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Long-run Heterogeneity in an Exchange Economy with Fixed-Mix Traders

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Long-run Heterogeneity in an Exchange Economy with Fixed-Mix Traders*

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Abstract

We consider an exchange economy where agents have heterogeneous beliefs and assets are long-lived, and investigate the coupled dynamics of assets prices and agents wealth. We assume that agents hold fixed-mix portfolios and invest on each asset proportionally to its expected dividends. Our main finding is that long-run coexistence of agents with heterogeneous beliefs is a generic outcome of the market selection that leads to assets' prices endogenous fluctuations. By using a direct approach that combines the intertemporal dynamics of wealth and prices via agents portfolios, we are able to work with both complete and incomplete markets.

Keywords: Market Selection Hypothesis; Heterogeneous Beliefs; Incomplete Markets; Asset Pricing; Generalized Kelly rule.

JEL Classification: C60, D52, D53, G11, G12

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1 Introduction

The Market Selection Hypothesis (MSH) applied to financial markets implies that traders' beliefs heterogeneity can be only a short-run phenomenon. In the long-run, the trader with the most accurate beliefs about asset's future dividends should gain all the wealth and price assets accordingly. Indeed, benchmark equilibrium asset pricing models, such as Lucas' model and the CAPM, just dismiss heterogeneity away and assume that all traders have correct beliefs about the assets' returns distribution. Despite these models provide an insightful characterization of the relation between assets equilibrium returns and risk preferences, they have not been validated by data[1]. In this paper we investigate whether relying on the MSH is a possible source of failure. Are financial markets likely to select for the most accurate beliefs?

The formal investigation of the MSH has started only many years after its formulation by Alchian (1950) and Friedman (1953). The seminal works by DeLong et al. (1991) and Blume and Easley (1992) have led to two strands of literature. In the first group, agents are expected utility maximizers, have rational price expectations, but disagree on the dividend process; see e.g. Sandroni (2000); Blume and Easley (2006); Jouini and Napp (2006) for discrete time models and Jouini and Napp (2007); Yan (2008); Cvitanić et al. (2012) for continuous time models. The main finding is that when markets are complete they do select for a unique trader. Both saving behavior and accuracy of beliefs are important and only the agent who maximizes a given survival index has positive wealth in the long-run.[2] Beliefs heterogeneity is only transient and assets are priced by the surviving agent.[3,4]

Another strand of literature has instead focused on market selection in economies where agents behavior can be modeled directly in terms of saving and portfolio rules, not necessarily coming from expected utility maximization under rational price expectations. These works contend that agents are able to coordinate on having perfect foresight on future prices, especially when they disagree on the dividend process, and prefer to assume that agents' investment strategies are given adapted process. The question is whether also in this more realistic set-up the

[1]For a list of puzzles and asset pricing anomalies see e.g. the entries "Financial Market Anomalies" and "Finance (new developments)" in the New Palgrave Dictionary of Economics.

[2]The survival index takes into account the trade-off between beliefs accuracy and saving behavior; see e.g. Yan (2008).

[3]See however, Cvitanić and Malamud (2011) for a distinction between the price and portfolio impact of a vanishing agent and Cvitanić et al. (2012) for an appraisal of the impact of vanishing agents on cumulated returns.

[4]Heterogeneity may be persistent when markets are incomplete, see e.g. the examples in Blume and Easley (2006) and their extension in Beker and Chattopadhyay (2010) and Coury and Sciubba (2012), when agents have recursive preferences (Borovička, 2015; Dindo, 2015), or when agents are ambiguity averse (Guerdjikova and Sciubba, 2015).

market selects for a unique agent. An interesting result is that when saving is homogeneous across agents there exists a portfolio rule that dominates against any other combination of adapted rules. This portfolio, named generalized Kelly after Kelly (1956), invests on each asset proportionally to its expected dividends, see e.g. Evstigneev et al. (2009) for a survey. In particular Evstigneev et al. (2008) establish the global dominance of the generalized Kelly rule in an i.i.d. exchange economy where agents can trade multiple long-lived assets. However, a characteristic of the generalized Kelly rule is that it relies on the exact knowledge of the dividend process. On the one hand, when the rule is used by some agents the market converges to a representative agent economy. Moreover, since when alone in the economy a generalized Kelly trader holds the same portfolio of an agent who has the same beliefs and maximizes an intertemporal expected log-utility, the economy converges to a Lucas' log-economy, validating the MSH. On the other hand, it is not known what happens when no agent with correct beliefs is in the market.[5]

In this work we investigate the implication of the MSH for pricing assets in a standard exchange economy with a finite number of agents having homogeneous saving rates[6] and heterogeneous portfolios. Agents can transfer consumption across time and states by means of long-lived assets. The states of the world follow an i.i.d. process and we consider both complete and (exogenously) incomplete markets. We assume that agents purchase assets according to the generalized Kelly rule of Evstigneev et al. (2008), that is, agents invest on each asset a fraction of wealth proportional to its expected dividends. Since we assume that both relative dividends and beliefs are i.i.d., each agent's portfolio is fixed-mix and invests a constant fraction of wealth on each given asset. Moreover, heterogeneous beliefs are chosen such that each agent holds a different portfolio.

Validating the MSH in this context would imply that, even when no agent knows the truth, only the agent with the most accurate beliefs has positive wealth in the long-run and prices assets. In particular, given that the asymptotic log-optimality of the generalized Kelly portfolio, the prediction of Lucas' model (with a log-utility maximizer) would be recovered in the limit. Otherwise, when more agents have positive wealth in the long-run, we shall show that beliefs heterogeneity cannot be ignored for characterizing assets' returns.

[5]Bottazzi and Dindo (2014) investigate the same issue in an economy with short-lived assets, finding that the MSH does not generally hold. Bektur (2013) shows that the agent whose rule is the "closest" to the generalized Kelly rule derived using correct beliefs survives almost surely. In a market for short lived assets, Lensberg and Schenk-Hoppé (2007) find that the generalized Kelly rule that relies on correct beliefs comes out as the asymptotic outcome of a Darwinian model of selection and reproduction implemented through genetic programming.

[6]As we shall see this can be related to the homogeneity of discount factors.

We assume that agents choose how to allocate their wealth across assets using a fixed-mix portfolio (such as the generalized Kelly rule) rather than using a portfolio that maximizes an intertemporal utility for three reasons. First, in a market for long-lived assets, the optimality of a trading rule relies on agents having perfect foresight on future prices, an hypothesis much stronger than that of agents not knowing the dividend process.[7] We view our fixed-mix rules as a first step in relaxing the hypothesis of rational price expectations. In fact, a generalized Kelly portfolio is still log-optimal, conditional on the beliefs used to derive it, in the limit when it determines assets' prices. As a result, if in our model an agent with accurate beliefs fails to dominate, it is not because, conditionally on her beliefs, she ends up using a non-optimal rule when he has gained a large amount of wealth but, rather, because of the non-optimality of the portfolio rules used by her opponents in this limit. Indeed, although one can question that 'smart' traders use non-optimal rules, we do not see strong arguments that prevent 'noise' traders to do so.

Second, simple trading rules have attracted the attention of several scholars in recent works. Using a genetic algorithm, Allen and Karjalainen (1999) prove that, after transaction costs, complicated technical trading rules do not earn superior out of sample returns with respect to a simple strategy that buys and hold the market portfolio. Gigerenzer and Brighton (2009) argue that when agents face uncertainty instead of risk, they tend to use simple heuristics such as fixed-mix rules. This view is confirmed in the work of Benartzi and Thaler (2007): people confronted with the decision of how to allocate their savings for retirement usually rely on naive diversification. That is, given N different alternatives, people tend to invest a fraction $1/N$ of their saving in each alternative. Brennan et al. (2005) consider the data of the Center for Research in Security Prices (CRSP) for the period from December 1925 to December 2003 and shows that investing in a monthly rebalanced portfolio which allocates 50% of the wealth in the market portfolio and 50% in the risk-less bond outperforms a strategy that buys and holds the market portfolio or that buys the market portfolio according to the so-called *Dollar Cost Averaging*.[8] DeMiguel et al. (2009) test naive diversification against sophisticated optimal rules and find that it does not under-perform. They use seven different database of financial prices and show that the fixed portfolio rule of investing $1/N$ of the wealth in each asset provides no-worst results than investment strategies derived from Mean-Variance optimization.[9].

[7] In particular, why should (endogenously determined) prices be easier to forecast than (exogenously given) dividends?

[8] The result can be reconciled with the one of Allen and Karjalainen (1999) noticing that Brennan et al. (2005) do not consider transaction costs. For an appraisal of buy and hold and fixed-mix portfolios see e.g. Perold and Sharpe (1988).

[9] This result is driven by the fact that a Mean-Variance strategy relies on the estimation of

Third, in an economy of fixed-mix traders it is already known that the generalized Kelly rule derived under correct belies dominates almost surely (Evstigneev et al., 2008). Moreover, when nobody plays the generalized Kelly rule with correct beliefs, if there is a rule properly "close" to it, then it survives almost surely (Bektur, 2013).[10] Finding that the rule with most accurate beliefs also dominates would indicate that the MSH should have asset pricing implications also in financial markets where portfolio rules rely neither on perfect foresight on future prices nor on the exact knowledge of the dividend process.

Working directly with fixed-mix rules has the advantage that the dynamics of wealth and asset prices can be derived from the intertemporal budget constraints and market clearing equations.[11] Although the two dynamics are coupled -since assets are long-lived their payoffs determine the new wealth distribution but the wealth distribution determines, through prices, assets payoffs- we are able to solve them explicitly and find an expression of assets payoffs that depends only on agents rules and wealth distribution. Long-run outcomes of the market dynamics can then be studied by means of the Martingale Converge Theorem. In particular, we provide sufficient conditions for a group of agents to have a positive, null, or unitary, fraction of wealth in the long-run. In the simplest case of a 2-agent economy, the sufficient conditions are also necessary (but for hairline cases) and particularly easy to check. Based on these results we are able to characterize when long-run heterogeneity occurs.

Our main finding is that the MSH does not hold generically. Depending on the initial agents' beliefs distribution there exist cases where agents with heterogeneous beliefs, and heterogeneous portfolios, have positive wealth in the long-run. When this is the case the relative wealth distribution changes over time, so that different portfolios have a different impact on asset prices in different periods. The distribution of risk neutral probabilities depends on the distribution of relative wealth and has as support the subset of the simplex defined by agents' beliefs. Moreover, we find that these cases occur in all economies, that is, no matter the exact asset structure and the number of agents, and are generic, that is, they do not disappear if agents beliefs are locally perturbed. We explore numerically the occurrence of long-run heterogeneity by analyzing some examples and find that

the covariance matrix over past returns. Such estimation can be heavily biased and this lets the portfolio be characterized by extremely unprofitable positions.

[10] "Close" for Bektur means component by component. As we shall show that the appropriate "distance" is instead the relative entropy of the whole portfolio.

[11] The use of rules has also disadvantages, for example the fact that one cannot rule out arbitrage by relying on agents demand being optimal. However we are still able to give conditions that exclude arbitrage by working directly with the (endogenously determined) payoff matrix. Generalized Kelly rules naturally satisfy these conditions, see Proposition 3.1.

the areas of beliefs combinations where heterogeneity occurs are large. It appears that the survival of different agents is related to portfolios and dividends being anti-correlated: if an agents invests more in the asset that pays more in one state while the other agent invests more in an asset that pays more in another state, then the outcome is long-run heterogeneity.[12]

The possibility to work with both complete and incomplete markets lets us study whether the survival of heterogeneous beliefs is more or less likely when markets are completed. We show that both cases occur. This result can be related to what Fedyk et al. (2013) argue about the welfare effects of enlarging the asset span. A simple example shows that when no one knows the truth and the investing rules are fixed-mix, adding assets may actually lead to long-run survival of multiple agents, and possibly increasing total welfare.

The contrast between our results and those of the general equilibrium literature with complete markets of Sandroni (2000) or Yan (2008) lies in the non-optimality of generalized Kelly rules. Consider an economy with two agents, i and j and assume that the beliefs of i are more accurate. When both agents hold log-optimal portfolios, agent i dominates and agent j vanishes. If instead we find that j does not vanish, then it must be that agent j non-optimal portfolio is "better" then the optimal portfolio derived under her beliefs, at least in the limit when agent i has most of the wealth. Thus, the non-optimality of agent j portfolio 'corrects' for the non-optimality of her beliefs, leading to "better" portfolios and to her survival.

