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Drift criteria for persistence of discrete stochastic processes on the line with examples of application

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Drift criteria for persistence of discrete stochastic processes on the line with examples of application

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Abstract

We provide sufficient conditions for the persistence or transience of stochastic processes on the real line based on the behavior of the first and second moment of their conditional increments at the boundaries. Our findings extend previous results in the literature (Lamperti, 1960) to the large class of discrete-time processes with bounded increments. We present some examples of application in the domain of economics.

Keywords: Discrete-time stochastic processes; asymptotic behavior; persistence and transience.

JEL Classification: C02

1 Introduction

Many economic dynamic phenomena can be represented by a real stochastic processes in discrete time. The dynamics of consumption, capital, and output in a growth model with shocks to the production process; the dynamics of equilibrium allocation and asset prices in stochastic exchange economy; the dynamics of optimal strategies, populations shares, and payoffs in repeated (evolutionary) games are understood examples. In all these processes, an exogenous state s_t with histories σ^t moves the endogenous state $x_t(\sigma^t)$ making it a process adapted to the filtration generated by the histories σ^t. This is typically the case in economies with heterogeneous agents, the standard in evolutionary game theory and now more and more popular also in macroeconomics and finance, where an important state variable is the relative consumption, wealth or payoffs of different groups of agents. Assessing the asymptotic dynamics of these quantities is instrumental to the characterization of the long-run behavior of the model.

The aim of this paper is to propose sufficient conditions for a process x_t on the real line to be persistent or transient. The process is persistent when with probability one it keeps visiting a bounded interval, it is transient when it diverges. We propose two sets of sufficient conditions. The first set of conditions is based on the sign of the asymptotic drift. The second set is more general and is based on a suitable quantity computed starting from the asymptotic first and second moment of the conditional increment. Thus, the second set of conditions can be applied also to the situation, often arising in practice, in which the drift is asymptotically zero. Both sets of conditions are inspired to and extend the seminal work of Lamperti (1960) and are presented in Section 3 for persistence and in Section 4 for transiency. As a novelty with respect to incumbent approaches, the proposed conditions work under minimal assumptions on the process. In particular, the Markov property is not required and they also apply to the case of a finite set of states.[1]

The first set of conditions, based on the asymptotic sign of the drift, are rather intuitive. In summary, if the drift conditional on a large enough positive state is negative and, at the same time, the drift conditional on a large enough negative state is positive, then the process is persistent. In this case either the process converges, to a deterministic value or to a random variable, or keeps fluctuating. Conversely, if the drift conditional on a large enough positive state is positive and, at the same time, the drift conditional on a large enough negative state is negative, then the process is transient. In this case, the process diverges, possibly displaying path dependency, that is, it could diverge to either extreme of the line depending on the initial condition and the realization of the underlying exogenous process. Finally, a positive (negative) drift for large enough, both positive and negative, states implies that the process is transient and diverges to plus (minus) infinity.

The second set of conditions relaxes the prescriptions on the asymptotic sign of the drift. For example, having a negative drift for large enough positive states can be replaced by having a drift that, for large enough positive states, approaches

[1] Meyn and Tweedie (1993) is the reference for Markov process in discrete time with a continuous support, as we have here. Another important reference with economic applications is Bhattacharya and Majumdar (2007).

zero at a rate which is fast enough if compared to the asymptotic behavior of the second moment of the conditional increment. Similar conditions can be derived for other limit values of the drift.

In Section 5, we apply the drift conditions to three specific examples. First, in Section 5.1, we consider the dynamics of population shares in an evolutionary game with stochastic payoffs (see e.g. Cabrales, 2000).[2] In this example we show how our sufficient conditions, not requiring the Markov property, can be used on aggregate variables, such as the ratio of the population shares of two groups.

Then, in Section 5.2, we consider a stochastic exchange economy with complete markets and two SEU-CRRA agents having heterogeneous beliefs, discount factor, and risk aversion. Here the endogenous process under study is the logarithm of relative consumption. The drift conditions are used to link survival and dominance to a survival index which is agent and history specific. The example belongs to the literature on the Market Selection Hypothesis (see e.g. Blume and Easley, 2010, for a survey) and extends Yan (2008) to general endowment processes. Although here we restrict to a 2-agent economy, the analysis can be easily extended to two groups of agents as done in the previous example.

