest psd matrix in the spectral norm (see Halmos, 1972). However, as Halmos (1972) is cast in a more general framework, we provide a proof. From the statement of Theorem 3.1 in Higham (1988), using the fact that $\mathbf{C} = \mathbf{0}$, the distance between the matrix $\ddot{\bar{F}}(\boldsymbol{\theta})$ and the set of psd matrices is:

$$\delta_2\left(\ddot{\bar{F}}(\boldsymbol{\theta})\right) = \min\left\{r \geq 0 : \ddot{\bar{F}}(\boldsymbol{\theta}) \geq r\mathbf{I}\right\} = \max\left\{0, -\lambda_{\min}\right\}.$$

Now, from the proof of Lemma 3.5 in Higham (1988), it is easy to see that $\left\| \ddot{\bar{F}}(\boldsymbol{\theta}) - \mathbf{U}\boldsymbol{\Lambda}_{+}\mathbf{U}' \right\|_2 = \max\left\{0, -\lambda_{\min}\right\}$, so that $\delta_2\left(\ddot{\bar{F}}(\boldsymbol{\theta})\right)$ is attained by $\mathbf{U}\boldsymbol{\Lambda}_{+}\mathbf{U}'$. QED

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