In order to provide an intuition of the interplay between non correct beliefs and non optimality for the formation of portfolios, we define the "effective" beliefs of an agent as those (time-varying) beliefs such that the generalized Kelly rule derived using the original beliefs coincides with the log-optimal portfolio rule derived using effective beliefs (and rational price expectations). Since a generalized Kelly portfolio is log-optimal in the limit of the agent using it having all the wealth, when an agent has most of the wealth effective beliefs and beliefs coincide. However, when assets' returns are determined by both agents, they differ. In particular, given two agents, the effective beliefs of each agent turn out to be a combination of his beliefs with the beliefs of the other agent. The larger the wealth share of one agent, the larger her impact on equilibrium returns, the larger the weight of her beliefs in determining both agents effective beliefs.[13] Long-run heterogeneity occurs when agent i effective beliefs are more accurate than agent j beliefs when assets' returns

[12]The result seems related to the analysis of the impact of pessimism and optimism on asset prices performed in Jouini and Napp (2010). Note however that their result is non-generic in that it holds only when agents' bias is equal, so that they have the same survival index. See also Blume and Easley (2009). Our results are instead non-generic.

[13]Given that the relative importance of capital gains and dividend yields for assets' returns depends on the market discount rate, the latter plays also a role for how much each agent effective beliefs incorporate the other agent beliefs.

are determined by agent j (because she holds most of the endowment) and, at the same time, agent j effective beliefs are more accurate than agent i beliefs when assets' returns are determined by agent i. The latter is typically the case when the truth is "in between" agent i and agent j beliefs.

The structure of the paper is as follows. In Section 2 we introduce the model. In Section 3 we characterize the coupled dynamics of asset prices and wealth shares and show how and why arbitrages are excluded. In Section 4 we investigate the long-run behavior of our exchange economy and provide sufficient conditions for an agent, or a group of agents, to gain all wealth in the long-run. In particular, in Section 4.2 we characterize long-run heterogeneity and show that it is a generic outcome of an economy with long-lived assets. In Section 5 we discuss our results in terms of effective beliefs and log-optimal portfolios and show, by mean of a numerical exploration, that long-run heterogeneity occurs for a wide range of the economy parameters. Section 6 concludes. All the proofs are collected in the Appendix.

2 The Model

Time is discrete and indexed by $t \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. At each date $t \in \mathbb{N}$ one of the possible $S = \{1, \ldots, S\}$ states of the world occurs.[14] We assume that at each date the state is drawn from the same distribution π over S, with $\pi(s) = \pi_s$, and that subsequent trials are independent. Without loss of generality $\pi \in \Delta^S_+ := \Delta^S \cap \mathbb{R}^S_{++}$.[15] We denote by $s_t \in \{1, \ldots, S\}$ the state that occurs in t, by $\sigma = (s_1, s_2, ..., s_t, ...)$ an entire realization, and by σ_t the partial history up to date t included. The set of all possible realizations is Σ. For each $t \in \mathbb{N}$, the σ-algebra generated by the realizations that share the same partial history till t is $\Im_t$ and $\Im_0 := \{\emptyset, \Sigma\}$. We name $\Im$ the smallest σ algebra that contains all $\Im_t$, so that $\{\Im_t\, ; t = 0, 1, \ldots\}$ is a well-defined filtration of $\Im$. The probability measure P on Σ is obtained as the product of all the measures π on S. The expected value operator $\mathrm{E}[\cdot]$ integrates with respect to the measure π or P depending on the context. $(\Sigma, \Im, \mathrm{P})$ is the probability space on which we construct our economy. It is understood that all the random variables on $(\Sigma, \Im)$ that we shall introduce (dividends, asset prices, portfolios, wealth...) are adapted to the filtration $\{\Im_t\}$. For this reason we may use $X_t(\sigma_t)$ in place of $X_t(\sigma)$. Unless otherwise noted, all our statements are true almost surely with respect to the probability measure P.

[14]Throughout the paper we use the same capital letter to denote a set and its cardinality, when finite.

[15]Given $\mathbb{R}^S$, Δ^S denotes its simplex, $\mathbb{R}^S_+$ is the subset of vectors with non-negative components (excluding the null vector), and $\mathbb{R}^S_{++}$ is the subset of vectors with all positive components.

We consider an exchange economy populated by I agents whose aggregate endowment in each date t is E_t units of the consumption good (apples, the numérarire of the economy). The aggregate endowment dynamics can be any adapted stochastic process on $(\Sigma, \Im, \mathrm{P})$, we only assume that $E_t(\sigma) > 0$, $\forall\, t \in \mathbb{N}$. Portions of the aggregate endowment can be traded by exchanging K long-lived assets (trees). Asset $k \in K$ traded in $t \in \mathbb{N}_0$ pays a dividend $D_{k,t'}$ in $t' > t$. Without loss of generality, each asset is in excess unitary supply and the dividend of the market portfolio is the aggregate endowment.

D1 $\sum_{k=1}^{K} D_{k,t}(\sigma_t) = E_t(\sigma_t)\,, \quad \forall\, t \in \mathbb{N}_0$.

Each agent is endowed in $t = 0$ with a quantity of goods and a portfolio of assets. In every period, agents consume dividends and trade assets to transfer future consumption across time and states. Let $h^i_t = (h^i_{1,t}, \ldots, h^i_{K,t})$ be the asset holding of agent i at time t. At the beginning of period t agent i holds as many assets as those purchased in the previous period $h^i_{t-1} = (h^i_{1,t-1}, \ldots, h^i_{K,t-1})$. Then a state of the world is realized, s_t, and agent i receives an amount of dividends equal to $h^i_{0,t} = \sum_{k=1}^{K} h^i_{k,t-1} D_{k,t}(\sigma_t)$, with $\sigma_t = (\sigma_{t-1}, s_t)$. After that agent i decides about her current consumption and portfolio holding, C^i_t and h^i_t respectively, and trades-in dividends $h^i_{0,t}$ and assets h^i_{t-1} to purchase them.[16] Denoting the vector of date-t asset prices as $P_t = (P_{1,t}, ..., P_{K,t})$, agent i budget constraint in $t \geq 1$ is thus[17]

$$C^i_t + \sum_{k=1}^{K} P_{k,t} h^i_{k,t} = h^i_{0,t} + \sum_{k=1}^{K} P_{k,t} h^i_{k,t-1}\,. \tag{1}$$

Asset prices are fixed in competitive markets. Having assumed that assets are in unitary supply, date-t asset-k market clearing condition reads

$$\sum_{i=1}^{I} h^i_{k,t} = 1\,. \tag{2}$$

The existence and uniqueness of positive clearing prices depends on agents' demands. We postpone to Section 3 the proof that under appropriate assumptions there exists a unique vector of arbitrage free prices such that (1) and (2) holds.

A central quantity to our analysis is agents' wealth. We define agent i wealth in t as her pre-consumption net worth

$$W^i_t = \sum_{k=1}^{K} P_{k,t} h^i_{k,t-1} + h^i_{0,t} = \sum_{k=1}^{K} h^i_{k,t-1}(P_{k,t} + D_{k,t})\,, \quad \forall\, i \in I\,. \tag{3}$$

[16]In period t agent i net demand for asset k is thus $h^i_{k,t} - h^i_{k,t-1}$.

[17]The budget constraint in $t = 0$ is similar but dividends and assets' holdings on the right hand side of (1) come from the initial endowment of, respectively, apples and trees.

$W_t = (W_t^1, \ldots, W_t^I)$ denotes the vector of agents wealth. Eqs. (1-2) can be re-written in terms of W_t^i and W_{t-1}^i. To this end, it is convenient to express each agent $i \in I$ consumption and portfolio decision in t as a function of her wealth W_t^i. We denote with δ_t^i the fraction of wealth she saves, so that $1 - \delta_t^i$ is the fraction of wealth she consumes, while $x_{k,t}^i$ is the fraction of the saved wealth which is used to purchase asset k. We obtain

$$C_t^i = (1 - \delta_t^i) W_t^i \,, \quad \text{and} \quad h_t^i = \frac{\delta_t^i x_{k,t}^i W_t^i}{P_{k,t}} \,. \tag{4}$$

The vector $x_t^i = (x_{1,t}^i, ..., x_{K,t}^i)$ is agent i portfolio rule and the vector $\alpha_t^i = \delta_t^i x_t^i$ is agent i investment rule. Given the budget constraint (1), $\sum_{k=1}^K x_{k,t}^i = 1 \ \forall\, i \in I$ and $\forall\, t \in \mathbb{N}_0$. Moreover, δ_t^i must be in $(0,1)$ to guarantee that consumption is positive in every period. These conditions are naturally satisfied given the choice of investment rule that we shall explicit in Section 2.2.

Using investment rules and agents wealth, budget constraints (1) and market clearing conditions (2) can be re-written for all agents and for all assets as

$$W_t^i = \sum_{k=1}^K (P_{k,t} + D_{k,t}) \frac{\delta_{t-1}^i x_{k,t-1}^i W_{t-1}^i}{P_{k,t-1}} \,, \quad \forall\, i \in I \,, \tag{5}$$

$$P_{k,t} = \sum_{i=1}^I \delta_t^i x_{k,t}^i W_t^i \,, \quad \forall\, k \in K \,. \tag{6}$$

Since assets are long-lived, the dynamics of agents' wealth and assets' prices is coupled. Before solving (5-6), we further characterize assets dividends and agents demands.

2.1 Assets

Together with **D1** we assume that each asset relative dividend process, $D_{k,t}/E_t$, does not depend on partial histories.[18] In other words there exists a $K \times S$ dividend matrix $D = [d_{k,s}]$ such that

D2 $D_{k,t}((\sigma_{t-1}, s_t)) = d_{k,s_t} E_t((\sigma_{t-1}, s_t))$, $\forall\, k \in K$ and $\forall\, t \in \mathbb{N}_0$.

The vectors d_s and d_k denote, respectively, the s-th column and the k-th row of D. The latter can also be viewed as a random variable on S.[19] By **D1**, $\sum_{k=1}^K d_{k,s} = 1$ for every $s \in S$. We also assume that dividends are non-negative and that every asset pays a positive dividend in at least some states

[18] Given our modeling assumption this is also a restriction on initial endowments.

[19] Thus, under **D2**, $\mathrm{E}^{\mathrm{P}}[D_{k,t}|\Im_{t-1}] = \mathrm{E}^{\pi}[d_k]\, \mathrm{E}^{\mathrm{P}}[E_t|\Im_{t-1}]$ for every $k \in K$ and $t \in \mathbb{N}_0$.

D3 $d_{k,s} \geq 0 \; \forall s \in S$ and $\mathrm{E}^{\pi}[d_k] > 0, \; \forall k \in K$;

then, we rule out the existence of redundant assets

D4 $\mathrm{Rank}(D) = K \leq S$.

As we shall show, the dividend matrix D, rather than the aggregate process $\{E_t\}$, is central to the analysis of agents' relative wealth dynamics. Some examples of dividend matrices follow.

Diagonal Dividends Assume that there are as many assets as states, $K = S$, and that the dividend of asset k in t is the entire aggregate endowment if and only if state $s_t = k$ is realized. Using our notation, and using $\delta_{i,j}$ for Kronecker's delta, asset k traded in t' pays the dividend

$$D_{k,t} = \delta_{k,s_t} E_t \quad \text{for all} \quad t > t' \, .$$

The dividend matrix D is just the $S \times S$ identity matrix, $D = I_s$ and **D2** – **D4** are satisfied. Asset k traded in t' is a bet on the occurrence of state $s_t = k$ for all $t > t'$. By construction assets dividends are anti-correlated.

Binomial Tree Here we construct the matrix D that replicates the simplest canonical model of financial markets. Assume that the aggregate endowment follows a geometric random walk:

$$E_t = \begin{cases} g_u E_{t-1} & \text{if } s_t = 1 \\ g_d E_{t-1} & \text{if } s_t = 2 \end{cases} ,$$

with $g_u > g_d$. Two assets in unitary supply are available. The first, $k = 1$, is risky and when purchased in t' has dividends in all $t > t'$ equal to

$$D_{1,t} = \begin{cases} (g_u - g_d) E_{t-1} & \text{if } s_t = 1 \\ 0 & \text{if } s_t = 2 \end{cases} .$$

The second asset is risk-free and has dividend in all $t > t'$ equal to $g_d E_{t-1}$ independently of the state s_t, like a perpetual bond with a time-varying coupon. Since the first asset is equivalent to a long position in the aggregate endowment and a short position in the second asset, the market is equivalent to one with a risk-free asset in zero supply and a risky asset, that pays the aggregate endowment as dividends, in unitary supply. The dividend matrix D is found by imposing **D1** – **D2**. Defining $r = g_d / g_u \in (0, 1)$:

$$D = \begin{bmatrix} 1 - r & 0 \\ r & 1 \end{bmatrix} .$$

It can be easily checked that also **D3** – **D4** are satisfied.

Trinomial Tree In both previous examples, market completeness relies on the full payoff matrix given by the sum of future dividends and prices. Thus, even if D is non-singular, the market might still be incomplete. However, when there are fewer assets than states, $K < S$, we know for sure that asset markets are incomplete. A strength of our approach is that we are able to analyze the long-run outcomes of the economy also for these incomplete markets.