In the last example, Section 5.3, we study the convergence of a Bayesian learner who under-reacts to information and has a misspecified prior support, combining Berk (1966) and Epstein et al. (2008). Under the special assumption that the support has only two models, we use the prior distribution as a state.

2 Persistency and Transiency

Consider a discrete process on the real line, $\{x_t\}$, and let $(P, \Sigma, \Im)$ be its underlying filtered probability space with Σ a subset of sequences of real numbers σ, $\{\Im_t\}$ a filtration of Borel σ-fields on Σ, and P an associated probability measure. We are interested to investigate whether, in the long run, the process persistently visits a finite set or, alternatively, it diverges to infinity. In particular, we want to investigate if the asymptotic sign of the conditional drift allows us to decide between to the two alternatives. The results we present concern specifically the following[3]

Definition 2.1. A real stochastic process $\{x_t\}$ is *persistent* if there exists a real

[2]Another source of randomness in evolutionary games is the (random) matching process when the population is finite (see e.g. Taylor et al., 2004; Traulsen and Hauert, 2009).

[3]In what follows, the expression *almost surely* (a.s.) means "apart from a set of histories of measure zero with respect to P".

interval $A = (a, b)$ such that for any t it is $\text{Prob}\{x_{t'} \in A \text{ for some } t' > t\} = 1$. If a process is not persistent, then it is *transient*.

A process is persistent when there exists a recurrent set A, a set that is visited in finite time with full probability. If a process is not persistent, then there is a positive probability that $\lim_{t\to\infty} |x_t| = +\infty$. Notice that, in general, it is not sufficient to characterize the supremum or infimum limit of a process to know if it is persistent or transient. This is immediately clear if one considers the following list of persistent processes:

- a convergent process, $\text{Prob}\{\lim_{t\to\infty} x_t = x^*\} = 1$ with x^* finite; any interval A that contains x^* can be used to show that Definition 2.1 applies. In this case it is $\text{Prob}\{\limsup_{t\to\infty} x_t = x^*\} = 1$ and $\text{Prob}\{\liminf_{t\to\infty} x_t = x^*\} = 1$;
- a process for which there exists a set $A \subset \mathbb{R}$ and a $T \in \mathbb{N}$ such that $\text{Prob}\{x_t \in A \,\text{when}\, t > T\} = 1$. In this case it is $\text{Prob}\{\limsup_{t\to\infty} x_t \leq \sup A\} = 1$ and $\text{Prob}\{\liminf_{t\to\infty} x_t \geq \inf A\} = 1$;
- the symmetric random walk on the line, $x_t = x_{t-1} + b_t$ where b_t is a Bernoulli variable taking values 1 and -1 with the same probability. In this case it is $\text{Prob}\{\limsup_{t\to\infty} x_t = +\infty\} = 1$ and $\text{Prob}\{\liminf_{t\to\infty} x_t = -\infty\} = 1$;
- a sub-martingale bounded from above; in this case the Martingale Converge Theorem guarantees that the process converges to a finite random variable $\hat{x}$. Any interval A that contains the support of the random variable applies.

Examples of transient processes are instead

- the asymmetric random walk on the line, $x_t = x_{t-1} + b_t$ where b_t is a Bernoulli variable taking values 1 and -1 with different probabilities; in this case $\text{Prob}\{\limsup_{t\to\infty} x_t = \liminf_{t\to\infty} x_t\} = 1$ and these limits are $\pm\infty$ depending on the sign of the drift;
- the process with exploding increments $x_t = x_{t-1} + 2^t b_t$, where b_t is a Bernoulli variable taking values 1 and -1 with fixed probabilities; in this case $\text{Prob}\{\limsup_{t\to\infty} x_t = \liminf_{t\to\infty} x_t\} = 0$ but $\text{Prob}\{\limsup_{t\to\infty} |x_t| = +\infty\} = 1$.

The last process, with exploding fluctuations, is somehow bizarre and we are happy to constrain our analysis to processes that comply with the following

Definition 2.2. A process $\{x_t\}$ has *bounded increments* if there exists a $B > 0$ such that $\text{Prob}\{|x_{t+1} - x_t| < B\} = 1$.

Definition 2.1 is different from the one provided in Lamperti (1960) for a recurrent process. The reason is that there only non-negative processes, $x_t \geq 0$, with $\text{Prob}\{\limsup_{t\to\infty} x_t = +\infty\} = 1$, are considered. While the two definitions are similar in spirit, here we consider more general real processes (albeit bounded).

3 Persistent processes

Let $\mu_t(x) = \text{E}[x_{t+1}|x_t = x, \Im_t] - x$ be the drift of the process in x, that is, conditional on the event $\{x_t = x\}$. The first result clarifies that a process that has a positive drift for sufficiently small realizations is bounded away from minus infinity.

Theorem 3.1. *Consider a bounded increments process $\{x_t\}$. If there exists $M > 0$ such that, for all $x < -M$ and definitely in t, it is $\mu_t(x) > 0$ almost surely, then* $\text{Prob}\{\limsup_{t\to\infty} x_t > -M\} = 1$.

Proof. Let $B > 0$ be such that $|x_{t+1} - x_t| < B$ almost surely. Without loss of generality we can take $M > B$. For any fixed integer T consider the process

$$Y_t^T = \begin{cases} x_{T+t} & \text{if} \quad x_l < -M \quad \text{for} \quad T \leq l \leq T+t-1 \,, \\ 0 & \text{otherwise} \,. \end{cases}$$

The state 0 is clearly absorbing, so that if $Y_t^T = 0$, then $Y_{t+1}^T = Y_t^T = 0$. If $Y_t^T < -M$ then $x_{T+t} = Y_t^T < -M$ and consequently $Y_{t+1}^T = x_{T+t+1} < 0$ almost surely. Let $I(.)$ be the indicator function, that is $I(x)$ is equal to 1 if $x > 0$ and 0 otherwise. On the events such that $Y_t^T = X_{T+t}$ one has

$$\text{E}[Y_{t+1}^T|Y_t^T, \Im_t] = \text{E}[I(-x_{T+t+1} - M)\, x_{T+t+1}|x_{T+t}, \Im_t] \geq \text{E}[x_{T+t+1}|x_{T+t} < -M, \Im_t].$$

The latter is greater than $x_{T+t} = Y_t^T$ by the assumption on the drift. Thus, $\text{E}[Y_{t+1}^T|Y_t^T, \Im_t] \geq Y_t^T$ and the process Y_t^T is a sub-martingale bounded from above by 0. By the Martingale Convergence Theorem there exists a finite random variable $\hat{Y}^T$ such that $\lim_{t\to\infty} Y_t^T = \hat{Y}^T$ almost surely.

Assume that for some T it is $\hat{Y}^T < -M$ with positive probability. Then on a positive measure set of realizations it would be $\{Y_t^T\} = \{x_{t+T}\}$ and $\lim_{t\to\infty} x_{T+t} = \hat{Y}^T < -M$. The latter is absurd given the strictly positive drift of the process when $x < -M$. It follows that for any T it is $\hat{Y}^T = 0$ with probability 1. This implies that for any T there exists a t such that $x_{T+t} > -M$ a.s. and proves the assertion. □

In other terms, the previous result proves that the event $x_t > -M$ is recurrent for the process $\{x_t\}$. The expression "definitely in t" means that the possible violation of the condition for a finite number of t's does not change the result.

It is also possible to find a weaker sufficient condition based on the second conditional moment of the process increment.[4]

Let $v_t(x) = \mathrm{E}\left[(x_{t+1} - x_t)^2 | x_t = x, \Im_t\right]$ be the second moment of the increment in x, that is, conditional on the event $\{x_t = x\}$.

Corollary 3.1. *Consider a bounded increments process $\{x_t\}$. If there exist $\epsilon, \delta > 0$ such that, definitely in t and almost surely, it is*

$$\liminf_{x \to -\infty} \mu_t(x) - \frac{v_t(x)}{2x} - \frac{\epsilon}{|x|^{2-\delta}} \geq 0 \,, \tag{1}$$

then there exists an $M > 0$ such that, definitely in t and almost surely, it is $\mathrm{Prob}\left\{\limsup_{t \to \infty} x_t > -M\right\} = 1$.