Consider for example an economy as the one just described in the previous paragraph but with $S = 3$ and three possible aggregate endowment growth rates: $g_u \geq g_m > g_d$. Only two assets are traded. As in the previous example the first contract is a long position in the aggregate endowment and a short position in the risk-free asset paying the dividend $g_d E_{t'-1}$ for all $t' > t$. The dividend matrix D is now

$$D = \begin{bmatrix} 1 - r_u & 1 - r_m & 0 \\ r_u & r_m & 1 \end{bmatrix},$$

where $r_u = g_d/g_u \leq r_m = g_d/g_m$ (also in this case also $\mathbf{D3} - \mathbf{D4}$ are satisfied). Assume now that the first contract is replaced by two contracts that can disentangle the position in the first and second state of the economy. Simple computations show that the dividend matrix is now complete and given by

$$D = \begin{bmatrix} 1 - r_u & 0 & 0 \\ 0 & 1 - r_m & 0 \\ r_u & r_m & 1 \end{bmatrix}.$$

2.2 Investment Rules

Although one could investigate the market dynamics with general investment rules (x_t^i, δ_t^i), throughout this work we concentrate on a special class of portfolios, fixed-mix portfolios. In particular we assume that agents derive their portfolios by using the generalized Kelly rule.[20]

R1 Each agent $i \in I$ has discount factor $\delta^i \in (0, 1)$, subjective i.i.d. beliefs $\pi^i \in \Delta^S$, and for all $t \in \mathbb{N}_0$ uses a fixed-mix investment rule $(x_t^i; \delta_t^i) = (x^i; \delta^i)$ with $x_k^i = \mathrm{E}^{\pi^i}[d_k]$ for all k.

Moreover, we further assume that each agent believes that all states are possible[21]

R2 $\pi^i \in \Delta_+^S \;\forall i \in I$.

[20]In so doing, we depart from the standard approach that derives consumption and portfolio decision from the maximization of an objective function subject to beliefs about the distribution of future assets payoffs (both dividend and prices)

[21]The same condition is assumed in the market selection general equilibrium literature (see e.g. Axiom 3 in Blume and Easley, 2006) to guarantee existence of a competitive equilibrium.

By choosing generalized Kelly rules, **R1**, we exclude that rules might depend on market prices or on agents' wealth. Moreover, since beliefs are fixed, rules do not depend neither on the history of assets' dividend nor on the price processes, and thus belong to the class of fixed-mix rules.[22] Given **R2**, each agent invests at least a positive amount of wealth in all assets. Rules do allow some form of short selling, as long as the aggregate position in the existing assets is positive (see the examples in the previous section). It is particularly important to realize that, given restrictions $\mathbf{R1} - \mathbf{R2}$, the set of consumption allocations that agent i can purchase by trading assets depends critically on D. In particular, given two different dividend matrices D and D', and two sequences of prices $\{P\}$ and $\{P'\}$ such that law of one price holds, there might not exist a pair of portfolio rules x and x' satisfying $\mathbf{R1} - \mathbf{R2}$ such that the stream of payoffs is the same with x under D and P and with x' under D' and P'. For this reason, the actual choice of D is relevant for the long-run dynamics.

Evstigneev et al. (2008) show that the generalized Kelly rule obtained under correct beliefs is a benchmark in that, when it trades with other fixed-mix rules, it gains all the aggregate endowment in the long run. In particular when $D = I_S$, the rule suggests to bet on the realization of state s proportionally to its underlying probability π_s, see also Kelly (1956); Evstigneev et al. (2009).

For the interpretation of the results, it is important to note that the generalized Kelly portfolio of agent i in **R1** coincides with the portfolio used on an equilibrium path by a representative agent that maximizes a geometrically discounted log-utility with discount factor δ^i and beliefs π^i. As a result the generalized Kelly rule of each agent is also optimal in an heterogeneous agent economy in the limit of that agent holding all the aggregate endowment.

3 Market Dynamics

In this section, we show that when agents use fixed-mix rules, inter-temporal budget constraints (5) and market clearing conditions (6) can be solved to give positive and unique market clearing prices P_t and, as a result, a well defined dynamics for agents wealth W_t and prices P_t. While working toward the solution of (5-6) we shall derive an explicit formulation for the payoff matrix, the sum of dividends and next period prices. We shall use the formula to show that in our framework equilibrium prices and payoffs exclude arbitrage.

Without loss of generality, we assume that each agent $i \in I$ starts with some

[22] Our analysis is also informative of the long-run behavior of a market where beliefs are not i.i.d. but are adapted to the information filtration of dividends and prices, provided that beliefs converge to some constant level as more and more information is gathered.

given positive wealth W^i_0.[23]

3.1 Representative agent

We start with the case where agent i posses all the aggregate endowment in $t = 0$, so that $W^j_0 = 0$ for all $j \neq i$. Straightforward computations lead to

$$W^i_t = \frac{E_t}{1-\delta^i}\,, \quad W^j_t = 0\,,\, j \neq i \quad \forall\, t \in \mathbb{N}_0\,, \tag{7}$$

$$P_{k,t} = \frac{\delta^i}{1-\delta^i}\,\mathrm{E}^{\,\pi^i}[d_k]E_t\,. \tag{8}$$

Asset k is priced as in a log-economy where the representative agent has beliefs π^i and discount factor δ^i. If the dividend matrix is non-singular the market is complete and risk neutral probabilities coincides with agent i beliefs. The Lucas' model is recovered.

3.2 Heterogeneous agents

Pricing is more interesting when agents have heterogeneous beliefs. Assume that there exist at least two agents i and j with $W^i_0 > 0$ and $W^j_0 > 0$ and $\alpha^i \neq \alpha^j$. By substituting (5) in (6) we get

$$\sum_{h=1}^{K}\left(\delta_{k,h} - \sum_{i=1}^{I}\frac{\alpha^i_k\alpha^i_h W^i_{t-1}}{P_{h,t-1}}\right)P_{h,t} = \sum_{h=1}^{K} d_{h,s_t}E_t \sum_{i=1}^{I}\frac{\alpha^i_k\alpha^i_h W^i_{t-1}}{P_{h,t-1}}\,. \tag{9}$$

The above expression can be conveniently written in matrix form. Consider the vector of price-rescaled investment fractions

$$\beta^i(W;\alpha) = \left(\alpha^i_1 / \sum_{j=1}^{I} W^j\alpha^j_1, \ldots, \alpha^i_K / \sum_{i=j}^{I} W^j\alpha^j_K\right)$$

and define the positive matrix

$$A(W;\alpha) = \sum_{i=1}^{I} W^i\,\alpha^i \otimes \beta^i(W,\alpha)\,.$$

Then (9) becomes

$$(I_K - A(W_{t-1};\alpha))\,P_t = A(W_{t-1};\alpha)\,d_{s_t}E_t \tag{10}$$

and one has

[23] This is equivalent to assume that agents start with an initial allocation of assets and consumption goods.

Lemma 3.1. *Under the assumption that rules satisfy* $\mathbf{R1} - \mathbf{R3}$*, the matrix* $I_K - A(W, \alpha)$ *is invertible for all* $W \in \mathbb{R}^I_+$*.*

From the previous Lemma and from (10) it follows that market clearing prices are uniquely defined for every $t \in \mathbb{N}_0$:

$$P_t(\sigma_{t-1}, s_t) = (I_K - A(W_{t-1}; \alpha))^{-1}\, A(W_{t-1}; \alpha)\, d_{s_t} E_t(\sigma_{t-1}, s_t) = \sum_{n=1}^{\infty} A^n(W_{t-1}(\sigma_{t-1}); \alpha) d_{s_t} E_t(\sigma_{t-1}, s_t)\,. \quad (11)$$

Given $W_{t-1}(\sigma_{t-1})$ and investment rules α for all agents i, there is a period t price vector P_t for every realization of the dividend process s_t. All assets with positive dividend in s_t contribute to the next period wealth distribution, which in turns determines prices. As a result, given a dividend matrix D and rules α, for every w there exists a matrix $P(W; \alpha, D)$, with the same dimension of D, such that $P_{k,t}(\sigma_{t-1}, s_t) = P_{k,s_t}(W_{t-1}; \alpha, DE_t)$. Since, by $\mathbf{D3}$ and $\mathbf{R2} - \mathbf{R3}$, $A(W; \alpha)$ is strictly positive and D is positive, the equation above shows that $P(W; \alpha, D)$ is strictly positive.[24] When the wealth distribution is degenerate, in that only agent j has positive wealth, it is $A = \delta^j x^j \otimes \mathbf{1}$ and (8) is recovered.

Long-lived assets prices and dividends D determine the payoff matrix

$$R(W; \alpha, D) = P(W; \alpha, D) + D = (I_K - A(W; \alpha))^{-1} D\,. \quad (12)$$

Since the payoff matrix depends, through prices, also on the wealth distribution W, it keeps changing as the wealth distribution evolves. Its rows $R_k(W; \alpha, D)$, with $k \in K$, are strictly positive random variables on S given by the sum of the two random variables $P_k(W; \alpha, D)$ and d_k. By substituting (12) in (5) one obtains the explicit evolution of the wealth distribution. By construction it is adapted to the information filtration. We can summarize the result of this section in the following proposition.[25]

Proposition 3.1. *Consider an exchange economy where* I *agents using rules obeying* $\mathbf{R1} - \mathbf{R2}$ *are trading* K *assets satisfying* $\mathbf{D1} - \mathbf{D3}$*. If* $W_0 \in \mathbb{R}^I_{++}$ *then for all* $t \geq 1$ *the process* $\{W_t\}$ *is* $\mathbb{R}^I_{++}$*, it is adapted to* $\{\Im_t\}$*, and evolves according to*

$$W^i_t(\sigma_{t-1}, s_t) = W^i_{t-1}(\sigma_{t-1}) \sum_{k=1}^{K} \beta^i_k(W_{t-1}(\sigma_{t-1}); \alpha)\, R_{k,s_t}(W_{t-1}(\sigma_{t-1}); \alpha, DE_t)\,, \quad \forall\, i \in I\,.$$

[24] However, it could still be the case that asset prices admits arbitrage. We rule out that arbitrage exists in equilibrium in Proposition 3.1.

[25] It is straightforward to see that the same proposition holds even when beliefs π^i_t are adapted to the information filtration generated by s_τ and $P_{\tau-1}$ for all $\tau \leq t$. Evstigneev et al. (2006) provides a different proof of the same result. In particular they do not explicitly characterize the payoff matrix R.

Moreover the sequence of wealth distributions $\{W_t\}$ is such that for all $t \geq 1$ market equilibrium prices $P_t = P_{s_t}(W_{t-1}; \alpha, DE_t)$ and payoffs $R_{t+1} = R(W_t; \alpha, DE_{t+1})$ do not admit arbitrage.

The coupled price-wealth dynamics is not only well defined but also prevents arbitrage opportunities to arise in equilibrium, so that state prices are always positive. The absence of arbitrage is non-trivial because, with long-lived assets, the dynamics of the wealth distribution determines both assets purchasing price and their payoffs. Under the standard utility maximization with unconstrained portfolios, arbitrage never occurs in equilibrium. In our model, however, assets' holdings are constrained and arbitrage might in principle occur. A sufficient condition to avoid arbitrages turns out to be that the vector of portfolio rules is in the interior of the cone generated by the S column of the matrix D (see the proof of Prop. 3.1 for more details). A condition that, given **R1**, is naturally satisfied by generalized Kelly rules.[26]

3.3 Relative wealth dynamics

If agents have a different saving rate, the agent who saves more is advantaged in terms of long-run wealth. If, for example, there are only two agents and they use the same portfolio rule but $\delta^1 > \delta^2$, then agent 1 wealth grows geometrically faster at the rate δ^1/δ^2 than the wealth of agent 2. When agents have different portfolios there is a trade-off between having a higher saving rate and a "better" portfolio. Although the trade-off is certainly interesting, here we concentrate on the heterogeneity of portfolio rules and assume homogeneous saving rates:

R3 $\delta^i = \delta \,, \ \forall\, i \in I \,.$

Under this assumption the relative wealth dynamics does not depend on the aggregate endowment process. Introducing normalized wealth and price

$$w_t^i = \frac{1-\delta}{E_t}\, W_t^i \quad \text{and} \quad p_{k,t} = \frac{1-\delta}{\delta\, E_t}\, P_{k,t}\,, \tag{13}$$

such that at any t its is $\sum_{i=1}^{I} w_t^i = 1$ and $\sum_{k=1}^{K} p_{k,t} = 1$, one can still apply Lemma 3.1 and Proposition 3.1 to normalized variables, provided $w_0 \in \Delta_+^I$. For this purpose the payoff matrix R defined in (12) should be replaced with

$$r(w; x, \delta, D) = [r_{k,s}] = (1-\delta)(I_K - \delta A(w; x))^{-1}\, D$$

[26]The condition is instead not satisfied by those fixed-mix rules that cannot be derived as a generalized Kelly rule given some beliefs.

where, in the definition of matrix A the normalized wealth replaces the original wealth, and portfolio shares x^i replaces investment shares α^i. When δ is close to zero normalized prices get very small and the (normalized) payoff matrix r is close to the dividend matrix d. Conversely when δ is close to one, normalized prices are much larger than dividends. When $\delta \to 1$, the payoff matrix becomes singular. An advantage of working with the normalized variables is that, since both states of the world and agents' beliefs are i.i.d., the relative wealth dynamics is a Markov process.