Proof. Let $B > 0$ be such that $|x_{t+1} - x_t| < B$ almost surely and let $F_{x,t}(u)$ denote the conditional distribution of the increment $u_t = x_{t+1} - x_t$. Consider the process

$$y_t = \begin{cases} \ln(1 + x_t) & x_t \geq 0 \,, \\ -\ln(1 - x_t) & x_t < 0 \,. \end{cases}$$

Notice that the process $\{y_t\}$ has bounded increments. Assume $x < -B$ and let $\mu^y_t(x) = \mathrm{E}\left[y_{t+1} - y_t | y_t = -\ln(1 - x), \Im_t\right]$. Using the Taylor expansion and the Lagrange form of the remainder one has

$$\mu^y_t(x) = \int_{-B}^{B} dF_{x,t}(u) \, \ln(1 - x) - \ln(1 - x - u) \geq \\ \frac{\mu_t(x)}{1 - x} + \frac{v_t(x)}{2(1 - x)^2} - \frac{B^3}{3\,(1 - x - B)^3} \,,$$

which by the inequality in (1), after a rearrangement of terms, implies, for negative and large enough value of x,

$$\mu^y_t(x) \geq \frac{\epsilon}{(1 - x)|x|^{2-\delta}} + \frac{v_t(x)}{2x(1 - x)^2} - \frac{B^3}{3\,(1 - x - B)^3} \,.$$

When $x \to -\infty$, the first term of the right hand side is the leading term and it is positive. This implies that there exists a sufficiently large M such that if

[4] The derivation is based on an intuition in Lamperti (1960) but the statement is made more general here.

$x < -M$, it is $\mu_t^y(x) > 0$ a.s. and definitely in t. Thus the process $\{y_t\}$ satisfies the hypothesis of Theorem 3.1 and Prob $\{\limsup_{t\to\infty} y_t > -\log(1+M)\} = 1$, hence the assertion. □

The previous result guarantees that the process is bounded away from minus infinity also when its conditional drift for large negative values is negative, provided it goes to zero sufficiently fast.

Along the same lines of Theorem 3.1, one can easily obtain an analogous result for the process at plus infinity.

Corollary 3.2. *Consider a bounded increments process* $\{x_t\}$*. If there exist* $M > 0$ *such that, for all* $x > M$ *and definitely in* t*, it is* $\mu_t(x) < 0$ *almost surely, then* Prob $\{\liminf_{t\to\infty} x_t < M\} = 1$.

Analogously, a weaker condition can be derived, based on the second moment of the conditional increment.

Corollary 3.3. *Consider a bounded increments process* $\{x_t\}$*. If there exist* $\epsilon, \delta > 0$ *such that, definitely in* t *and almost surely, it is*

$$\limsup_{x\to+\infty} \mu_t(x) - \frac{v_t(x)}{2x} + \frac{\epsilon}{|x|^{2-\delta}} \leq 0\,, \tag{2}$$

then there exists an $M > 0$ *such that, definitely in* t *and almost surely, it is* Prob $\{\limsup_{t\to\infty} x_t < M\} = 1$.

Proof. Consider the process $\{y_t\}$ as defined in the proof of Cor. 3.1. For sufficiently large x its drift read $\mu_t^y(x) = \mathrm{E}\left[y_{t+1} - y_t | y_t = \ln(1+x), \Im_t\right]$ and one has

$$\mu_t^y(x) = \int_{-B}^{B} dF_{x,t}(u) \ln(1+x+z) - \ln(1+x) \leq$$

$$\frac{\mu_t(x)}{1+x} - \frac{v_t(x)}{2(1+x)^2} + \frac{B^3}{3\,(1+x-B)^3}\,.$$

Substituting the inequality in (2), one observes that the leading term of the expansion is negative and $\limsup_{x\to+\infty} \mu_t^y(x) < 0$. Following a reasoning analogous to the one in the proof of Cor. 3.1, one easily derives the statement. □

Starting from the previous results, a set of sufficient conditions for the persistence of the process can be derived. First, we propose a theorem based on the asymptotic sign of the conditional drift, based on the results of Theorem 3.1 and Corollary 3.2.

Theorem 3.2. *Consider a bounded increments process* $\{x_t\}$*. If, definitely in* t *and almost surely, for sufficiently large positive values of* x *it is* $\mu_t(x) < 0$ *and for sufficiently large negative values of* x *it is* $\mu_t(x) > 0$*, then the process is persistent.*

Proof. Let $M > 0$ be such that it is $\mu_t(x) < 0$ if $x > M$ and $\mu_t(x) > 0$ if $x < -M$, for any t and almost surely. Let $B > 0$ be such that $|x_{t+1} - x_t| < B$ almost surely. Without loss of generality we can take $M > B$. For any positive integer T define the process

$$Y_t^T = \begin{cases} |x_{T+t}| & \text{if} \quad |x_l| > M \quad \text{for} \quad T \leq l \leq T+t-1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