Corollary 3.1. *Under the assumptions of Proposition 3.1, the normalized wealth w_t follows a Markov Process on Δ^I_+ such that for every $t \geq 1$ with probability π_s the relative wealth vector w_{t-1} evolves into*

$$w^i_t = w^i_{t-1} \sum_{k=1}^{K} \beta^i_k(w_{t-1}; x)\, r_{k,s}(w_{t-1}; x, \delta, D) \quad \forall\, i \in I\,. \tag{14}$$

4 Market Selection and Long-run Heterogeneity

In the rest of the paper, we characterize the long-run behavior of the relative wealth dynamics (14). In particular, we focus our attention on the long-run performance of groups of agents. For any proper subset $J \subset I$, we denote the sum of period t wealth of agents in J as w^J_t, so that $1 - w^J_t$ is the sum period t wealth of agents in $I \setminus J$. The aggregate portfolio rule of the two groups of agents at time t are, respectively,

$$x^J_k(w_t; x) = \sum_{j \in J} x^j_k \frac{w^j_t}{w^J_t} \quad \text{and} \quad x^{-J}_k(w_t; x) = \sum_{j \in I \setminus J} x^j_k \frac{w^j_t}{1 - w^J_t}\,. \tag{15}$$

With usual notation we set $\beta^J = x^J/p = (x^J_1/p_1, \ldots, x^J_K/p_K)$ and $\beta^{-J} = x^{-J}/p = (x^{-J}_1/p_1, \ldots, x^{-J}_K/p_K)$. In this section, we provide sufficient conditions for the survival or dominance of a generic group J of agents. The next definition makes precise what we mean by dominance, survival, and vanishing.

Definition 4.1. We say that group J dominates on a realization σ if

$$\lim_{t \to \infty} w^J_t(\sigma) = 1\,. \tag{16}$$

Group J survives on a realization σ if

$$\limsup_{t \to \infty} w^J_t(\sigma) > 0\,. \tag{17}$$

If a group does not survive on σ, we say that it vanishes on that realization. We say that group J dominates or survives if (16) or (17) hold P-a.s.. Group J vanishes if it survives on a set of realizations with P-measure zero.

If a group composed by only one agent dominates, heterogeneity is only a transient property and the economy converges with probability one to a representative agent limit. If instead more than one agent survives, then the economy exhibits long-run heterogeneity.

Definition 4.2. An I-agent asset market economy exhibits long-run heterogeneity if there exists a proper subset of traders $J \subset I$ such that both the group J and the group $I \setminus J$ survive.

Dominance of a single agent is the unique possible long-run outcome when markets are complete and all agents maximize an expected log-utility given their beliefs and discount factors, see e.g. Sandroni (2000); Yan (2008).[27] Conversely, long-run survival of more than one agent is a generic outcome of our model with 'quasi-optimal' log-rules.

In order to characterize the relative performance of group J, we use the difference between the conditional expected log-growth rate of agents in J and the conditional expected log-growth rate of the other agents. Corollary 3.1 implies that this quantity depends on the history of states prior to period t only through the wealth distribution w at period t. Formally

$$\begin{aligned}\mu_t^J(w) &= \mathrm{E}^{\,\mathrm{P}}\left[\log \frac{w_{t+1}^J}{w_t^J} - \log \frac{1-w_{t+1}^J}{1-w_t^J}\middle|\Im_t \text{ s.t. } w_t = w\right] \\ &= \mathrm{E}^{\,\pi}\left[\log \frac{\sum_{k=1}^K \beta_k^J(w_t;x) r_k(w_t,;x,\delta,D)}{\sum_{k=1}^K \beta_k^{-J}(w_t;x) r_k(w_t;x,\delta,D)}\right] .\end{aligned} \tag{18}$$

The sign of $\mu_t^J(w)$ reveals if the expected aggregate wealth of the agents in J grows or shrinks. It turns out that sufficient conditions for survival or dominance of group J can be derived studying the sign of $\mu_t^J(w)$ when the relative wealth w^J is very large or very small.

For a proper subset J and for all $v \in [0,1]$ consider the quantity

$$\overline{\mu}^J(v) = \max\left\{\mu^J(w) \quad | \quad w \in \Delta^K \,,\, w^J = v\right\} \,,$$

and

$$\underline{\mu}^J(v) = \min\left\{\mu^J(w) \quad | \quad w \in \Delta^K \,,\, w^J = v\right\} \,.$$

The definition is meaningful because the function μ^J is continuous in w and the extrema are computed on compact sets. Since these sets are continuous in v (both

[27] Long-run heterogeneity is possible in log economies but, first, it is non-generic in that it occurs when agents' beliefs have the same relative entropy with respect to the truth, and, second, it is not robust to a perturbation of beliefs, see Blume and Easley (2009).

upper and lower hemicontinuous) the quantities $\underline{\mu}^J$ and $\overline{\mu}^J$ are continuous function of their argument.

The next Proposition exploits the Martingale Converge Theorem (see the proposition's proof for details) to characterize long-run survival.

Proposition 4.1. *Consider an exchange economy with I agents using rules obeying* **R1** $-$ **R3** *and trading K assets satisfying* **D1** $-$ **D4***:*

i) If $\underline{\mu}^J(0) > 0$, then group J survives.

ii) If $\overline{\mu}^J(1) < 0$, then group $I \setminus J$ survives.

Under *i*) and *ii*), both group J and $I \setminus J$ survive and the market exhibits log-run heterogeneity. Propositions 4.1 provides only sufficient conditions because by considering all possible wealth distributions w we are taking into consideration wealth distributions that cannot be realized with positive probability. It is however clear that if the conditions on the conditional drift in Proposition 4.1 are realized, they are *a forziori* true for (almost) all possible trajectories of the system.

Long-run heterogeneity is not the only possible market outcome. In order derive other results, such as dominance of a group, we shall need to assume that agents J aggregate rule x^J cannot be replicated by a combination of other agents rules. We assume the following.

R4 There exists a hyper-plane in R^K which separates the rules of agents in J from the rules of agents in $I \setminus J$.

Since the aggregate rules x^J and x^{-J} belong to the convex cone generated by the strategies of agents in J and $I \setminus J$ respectively, condition **R4** is sufficient to guarantee that they can never be equal, irrespective of the wealth distribution. Notice that, as long as individual rules are all different and there are no less assets than agent, $K \geq I$, condition **R4** is satisfied for any group J.

Assumption **R4**, combined with the absence of redundant assets, is sufficient to prove that, for all t, there is a positive probability that wealth distribution between group J and group $I \setminus J$ changes.

Lemma 4.1. *If the set of rules are not overlapping,* **R4***, and if there are no redundant assets,* **D4***, then there exists a $\gamma > 0$ such that*

$$\text{Prob}\left\{ \left| \log \frac{w^J_{t+1}}{w^{-J}_{t+1}} - \log \frac{w^J_t}{w^{-J}_t} \right| > \gamma \middle| \Im_t \right\} > \gamma\,. \tag{19}$$

The Lemma implies that, as long as two groups aggregate rules are separated, there never exists a stable wealth distribution between the two groups. However, it does not exclude the possibility that one group grows more than the other with probability one. As we shall prove, such occurrence is excluded by the absence of arbitrage. As a result, all t, σ_t, the ratio w_t^J/w_t^{-J} both increases and decreases with positive probability.

Lemma 4.2. *Under* **R4**, *if market equilibrium prices p_t and assets payoffs r_{t+1} do not admit arbitrages, then for all groups J and all $t \in \mathbb{N}_0$ there exist $\epsilon > 0$ such that*

$$\mathrm{Prob}\left\{\frac{w_{t+1}^J}{w_t^J} > \frac{w_{t+1}^{-J}}{w_t^{-J}} \,\Big|\, \Im_t\right\} > \epsilon \quad \textit{and} \quad \mathrm{Prob}\left\{\frac{w_{t+1}^J}{w_t^J} < \frac{w_{t+1}^{-J}}{w_t^{-J}} \,\Big|\, \Im_t\right\} > \epsilon\,. \tag{20}$$

Both Lemma allow us to prove the following.

Proposition 4.2. *Consider an exchange economy with I agents using rules obeying* **R1** $-$ **R4** *and trading K assets satisfying* **D1** $-$ **D4***:*

i) If $\underline{\mu}^J(0) > 0$ and $\underline{\mu}^J(1) > 0$, then group J dominates;

ii) If $\overline{\mu}^J(0) < 0$ and $\overline{\mu}^J(1) < 0$, then group J vanishes.

iii) If $\underline{\mu}^J(0) > 0$ and $\overline{\mu}^J(1) < 0$, then both groups survive and for both groups $G = J, I \setminus J$

$$\mathrm{Prob}\{\liminf_{t\to\infty} w_t^G = 0 \textit{ and } \limsup_{t\to\infty} w_t^G = 1\} = 1$$

Proposition 4.2 provides sufficient conditions for a group to dominate or vanish and complements Proposition 4.1: if the rules used by each group are always different, **R4**, then, due to Lemma 4.2, point *iii*) of Proposition 4.2 holds. The relative wealth shares keep fluctuating in the interval $(0, 1)$ and assets' prices keep fluctuating in between the two groups evaluations. In general terms, lack of arbitrage ensures that Lemma 4.2 holds and each group's relative wealth increases and decreases with positive probability. In cases *i*) and *ii*), this is important because even if the asymptotic drift conditions point to dominance of a trader, a limited arbitrage in favor of the trader with asymptotically "worst" portfolio rule could occur when $w \in (0, 1)$, thus preventing the trader with asymptotically "better" rules to dominate. In case *iii*), knowing that the relative wealth keeps fluctuating implies that its effective domain is the interval $(0, 1)$ and that asset prices are in between the two groups evaluations (which by assumptions are separated).

4.1 2-agent economy

In a two-agent economy we can limit the study to the relative wealth dynamics of one agent, say agent 1. Given the wealth normalization, the conditional drift (18) can be written as function of $w_t^1 = w$ only, giving for $w = (w, 1-w)$

$$\mu_t^1(w) = \mu(w) = \mathrm{E}^{\pi}\left[\log \frac{\sum_{k=1}^K \beta_k^1(w;x) r_{k,s}(w;x,\delta,D)}{\sum_{k=1}^K \beta_k^2(w;x) r_{k,s}(w;x,\delta,D)}\right].$$

As a result, $\mu(w) = \underline{\mu}^1(w) = \overline{\mu}^1(w)$ for all $w \in [0,1]$.

In what follows it is convenient to define the Kullback Leibler divergence, or relative entropy, of rules x^i with respect to the reference rule x^* with $x_k^* = \mathrm{E}^{\pi}[d_k]$ as

$$D(x^*||x^i) = \mathrm{E}^{x^*}\left[\log\left(\frac{x^*}{x^i}\right)\right].$$

and the difference of relative entropies as

$$\Delta_{x^*}(x^2||x^1) = D(x^*||x^2) - D(x^*||x^1)\,.$$

The following lemma provides an ordering of asymptotic drifts with respect to difference of rules relative entropies.

Lemma 4.3. *Consider an exchange economy with $I = 2$ agents using rules obeying* **R1 – R4** *and trading K assets satisfying* **D1 – D4***, then*

$$\mu(0) > (1-\delta)\Delta_{x^*}(x^2||x^1) > \mu(1)\,.$$

Lemma 4.3 excludes the case in which both agents are better off, in expectations, when they have most of the wealth.[28] Coupling this result with a straightforward application of Proposition 4.2 leads to the following set of sufficient and, apart from hairline cases, necessary conditions for the long-run outcomes of a two-agent economy.

Proposition 4.3. *Consider an exchange economy with $I = 2$ agents using rules obeying* **R1 – R4** *and trading K assets satisfying* **D1 – D4***. Provided both $\mu(0)$ and $\mu(1)$ have a definite sign, one of the following occurs*

i) If $\mu(1) > 0$, then agent 1 dominates and 2 vanishes;

ii) If $\mu(0) < 0$, then agent 2 dominates and 1 vanishes;

[28] This is in contrast with what has been shown in Bottazzi and Dindo (2014) for a market of short-lived assets. The key difference is that Bottazzi and Dindo allow portfolio rules to depend also on prices.

iii) If $\mu(0) > 0$ and $\mu(1) < 0$, then both agents survive and for all assets $k \in K$

$$\text{Prob}\left\{\liminf_{t\to\infty} p_{k,t} = \min_{i=1,2}\{\mathrm{E}^{\pi^i}[d_k]\} \text{ and } \limsup_{t\to\infty} p_{k,t} = \max_{i=1,2}\{\mathrm{E}^{\pi^i}[d_k]\}\right\} = 1\,.$$

Long-run heterogeneity occurs when both agents have a higher wealth growth rates at the returns determined by the other agent[29].

As in Proposition 4.2, long-run heterogeneity amounts to a relative wealth that keeps fluctuating when agents portfolio rules can be separated (otherwise, if **R4** does not hold agents have the same demand for assets and their relative wealth is constant). With only two agents the result has direct implications for asset prices dynamics: prices keep fluctuating between the two agents evaluations. Moreover, contrary to Propositions 4.1 and 4.2, $\mu(0)$ and $\mu(1)$ can be computed easily, making the 2-agent economy particularly tractable and amenable to applications.

Another advantage of a two-agent market is that no other cases than those of Proposition 4.3 can occur. Indeed, leaving out the non generic cases when asymptotic drifts are zero, only three cases are possible. This is due to the result provided by Lemma 4.3: if an agent has a favorable drift when she has most of the wealth, then she has a favorable drift also when she has little wealth. Conversely, if she faces an unfavorable drift when she has little wealth, the drift would be against her also if she possessed almost all the wealth. As a result $\mu(1) > 0$ ($\mu(0) < 0$) is sufficient to prove that agent 1 (2) dominates. The third possibility is that $\mu(0) > 0$ and $\mu(1) < 0$, in this case both agent 1 and 2 survive, and none dominates. We concentrate on proving that such cases do always exist and are robust to perturbations of the beliefs in the next section.

Together with Lemma 4.3, Proposition 4.3 implies the survival of the agent whose beliefs are such that the portfolio rule she uses is the 'closest', in terms of relative entropy, to the Generalized Kelly rules derived under correct beliefs. This extends the result of Bektur (2013) who shows that if a rule is the closest to x^* coordinate-wise then it survives. Whether the agent also dominates, or both agents survive, it depends on his performance when he has most of the aggragte endowment. As shown by Evstigneev et al. (2008), the generalized Kelly trader who uses correct beliefs dominates against any other generalized Kelly trader. Summarizing, we have the following.

Corollary 4.1. *Consider an exchange economy with $I = 2$ agents using rules obeying* **R1** − **R4** *and trading K assets satisfying* **D1** − **D4**. *If agents beliefs are such that*

$$D(x^*||x^2) > D(x^*||x^1)\,,$$

[29] As we shall discuss in Section 5 the result can be given in term of "effective" beliefs and log-optimal rules.

then agent 1 *survives. If, moreover, agent* 1 *beliefs are correct so that* $x^1 = x^*$, *then agent* 1 *dominates.*

In particular, when $D = I_S$, the above inequality can be written in terms of beliefs and becomes

$$D(\pi||\pi^2) > D(\pi||\pi^1)\,,$$

In this case agent 1 survives when she has more accurate beliefs. With more general dividend matrices D and, possibly, incomplete markets it is the relative entropy of rules, rather than of beliefs that guarantees survival to the most "accurate" trader. Having correct beliefs is instead always sufficient for dominance.[30]

4.2 Long-run Heterogeneity

Having defined sufficient conditions for long-run heterogeneity we turn to generality and existence. First, we show that when long-run heterogeneity occurs it is also generic, in that perturbations of beliefs do not lead to dominance of any of the surviving agent. Second, we show that for any asset structure D there exist beliefs for which heterogeneity is indeed the long-run outcome. In both cases, we restrict our analysis to an economy with 2 agents and assume that both agents do not know the truth, $\pi^i \neq \pi$.

The next proposition states that if the conditions for persistent heterogeneity of Proposition 4.3 apply, then there exist perturbations of agents' beliefs such that heterogeneity is still the long-run outcome.

Proposition 4.4. *If an economy with* 2 *agents having beliefs* $\bar{\pi}^1, \bar{\pi}^2$ *and rules satisfying* $\mathbf{R1} - \mathbf{R4}$ *exhibits long-run heterogeneity, then there exist vectors* $\epsilon^1, \epsilon^2 \in \mathbb{R}^S$ *with components* $\epsilon^i_s \in [-\varepsilon, \varepsilon]$, $\varepsilon > 0$, *and* $\sum_{s=1}^{S} \epsilon^i_s = 0$ *for* $i = 1, 2$, *such that under beliefs* $\bar{\pi}^1 + \epsilon^1$ *and* $\bar{\pi}^2 + \epsilon^2$ *the economy still exhibits long-run heterogeneity.*

To prove the above not that sufficient conditions for long-run heterogeneity involve strict inequalities. Since conditional drift are continuous functions of beliefs (via the portfolio rules), then there exists perturbation of beliefs such that conditional drift satisfy the same inquality.

Having shown that heterogeneity, when it occurs, is generic we address a different issue. Given Generalized Kelly traders and a market for long-lived assets satisfying $\mathbf{D1} - \mathbf{D4}$, is it always possible to find some beliefs such that heterogeneity occurs?

The result of Lemma 4.3 together with condition *iii*) of Proposition 4.3 imply that if two agents have beliefs such that the corresponding rules have the same

[30]The dominance of the agent with correct beliefs holds also for I-agents economies, as shown by Evstigneev et al. (2008).

relative entropy with respect to x^*, then they both survive. Thus, in order to prove the existence of long-run heterogeneity for every admissible choice of the matrix D, we have to find beliefs for agents 1 and 2 such that the corresponding Generalized Kelly rules have the same relative entropy. At this purpose, define $\bar{\Delta}$ as the open set $\{x' \in \Delta^K : x' = E^{\pi'}[D], \pi' \in \Delta^S_+\}$ and call $\partial(\bar{\Delta})$ its frontier.

Proposition 4.5. *Given a dividend matrix D and beliefs $\pi^1 \neq \pi$ such that $D(x^*||x^1) < \mathcal{K}$ with $\mathcal{K} = \min_{\pi'}\{D(x^*||x')\}\, s.t.\, x' \in \partial(\bar{\Delta})\}$, there exists a non-empty set of beliefs $\Pi \subset \Delta^S_+$ with $\pi^1 \in \Pi$ such that for all $\pi^2 \in \Pi$ the asset market economy with generalized Kelly traders having beliefs π^1 and π^2 exhibits long-run heterogeneity.*

The fundamental ingredients for proving Proposition 4.5 are the properties of the relative entropy. Its continuity, strict convexity and the fact that it has a minimum equal to zero in x^* are sufficient to show the existence of Π. Indeed, to build Π it is enough to fix π^1 and take the set of beliefs such that the Generalized Kelly rules they generate have all the same relative entropy with respect to x^*.[31] Note that, thanks to Proposition 4.4, one can also expand such set including the neighborhood of all these beliefs.

Finally note that equality of beliefs relative entropy implies long-run heterogeneity also in market economies where agents are expected utility maximizers and assets markets are (dynamically) complete, see e.g. Blume and Easley (2009) and Jouini and Napp (2010). There is however an important difference with the model presented here. Whereas heterogeneity is generic in our market, see Proposition 4.4, it is non-generic in the latter cases. Any small perturbation of an agent beliefs or bias will break the tie of relative entropies and thus lead to dominance of the agent whose beliefs turn out to be "closest" to the truth.

5 Discussion and Examples

We begin this section by providing an intuition on the source of long-run heterogeneity based on the comparison between Generalized Kelly rules and log-optimal rules. Then, we explore market selection outcomes in I-agent economies for specific choices of the dividend matrix D.

[31] The fact that $\Pi \subset \bar{\Delta}$ depends on the technical condition $D(x^*||x^1) < \mathcal{K}$. Otherwise the set Π could encompass rules that are not generated by any belief.

5.1 Effective Beliefs

What is the intuition behind the occurrence of long-run heterogeneity? In a similar asset market economy, if portfolios are log-optimal and the asset market is complete, then the agent with the most accurate beliefs dominates, see e.g. Sandroni (2000) and Yan (2008). However, we find that when agents use generalized Kelly rules the accuracy of beliefs is not directly related to dominance. Provided D is diagonal, Corollary 4.1 proves only a weaker result, that is, accuracy of beliefs is sufficient for survival. Since a generalized Kelly rule is log-optimal in the limit when the agent using it has most of the wealth, failure to dominate must be caused by the portfolio of the agent with less accurate beliefs being particularly "good" in this limit. Non accuracy of beliefs and non log-optimality of the portfolio must compensate each other.

In order to establish how, and when, the compensation occurs, we use the concept of "effective beliefs". Given asset prices in t and payoffs in $t+1$, we define an agent i effective beliefs in t, $\bar{\pi}^i_t$, as the beliefs such that the generalized Kelly rule x^i derived from π^i is log-optimal in t. More specifically, to compute "effective beliefs" we proceed as follows. Given agents beliefs, discount factors, and a dividend matrix D, for every value of the relative wealth distribution w_t there correspond both a vector of prices p_t and a payoff matrix r_{t+1} (see Section 3.3 for details). Thus for every w_t one can find the "effective beliefs" of agent i as those beliefs $\bar{\pi}^i_t$ such that the portfolio rule x^i is log-optimal given prices p_t and payoffs r_{t+1}.[32] As a result, for each agent i, we derive a function $\bar{\pi}^i : \Delta^I \to \Delta^S_+$ such that $\bar{\pi}^i_t = \bar{\pi}^i(w_t; \pi, \delta, D)$. Note that the function depends on all agents rules (and thus beliefs), on the discount factor δ, and on the dividend matrix D. Given the log-optimality of the generalized Kelly rule when it has all wealth, we have $\bar{\pi}^i((0, \dots, w^i = 1, \dots, 0); \pi, \delta, D) = \pi^i$ for all $i \in I$ independently from δ, D, and other agents beliefs.

Effective beliefs enable us to view the economy with generalized Kelly traders as an economy with log-optimal traders using effective beliefs. The general equilibrium literature tells us that, provided the asset market is complete, an agent survives only when her beliefs are, on average, as accurate as prices (see Massari, 2014). As a result whenever we find that long-run heterogeneity is the long-run outcome, agents effective beliefs must be, on average, equally accurate. Moreover, along the lines of Propositions 4.2 and 4.3, one can prove that the sufficient conditions that characterize long-run outcomes can be given in terms of 'asymptotic'

[32] Prices in t and payoffs in $t+1$ rely on agents using fixed-mix rules x both in t and in $t+1$. As a result, the equivalence between the market dynamics under generalized Kelly traders with fixed beliefs and the one under log-optimal traders with effective beliefs can only be established for every t (and thus in an infinite horizon economy). Equilibrium prices and returns are the same in this second model since the actual portfolio rule used by agents is still x by construction.

effective beliefs accuracy instead that in terms of 'asymptotic' growth rates μ. In fact, the following proposition shows that the relative accuracy of effective beliefs can be used to characterize the value of asymptotic drifts $\mu(0)$ and $\mu(1)$.[33]

Proposition 5.1. *Consider an exchange economy with $I = 2$ agents using rules obeying* $\mathbf{R1} - \mathbf{R4}$ *and trading a (dynamically) complete set of assets with dividend matrix D satisfying* $\mathbf{D1} - \mathbf{D4}$*, then*

$$\mu(0) = \Delta_\pi(\pi^2 || \bar{\pi}^1((0,1);\delta, D)) \quad \textit{and} \quad \mu(1) = \Delta_\pi(\bar{\pi}^2((1,0);\delta, D))||\pi^1)\,.$$

In a two-agent economy, long-run heterogeneity occurs when, for both $i = 1, 2$, agent i effective believes are more accurate than agent $j \neq i$ (effective) believes when agent j sets prices and payoffs.

Fig. 1 shows effective beliefs in a two-agent economy with complete markets, two states, and diagonal dividend matrix D. Effective beliefs depend on the value of w^1. By construction, effective beliefs and beliefs coincide when an agent has most of the wealth. However, beliefs and effective beliefs differ when both agents have positive wealth, in that assets' payoffs are determined by both agents. In particular, given two agents, the effective beliefs of each agent are a combination of his beliefs with the beliefs of the other agent. The larger the wealth share of one agent, the larger her impact on equilibrium returns, the larger the weight of her beliefs in determining both agents effective beliefs. Discount rates are also important because, by setting the interest rate and thus the level of asset prices, they determine the relative importance of dividends and prices in the total payoff matrix.

[33]The proposition generalizes to I-agent economies by taking all the possible combinations of the two groups effective beliefs.