The state 0 is clearly absorbing so that if $Y_t^T = 0$, then $Y_{t+1}^T = Y_t^T = 0$. Let $I(.)$ be the indicator function, that is $I(x)$ is equal to 1 if $x > 0$ and 0 otherwise. If $Y_t^T > M$, then either $x_{T+t} > M$ or $x_{T+t} < -M$. In the first case, it is $x_{T+t+1} > 0$ almost surely and on the events such that $Y_t^T = x_{T+t}$ one has

$$\begin{aligned} \mathrm{E}[Y_{t+1}^T | Y_t^T, \Im_t] &= \mathrm{E}[I(x_{T+t+1} - M)\, x_{T+t+1} | x_{T+t}, \Im_t] \\ &\leq \mathrm{E}[x_{T+t+1} | x_{T+t}, \Im_t] < x_{T+t} = Y_t^T\;. \end{aligned}$$

In the second case, it is $x_{T+t+1} < 0$ almost surely and on the events such that $Y_t^T = x_{T+t}$ one has

$$\begin{aligned} \mathrm{E}[Y_{t+1}^T | Y_t^T, \Im_t] &= -\,\mathrm{E}[I(-x_{T+t+1} - M)\, x_{T+t+1} | x_{T+t}, \Im_t] \\ &\leq -\,\mathrm{E}[x_{T+t+1} | x_{T+t}, \Im_t] < -x_{T+t} = Y_t^T\;. \end{aligned}$$

Summarizing, it is always the case that $\mathrm{E}[Y_{t+1}^T | Y_t^T, \Im_t] \leq Y_t^T$. Hence Y_t^T is a super-martingale bounded from below by 0 and by the Martingale Convergence Theorem there exists a random variable $\hat{Y}^T$, such that $\lim_{t\to\infty} Y_t^T = \hat{Y}^T$ almost surely.

If for some T it is $\hat{Y}^T > M$ with finite probability, then on a positive measure set of realizations it would be $\lim_{t\to\infty} |x|_{T+t} = \hat{Y}^T$. The latter is absurd given the non-zero drift of the process in those regions. Thus $\hat{Y}^T = 0$ with probability 1. This means that for any T there exists a t such that $y_{T+t} \in (-M, M)$ and the assertion is proved. □

The second set of sufficient conditions can be obtained from Corollaries 3.1 and 3.3.

Corollary 3.4. *Consider a bounded increments process* $\{x_t\}$*. If there exist* $\epsilon, \delta > 0$ *such that, definitely in* t *and almost surely,*

$$\begin{aligned} \liminf_{x \to -\infty} \mu_t(x) - \frac{v_t(x)}{2x} - \frac{\epsilon}{|x|^{2-\delta}} \geq 0 \ \textit{and} \\ \limsup_{x \to +\infty} \mu_t(x) - \frac{v_t(x)}{2x} + \frac{\epsilon}{|x|^{2-\delta}} \leq 0 \,, \end{aligned} \tag{3}$$

then the process is persistent.

Proof. Consider the process $\{y_t\}$ as defined in the proof of Cor. 3.1. Derive the expressions of the left and right asymptotic conditional drift of $\{y_t\}$, as in Cor. 3.1 and 3.3 respectively. If (3) applies, then the process $\{y_t\}$ fulfills the hypothesis of Theorem 3.2, whence the assertion. □

4 Transient processes

Having derived sufficient conditions for the process to be persistent, we want to do the same for transience. For this purpose, we need to restrict our investigation to processes that do not have finite fixed points.

Definition 4.1. A process $\{x_t\}$ has *finite positive increments* if there exists a $\epsilon_L > 0$ such that a.s. $\text{Prob}\{x_{t+1} > x_t + \epsilon_L\} > \epsilon_L$. A process $\{x_t\}$ has *finite negative increments* if there exists a $\epsilon_L > 0$ such that a.s. $\text{Prob}\{x_{t+1} < x_t - \epsilon_L\} > \epsilon_L$.

For the present discussion, the essential feature of a process with finite positive increments is that its asymptotic supremum (limsup) cannot be a finite number, apart, possibly, for a zero-measure set of realizations. In the case of a process with finite negative increments, the same is true for the asymptotic infimum (liminf).

The following result shows that a bounded process with finite positive increments and positive drift outside a finite set diverges to positive infinity and, consequently, is transient.