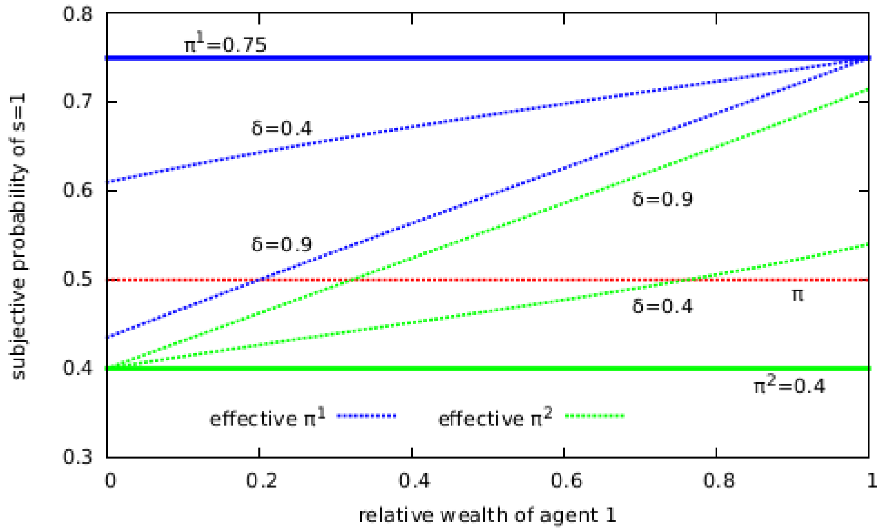

Figure 1: Effective beliefs of two agents with different values of discount rate δ in a complete markets of two assets with $D = I_2$. Since the truth is $\pi = (1/2, 1/2)$, the relative entropy is a symmetric function so that the euclidean vertical distance between a belief $\bar{\pi}^i$ and $\pi = (1/2, 1/2)$ can be used directly to appraise $D(\pi||\bar{\pi}^i)$.

In the example of Fig. 1, agents beliefs are at the opposite side of the truth and agent 2 has more accurate beliefs than agent 1. Since D is diagonal, by Corollary 4.1 agent 2 never vanishes. Whether she dominates or also agent 1 survives it depends on the discount rate, that is on how much non-accuracy of beliefs and non-optimality of the generalized Kelly rule influence each others. Simple calculations (and our numerical exploration of Section 5.2) show that agent 2 dominates when $\delta = 0.4$ whereas both agents survive when $\delta = 0.9$. Effective beliefs confirm this outcome. When $\delta = 0.4$ agent 2 has better effective beliefs both when she has most of the wealth and when she has none, and thus dominates. In fact, she has most accurate effective beliefs for all possible wealth distributions. When $\delta = 0.9$, however, each agent has most accurate effective beliefs when the other agent sets assets returns, so that both agents survive.

The graphical representation clarifies also why, for all $\delta \in (0, 1)$, long-run heterogeneity is the long run outcome when rules have the same relative entropy, see Proposition 4.4. Assume that the beliefs of agent 1 are $\pi^2 = 0.6$ instead of $\pi^2 = 0.75$, so that $\Delta_\pi(\pi^2||\pi^1) = 0$. Effective beliefs, by laying between the two agents beliefs, are such that the condition for long-run heterogeneity is satisfied for all $\delta > 0$.

5.2 2-Generalized Kelly agents economies

In what follows we numerically explore the occurrence of long-run heterogeneity. We start with 2-agent economies.

Diagonal assets Consider an economy with two states of the world where two Generalized Kelly agents trade two assets. Assume $D = I_2$ and fix $\pi = (0.5, 0.5)$. We use the conditions of Proposition 4.3 to characterize long-run outcomes for different values of the economy parameters.

In the left plot of Figure 2, $\delta = 0.8$ and all possible combinations of agents' beliefs are considered. In the right plot we instead set $\pi^2 = (0.6, 0.4)$, and vary the beliefs of agent 1 and the value of δ.

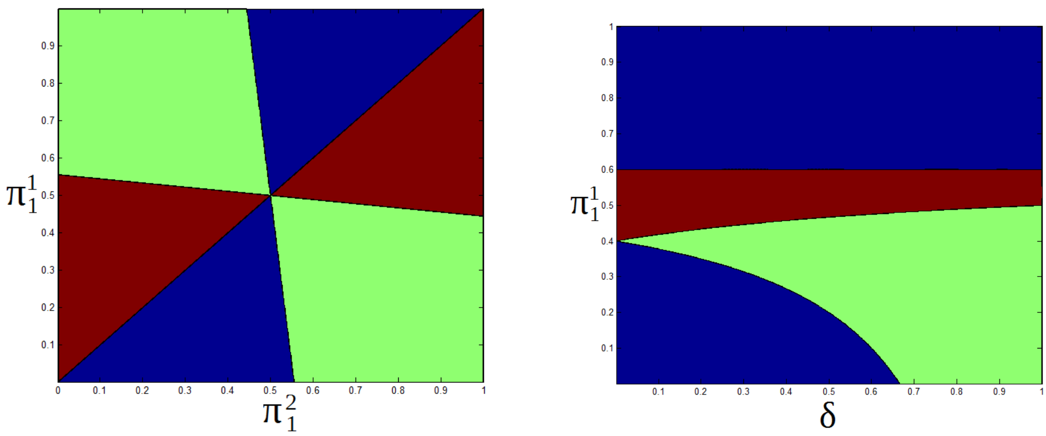

Figure 2: Areas of dominance and survival. Red: agent 1 dominates, blue: agent 2 dominates, green: long-run heterogeneity.

Consistently with the derivation of effective beliefs, long-run heterogeneity occurs only for beliefs that are anti-correlated, that is, when one agent believes that asset 1 pays with probability greater than 1/2 while the other believes the opposite. The figure also confirms the result of Corollary 4.1 for diagonal dividends matrices: the agent with beliefs farthest from the truth never dominates.

In the plot on the right one can notice how the area of long-run heterogeneity shrinks for low values of δ until it disappears when $\delta = 0$. In that limit effective beliefs coincide with beliefs for all values of the wealth distribution so that log-run heterogeneity is only a non-generic phenomenon that takes place when beliefs have the same relative entropy with respect to the truth. The discount factor, by determining the relative size of dividends and prices, also determines the relative importance of other agent beliefs in determining effective beliefs.

To give an idea of how a particular trajectory of the stochastic system looks like, we keep $\delta = 0.8$ and $\pi = (0.5, 0.5)$ while we set $\pi^1 = (0.45, 0.55)$ and $\pi^2 = (0.6, 0.4)$. In Figure 3 we plot the evolution of wealth shares for $T = 1000$ periods when $w_0 = 0.5$. When the wealth share of an agent approaches low values then it is bounced back and, eventually, wealth shares are re-balanced.

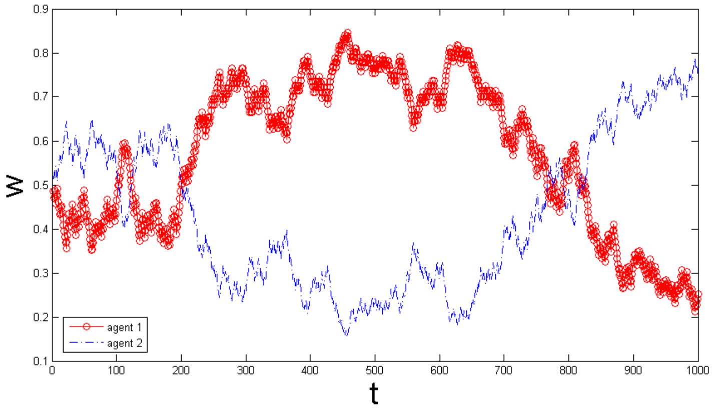

Figure 3: Dynamics of agents' relative wealth shares for $T = 1000$ and a given σ and $w_0 = 0, 5$.

Asset prices (and thus state prices) follow a similar pattern where the bounds are not zero and one but each agent evaluation of the asset stream of dividends given by (8). Figure 4 illustrates, for the same sequence σ, the evolution of risk neutral probabilities.

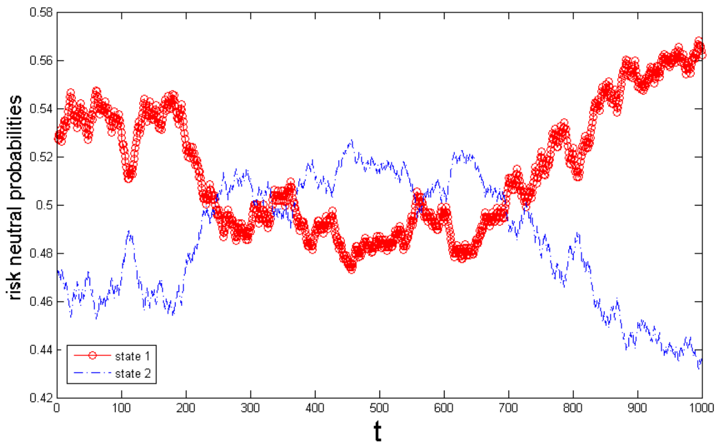

Figure 4: Dynamics of risk neutral probabilities for $T = 1000$ and the same σ and initial condition of Fig. 3.

Averaging over $T = 100000$ iterations, Figure 5 shows the frequency of observation of risk neutral probabilities in the interval of agents' beliefs.[34]

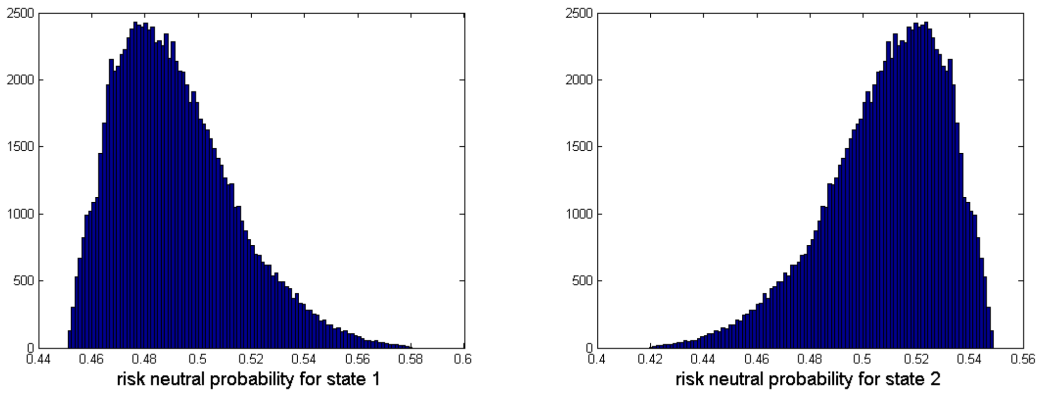

Figure 5: Frequency of risk neutral probabilities for $T = 1000$ and the same σ and initial condition of Fig. 3.

Binomial Tree Consider now the case of the binomial tree economy of Section 2.1 with $r = g_d/g_u = 0.2$, $\pi = (0.5, 0.5)$. As before we can use our conditions to establish what happens for all the possible combinations of beliefs when

[34]Given that there is not aggregate risk in this example, risk neutral probabilities coincides with the belief of the representative agent.

$\delta = 0.8$, left panel of Figure 6, and for all possible combinations of δ and π^1 when $\pi^2 = (0.6, 0.4)$, right panel of Figure 6.

As one can notice from the comparison between Figure 2 and Figure 6, the plots are quite similar, the only difference is that the areas of long-run heterogeneity slightly increase. Moreover, with non-diagonal assets there exist cases of long-run heterogeneity even in the limit of $\delta = 0$.

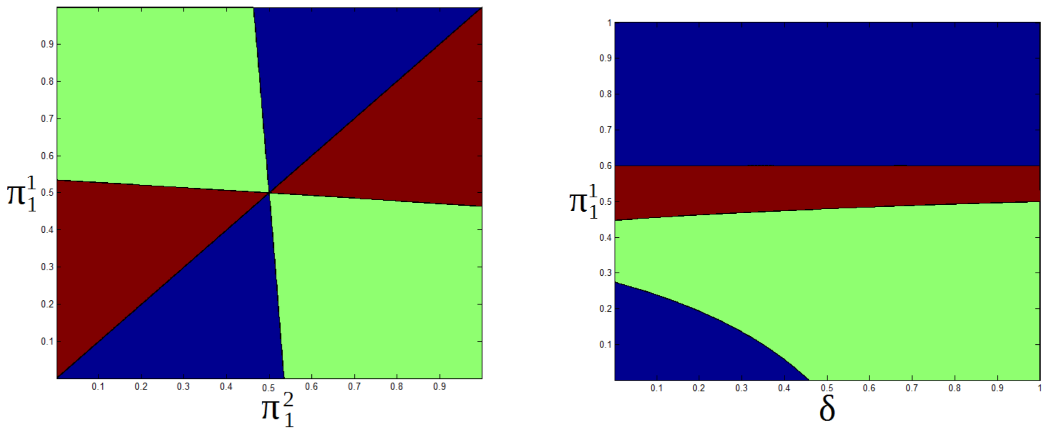

Figure 6: Areas of dominance and survival. Red: agent 1 dominates, blue: agent 2 dominates, green: long-run heterogeneity. $r = 0.2$

In this example, the selection process is less sharp since investing in the second asset is a quite safe way to survive. To shed light, we investigate what happens when we change the value of r. In Figure 7 we plot the areas of dominance and survival for all the possible combinations of beliefs of agent 1 and the parameter r when $\delta = 0.5$ and $\pi^2 = (0.6, 0.4)$.

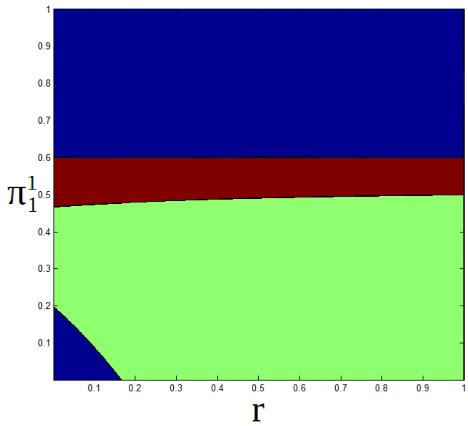

Figure 7: Areas of dominance and survival. Red: agent 1 dominates, blue: agent 2 dominates, green: long-run heterogeneity. $\delta = 0.5$

We continue by exploring the outcomes of market selection under complete and incomplete markets. For this purpose, we take the market structure with two assets and three states of the world shown in Section 2.1 with $r_u = r_m = 0.2$. We also choose $\pi = (1/3, 1/3, 1/3)$, $\delta = 0.5$, $\pi^1 = (3\pi^1_{1,2}/4, \pi^1_{1,2}/4, 1 - \pi^1_{1,2})$ and $\pi^2 = (\pi^2_{1,2}/4, 3\pi^2_{1,2}/4, 1 - \pi^2_{1,2})$.

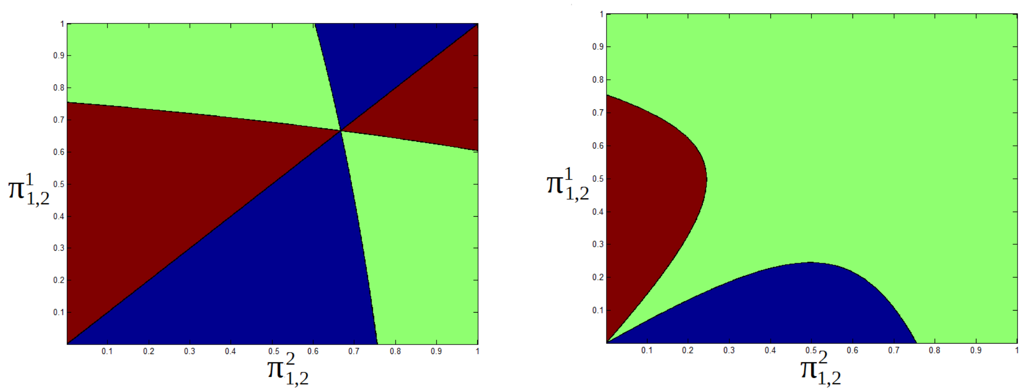

Figure 8: Areas of dominance and survival. Red: agent 1 dominates, blue: agent 2 dominates, green: long-run heterogeneity. Left plot: incomplete markets. Right plot: complete markets.

In the left plot one can observe how the shape of the areas of dominance and survival is similar to those in the first plot of Figure 6, the only difference is that now the truth corresponds to the sum of probabilities π_1 and π_2, hence

2/3. Instead in the second plot (complete markets) the situation is dramatically different: the area of long-run heterogeneity occupies a large portion of the space. That is, completing the market offers a way for agents with anti-correlated beliefs for states 1 and 2 to speculate against each-others. This ends up in increasing the combinations of beliefs that produce long-run heterogeneity.

Obviously the role played by the choice of the belief structure is fundamental. To this regard consider a slightly different belief structure only for agent 1: $\pi^1 = (\pi^1_{1,2}/2, \pi^1_{1,2}/2, 1 - \pi^1_{1,2})$. In this situation agent 1 should be favored since she can distribute more evenly (hence in accordance with the underlying stochastic process) his wealth among assets. Indeed when $\pi^1_{1,2} = 2/3$, she plays the generalized Kelly rule with correct beliefs, hence she dominates in the market no matter what is the value of $\pi^2_{1,2}$. Figure 9 confirms the intuition, the area where agent 1 dominates increases and occupies a large portion of the plot.

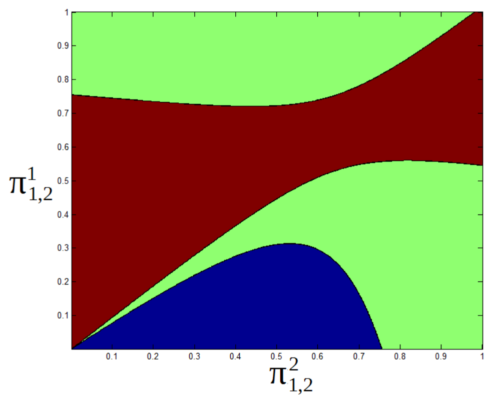

Figure 9: Areas of dominance and survival. Red: agent 1 dominates, blue: agent 2 dominates, green: long-run heterogeneity. Complete markets.

This exercise provides a link with the work of Fedyk et al. (2013) about the welfare effect of enlarging the asset span. Considering a general equilibrium model where one agent has correct beliefs and one has incorrect beliefs, the authors show that the possibility of trading several risky assets does not increase welfare in general while, in most of the cases, it causes a severe welfare loss. A basic feature of their model is that the agent with correct beliefs dominates no matter how many assets are traded, thus the divergence in terms of welfare is triggered by the speed at which the inaccurate agent loses everything. Figures 8 and 9 show, instead, how in our model there exists combinations of beliefs such that, when the asset span increases, the dominant agent changes from agent 2 to agent 1 or we pass from the dominance of one of the two agents to the survival of both. Hence establishing whether a larger asset span can cause a welfare loss becomes much more complex.

Indeed it could be the case that enlarging the asset span increases total welfare.[35]

5.3 3-Generalized Kelly agents economy

In this section, we use our criteria to investigate the market selection outcomes in an economy with three states of the world, complete markets, and three Generalized Kelly agents . Assume $D = I_3$ and $\pi = (1/3, 1/3, 1/3)$. We fix $\pi^1 = (\pi^1_1, (1-\pi^1_1)/2, (1-\pi^1_1)/2)$, $\pi^2 = (1/4, 1/2, 1/4)$ and $\pi^3 = (1/4, 1/4, 1/2)$. We plot the outcome of our criteria for all possible combinations of δ and π^1_1.

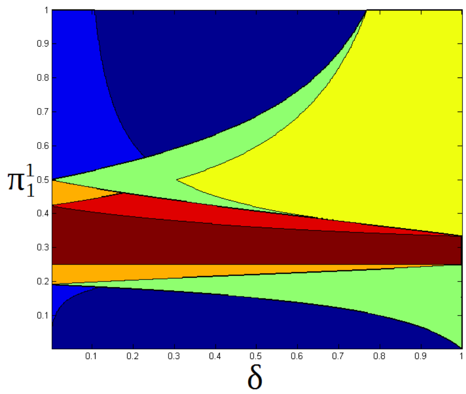

Figure 10: Areas of dominance and survival. Red: agent 1 survives; Dark Red: agent 1 dominates; Blue: agent 1 vanishes; Dark Blue: only agent 2 and agent 3 survive; Green: at least two agents survive; Yellow: all three agents survive; Orange: unknown.

Compared with two-agent economies, in a three-agent economy our sufficient conditions are not tight. Thus, there exists combinations of π^1_1 and δ for which we cannot characterize market selection long-run outcomes. Consider the red regions, for these combinations of π^1_1 and δ agent 1 survives, indeed choosing the group $J = \{1\}$ we have $\underline{\mu}^J(0) > 0$. In the dark red region around the truth, we also have $\underline{\mu}^J(1) > 0$ so that agent 1 dominates. In the blue areas, $\overline{\mu}^J(0) < 0$ and $\overline{\mu}^J(1) < 0$: agent 1 vanishes and group $I \setminus J = \{2, 3\}$ dominates. Regrading the fate of agents 2 and 3, both 2 and 3 can survive or one of the two dominates. Define $J' = \{2\}$ and $J'' = \{3\}$, in the dark blue regions $\underline{\mu}^{J'}(0) > 0$ and $\underline{\mu}^{J''}(0) > 0$, thus both agent 2 and agent 3 survive. In the light blue regions nothing more can be said.

Continuing the analysis, in the yellow region $\underline{\mu}^J(0) > 0$, $\underline{\mu}^{J'}(0) > 0$, and $\underline{\mu}^{J''}(0) > 0$, hence all agents survive. In the green areas $\overline{\mu}^J(1) < 0$, $\overline{\mu}^{J'}(1) < 0$

[35] It remains the difficulty to measure welfare in a framework such ours where rules are not explicitly derived from an utility maximization.

and $\overline{\mu}^{J''}(1) < 0$, so that no one dominates. This is equivalent to say that at least two agents survive. Finally, there also exists a region, the orange one, where our sufficient conditions are too weak to characterize the market selection outcome.

From the previous figure, if we set $\pi^1 = (0.6, 0.2, 0.2)$, $\delta = 0.8$ and $w_0^1 = w_0^2 = w_0^3 = 1/3$ then all agents survive. Simulating the model for a particular realization of the underlying stochastic process we have an example of the dynamics of agents' relative wealth, see Figure 11. Around the period $t = 150$, and again at $t \approx 550$, agent 1 has lost almost all his wealth. However, in later periods, she has still a substantial share of the aggregate output. Notice also that agents relative wealth has not a stable ordering.

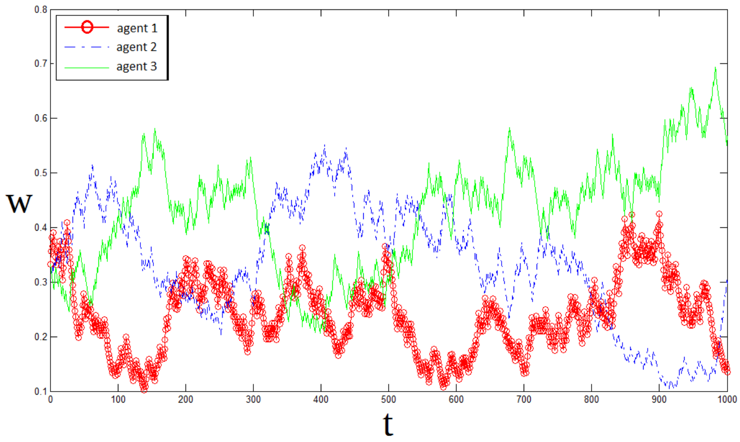

Figure 11: Simulation of the evolution of wealth shares for 1000 periods.

6 Conclusion

In this paper we investigate the MSH in an exchange economy with long-lived assets where agents have homogeneous discount factors, heterogeneous beliefs, and employ generalized Kelly rules, a particular type of fixed-mix portfolios. In this framework Evstigneev et al. (2008) proves that if there exists an agent with correct beliefs, then she gains all the wealth in the long-run. Asset prices converge to those of a Lucas' model where the representative agent has logarithmic preferences. We instead focus on an economy where agents have heterogeneous, and not correct, beliefs, and provide sufficient conditions for an agent to have a positive, null, or unitary fraction of wealth in the long-run.

Our main finding is that there exist initial distributions of beliefs such that beliefs heterogeneity, rather then convergence to the most accurate beliefs, is the long-run outcome. Moreover this result is generic and robust to local perturbation of beliefs. We show that our results are due to the non-optimality of fixed-mix rules in the limit of an agent having a negligible share of the total wealth. Non-optimality of beliefs and non-optimality of the rules balance each other and lead to survival instead than to vanishing.

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A Proofs of Theorems and Lemmas

A.1 Proof of Lemma 3.1

Proof. Let $\bar{\alpha} = \max_{i \in I}\{\alpha^i_h\}$ and $\bar{\delta} = \max_{i \in I}\{\sum_{k=1}^K \alpha^i_k\}$. From **R1** and **R2** it is immediate to see that $0 < A_{h,h} < \bar{\alpha} < 1$ and $0 < \sum_{k=1}^K A_{k,h} < \bar{\delta} < 1$. Then

$$\sum_{k=1}^{K} (I_K - A)_{k,h} = |I_K - A|_{h,h} - \sum_{k=1, k \neq h}^{K} |I_K - A|_{k,h}$$

but at the same time $\sum_{k=1}^K (I_K - A)_{k,h} = 1 - \sum_{k=1}^K A_{k,h} > 1 - \bar{\delta} > 0$ so that the matrix $I_k - A$ is column strictly diagonally dominant and, by the Levy-Desplanques theorem (Taussky, 1949), it is invertible. □

A.2 Proof of Proposition 3.1

Proof. The first part of the statement follows from Lemma 3.1 and from the derivation in the text before the proposition.

Regarding the absence of arbitrages consider the following. According to Stiemke's Lemma, the absence of arbitrage is equivalent to the existence of a vector $q \in \mathbb{R}^S_{++}$ such that $R(w; \alpha, D)q = P$ or, with (12), $Dq = (I_K - A(w; \alpha))P$. The k^{th} component of $(I_K - A(w; \alpha))P$ reads $\sum_{i=1}^I [(1 - \delta^i)\delta^i W^i] x^i_k$. It follows that if for all $i \in I$ x^i belongs to the interior of the convex cone generated by the columns of D, also the vector $(I_K - A(\alpha, w))P_t$ belongs to it and there are no arbitrage. Provided agents beliefs satisfy **R2** generalized Kelly rules belong to the interior of the convex cone generated by the columns of D. □

A.3 Proof of Propositions 4.1

As we shall show, the stochastic process that corresponds to the two groups' relative wealth dynamics has bounded increments. As a result we can prove the proposition by applying Theorem 2.1 in Bottazzi and Dindo (2015).

Consider the variable

$$z^J_t = \log \frac{w^J_t}{1 - w^J_t} \tag{21}$$

such that $z^J_t = z^J_{t-1} + g^J(\sigma_t)$, with $g^J(\sigma_t) = \log G^J(\sigma_t)$ and

$$G^J(\sigma_t) = \frac{\sum_{k=1}^K r_{k,s_t}(w_{t-1}; x, \delta, D) x^J_k(w_{t-1}; x)/p_k(w_{t-1}; x)}{\sum_{k=1}^K r_{k,s_t}(w_{t-1}; x, \delta, D) x^{-J}_k(w_{t-1}; x)/p_k(w_{t-1}; x)} \; . \tag{22}$$

One has the following

Lemma A.1. *The process z_t^J has bounded increments, that is, there exists an $B \in \mathbb{R}$ such that $|z_t^J - z_{t-1}^J| < B$ P-a.s..*

Proof. By **R3** there exists a small enough $\varepsilon > 0$ such that $\varepsilon \leq x_k^J \leq 1 - \varepsilon \ \forall i, k$. Then for any asset k and any time t

$$\varepsilon \leq p_k(w_t; x) \leq 1 - \varepsilon$$

and for any agent i and state s

$$\frac{\varepsilon}{1-\varepsilon} \leq \sum_{k=1}^{K} r_{k,s}(w_{t-1}; x, \delta, D) \frac{x_k^J}{p_k(w_{t-1}; x)} \leq \frac{1-\varepsilon}{\varepsilon} \ .$$

By direct algebraic substitution it is straightforward to verify that

$$2 \log \frac{\varepsilon}{1-\varepsilon} \leq z_t^J - z_{t-1}^J \leq 2 \log \frac{1-\varepsilon}{\varepsilon}$$

and the statement is proven. □

In order to prove Proposition 4.1, note that $\mathrm{E}[g(\sigma_t)] = \mu^J(w_t)$. We shall start from the first statement. If $\underline{\mu}^J(0) > 0$, then, given the continuity of the function, there is a neighborhood of $-\infty$ in which, almost surely, $\mu^J(w_t) > \underline{\mu}^J(0) > 0$. Since z_t^J has bounded increments the Theorem 2.1 in Bottazzi and Dindo (2015) applies and $\mathrm{Prob}\left\{\limsup_{t\to\infty} z_t^J > -\infty\right\} = 1$. The same reasoning applies to the second statement, see also the Corollary 2.1 of Bottazzi and Dindo (2015).

A.4 Proof of Lemma 4.1

Let us consider the process z_t^J in (21) and all the other quantities defined at the beginning of appendix A.3. We begin with the following Lemma.

Lemma A.2. *If the set of rules are not overlapping,* **R4***, and if there are no redundant assets,* **D4***, then z_t^J does not possess any deterministic fixed point, that is $\nexists\, z$ s.t. $\mathrm{P}(z_t^J = z | z_{t'}^J = z) = 1 \ \forall t > t'$.*

Proof. Suppose such z exists and at a certain time $t-1$ it is $z_{t-1}^J = z$. Then, by definition, it holds that $z_t^J - z_{t-1}^J = 0$ for all the possible states of the world $s = 1, 2, ..., S$, so that

$$\sum_{k=1}^{K} r_{k,s}(w_{t-1}; x, \delta, D) \left(\beta_k^J(w_{t-1}; x) - \beta_k^{-J}(w_{t-1}; x)\right) = 0 \quad \forall s = 1, 2, ..., S \ .$$

That is

$$\left(\beta^J(w_{t-1};x) - \beta^{-J}(w_{t-1};x)\right)\,\left((I - \delta A(w_{t-1};x))^{-1} D\right) = \mathbf{0}\ .$$

The trivial solution $\beta^J = \beta^{-J}$ is excluded by **R4** and according to Proposition 3.1 the kernel of $(I - \delta A(w_{t-1}, x))^{-1} D$ is zero, implying that the system of equations has no solution and the statement is proven. $\square$

The proof proceeds by noticing that G^J in (22) depends on history σ_t thorough the wealth distribution w_t and the last realizes state s_t. Given the distribution $w \in \Delta^I$ define

$$\bar{G}^J(\sigma_t) = \max_{s=1,\dots,S} \{|G^J(w, s_t)|\}\ ,$$

which, being the upper envelope of continuous functions, is a continuous function on the compact set Δ^I. Then, by the Weierstrass theorem, it has a minimum $\underline{G}$. Moreover it is $\underline{G} > 0$ because, otherwise, z_t^J would have a deterministic fixed point, which is not possible by Lemma A.2. Then

$$\text{Prob}\left\{|z_t^i - z_{t-1}^i| \geq \underline{g}\,|\,\Im_{t-1}\right\} \geq \rho = \min\{\pi_1, \dots, \pi_S\}\ .$$

and by taking $\gamma = \min\{\underline{g}, \rho\}/2$ the assertion is proved.

A.5 Proof of Lemma 4.2

Let us consider the first statement. If it is wrong then

$$\frac{w_{t+1}^{-J}}{w_t^{-J}} - \frac{w_{t+1}^{J}}{w_t^{J}} = \sum_{k=1}^{K} (\beta_k^{-J} - \beta_k^{J})\, r_{k,s}(w_{t-1}; x, \delta, D) \geq 0 \quad \forall s$$

and since the process does not admit any deterministic fixed point (c.f. Lemma A.2), the inequality is strict for some s'. For construction it is

$$\sum_{k=1}^{K} (\beta_k^{-J} - \beta_k^{J}) p_k(w_{t-1}; x) = 0\,,$$

so that $\beta^{-J}(w_{t-1}; x) - \beta^J(w_{t-1}; x)$ would be a weak arbitrage, which contradicts the hypotheses. For the second statement one can reason following the same lines and, in order to complete the proof, it is enough to choose $\epsilon = \min_s\{\pi_s\}/2$.

A.6 Proof of Proposition 4.2

Consider the process z_t^J in (21) and all the other quantities defined at the beginning of appendix A.3. If the set of rules used by the agents belonging to group J and those used by the agents in $I \setminus J$ are separated, **R4**, then it follows from Lemma 4.1 that z_t^J has always a finite probability to have a finite jump.

By Lemma 4.2, the lack of weak arbitrage implies, in turn, that the process cannot have a deterministic drift. Thus, Lemmas 4.1 and 4.2 together imply that the process z_t^J jumps to the left and to the right of finite amount with a finite probability.

As show in Lemma A.1, z_t^J has finite increments (in both directions). The condition $\underline{\mu}^J(0) > 0$ and $\underline{\mu}^J(1) > 0$ implies that $\mu^J(w) > 0$ for sufficiently small and sufficiently large values of w. Using Theorem 3.1 from Bottazzi and Dindo (2015) it follows that $\text{Prob}\,\{\lim_{t\to\infty} z_t^J = +\infty\} = 1$, group J dominates. Conversely, from the condition $\overline{\mu}^J(0) < 0$ and $\overline{\mu}^J(1) < 0$ we have that $\mu^J(w) < 0$ for sufficiently small and large values of w. Using Corollary 3.1 of Bottazzi and Dindo (2015), it follows that $\text{Prob}\,\{\lim_{t\to\infty} z_t^J = -\infty\} = 1$ and the group J vanishes.

A.7 Proof of Lemma 4.3

From the definition of conditional drift

$$\mu(0) = \sum_{s=1}^{S} \pi_s \log\left(\delta + (1-\delta)\sum_{k=1}^{K} d_{k,s}\frac{x_k^1}{x_k^2}\right),$$

$$\mu(1) = -\sum_{s=1}^{S} \pi_s \log\left(\delta + (1-\delta)\sum_{k=1}^{K} d_{k,s}\frac{x_k^2}{x_k^1}\right).$$

Exploiting the concavity of the logarithmic function and considering that $0 \leq d_{k,s} \leq 1$ for all s,k and that $\sum_{k=1}^{K} d_{k,s} = 1$ for all s, we can see that

$$\mu(0) > (1-\delta)\sum_{s=1}^{S} \pi_s \log\left(\sum_{k=1}^{K} d_{k,s}\frac{x_k^1}{x_k^2}\right) \geq (1-\delta)\sum_{k=1}^{K} x_k^* \log\left(\frac{x_k^1}{x_k^2}\right).$$

At the same time

$$\mu(1) < -(1-\delta)\sum_{s=1}^{S} \pi_s \log\left(\sum_{k=1}^{K} d_{k,s}\frac{x_k^2}{x_k^1}\right) \leq (1-\delta)\sum_{k=1}^{K} x_k^* \log\left(\frac{x_k^1}{x_k^2}\right).$$

Putting together the two inequalities proves that $\mu(0) > (1-\delta)\Delta_{x^*}(x^2||x^1) > \mu(1)$.

A.8 Proof of Proposition 4.3

The statement follows from the particular case of Proposition 4.2 when $I = 2$ together with Lemma 4.3 and the definition of normalized prices.

A.9 Proof of Corollary 4.1

The proof of the survival of agent 1 follows from Lemma 4.3 and Proposition 4.3. When agent 1 beliefs are correct, so that $x^1 = x^*$, she also dominates since, exploiting the strict convexity of $-\log(\cdot)$,

$$\mu(1) = \sum_s \pi_s \left(-\log\left(\delta + (1-\delta)\sum_{k=1}^{K} d_{k,s}\frac{x_k^2}{x_k^*}\right)\right) > \\ -\log\left(\delta + (1-\delta)\sum_{k=1}^{K}\frac{x_k^2}{x_k^*}\sum_s \pi_s d_{k,s}\right) = -\log\left(\delta + (1-\delta)\sum_{k=1}^{K}\frac{x_k^2}{x_k^*}x_k^*\right) = 0\,.$$

A.10 Proof of Proposition 4.4

Notice that the conditional drift in the case of Generalized Kelly agents is a continuous function of agents' belief, that is

$$\mu(w^1) = \mu(w^1; \pi^1, \pi^2)\ .$$

Thus the result simply follows applying the standard definition of continuity.

A.11 Proof of Proposition 4.5

The statement follows from the properties of the function $D(x^*||x) : \Delta^K \to \mathbb{R}_+$, $x \mapsto D(x^*||x)$. In particular it is a continuous strictly convex function with a minimum equal to zero in $x = x^*$. Thus it is defined over the compact set $\partial(\bar{\Delta})$ and there exists a minimum over this set because of the Weirstrass theorem. The strict convexity of $D(x^*||x)$ implies that it is also strictly quasi convex. This property together with the fact that $x^* \in \bar{\Delta}$ implies $\{x : D(x^*||x) < \mathcal{K}\} \subseteq \bar{\Delta}$. Hence, it is possible to choose a $\pi^1 \neq \pi$ such that $D(x^*||x^1) = m < \mathcal{K} - \epsilon$ with $\epsilon > 0$ and small enough. Then, one can easily define the set $\Pi = \{\pi' : \pi' \in \Delta_+^S, D(x^*||x') = m\}$ which has always at least two elements[36]. Choosing x^1 and x^2 such that $\pi^1, \pi^2 \in \Pi$ it is $\Delta_{x^*}(x^2||x^1) = 0$.

[36] The worst case scenario is $S = 2$ and in this case the cardinality of Π' is two.

A.12 Proof of Proposition 5.1

An asset market economy with agents trading according to generalized Kelly rules and an asset market economy with agents maximizing expected log-utilities under effective beliefs have, by construction, the same relative wealth dynamics. When markets are dynamically complete and agents maximize an expected log-utility, there is no loss of generality, in assuming that they are trading all possible contingent commodities in date zero. In fact, all assets structure, as long as they are complete, allow agents to achieve the same consumption allocation, so that the relative wealth dynamics does not on the asset structure. Under time-zero trading it is well known that agents allocate in each state contingent good a fraction of wealth proportional to the state likelihood. In a two-agent economy, the relative wealth dynamics can thus be re-written as

$$\frac{w^1_{t+1}(\sigma_t, s_{t+1})}{w^2_{t+1}(\sigma_t, s_{t+1})} = \frac{\bar{\pi}^1_{s_{t+1}}(w_t; \delta, D)}{\bar{\pi}^2_{s_{t+1}}(w_t; \delta, D)} \frac{w^1_t(\sigma_t)}{w^2_t(\sigma_t)} \quad \forall \sigma_t, s_{t+1}, t\,.$$

Applying the definition of $\mu(\cdot)$ and remembering that the relative wealth dynamics just found is the same as the original one with long-lived assets and generalized Kelly traders, leads to the result.