Theorem 4.1. *Consider a bounded increments process* $\{x_t\}$ *with finite positive increments. If there exists* $\epsilon, \delta > 0$ *such that, definitely in* t *and almost surely,*

$$\liminf_{x \to \pm\infty} \mu_t(x) - \frac{v_t(x)}{2\,x} - \frac{\epsilon}{|x|^{2-\delta}} \geq 0 \tag{4}$$

then the process is transient and $\text{Prob}\{\lim_{t \to \infty} x_t = +\infty\} = 1$.

Proof. Since the process satisfies the condition of Cor. 3.1, there exists an M' such that $\text{Prob}\{\limsup_{t\to\infty} x_t > -M'\} = 1$. Together with the fact that the process has finite positive increments, the latter implies that $\limsup_{t\to\infty} x_t = +\infty$. Let $B > 0$ be such that $|x_{t+1} - x_t| < B$ almost surely and consider a positive real number $M > \max\{B, 1\}$. Define the process

$$y_t = \begin{cases} 1 - \frac{1}{x_t} & \text{if} \quad x_t > M \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Since $M \geq 1$, the process $\{y_t\}$ is bounded in $[0, 1]$. Moreover one has that $\text{Prob}\{\limsup_{t\to\infty} y_t = 1\} = 1$. Let $K \geq 1 - 1/(M + B)$. If $y_t > K$, then $x_t > 1/(1 - K) \geq M + B$ and, with probability one, $x_{t+1} > M$, thus

$$\text{E}[y_{t+1} - y_t | y_t > K, \Im_t] = \int_{-B}^{B} dF_{x_t,t}(u) \, \frac{1}{x_t} - \frac{1}{x_t + u} \geq \frac{\mu_t(x_t)}{x_t^2} - \frac{v_t(x_t)}{2x_t^3} - \frac{B^3}{3\,(x - B)^4} \ ,$$

where $F_{x,t}(u)$ denote the conditional distribution of the increment $u_t = x_{t+1} - x_t$ if $x_t = x$. For sufficiently large x_t, substituting inequality (4), one gets

$$\text{E}[y_{t+1} - y_t | y_t > K, \Im_t] \geq \frac{\epsilon}{x_t^{4-\delta}} - \frac{B^3}{3\,(x_t - B)^4} \ .$$

Upon choosing K large enough, the above condition applies, so that $\text{E}[y_{t+1} - y_t | y_t > K, \Im_t] > 0$. Thus, the process $\{y_t\}$ satisfies all the requirements of Theorem 2.2 in Lamperti (1960) and, consequently, $\text{Prob}\{\lim_{t\to\infty} y_t = 1\} = 1$. The assertion immediately follows. □

Along the same lines it is possible to prove the divergence to negative infinity of a process with finite negative increments and asymptotically negative drift.

Corollary 4.1. *Consider a bounded increments process $\{x_t\}$ with finite negative increments. If there exist $\epsilon, \delta > 0$ such that, definitely in t and almost surely,*

$$\limsup_{x\to\pm\infty} \mu_t(x) - \frac{v_t(x)}{2\,x} + \frac{\epsilon}{|x|^{2-\delta}} \leq 0 \tag{5}$$

then the process is transient and $\text{Prob}\{\lim_{t\to\infty} x_t = -\infty\} = 1$.

The case of a homogeneous random walk with non-negative drift falls in one of the two cases of Theorem 4.1 or Corollary 4.1, with $\delta = 2$. In general, if the process has asymptotic drifts at plus and minus infinity with definite signs, it is transient. When the drifts of the process outside a bounded set have a different sign, as long as they point away from the origin, the process is still transient as clarified by the following

Theorem 4.2. *Consider a bounded increments process x_t with positive and negative finite increments. If there exist $\epsilon, \delta > 0$ such that, definitely in t and almost surely*

$$\begin{aligned} \limsup_{x\to-\infty} \mu_t(x) - \frac{v_t(x)}{2\,x} + \frac{\epsilon}{|x|^{2-\delta}} &\leq 0 \text{ and} \\ \liminf_{x\to+\infty} \mu_t(x) - \frac{v_t(x)}{2\,x} - \frac{\epsilon}{|x|^{2-\delta}} &\geq 0\,, \end{aligned} \tag{6}$$

then the process is transient and, with probability one, either $\lim_{t\to\infty} x_t = +\infty$ *or* $\lim_{t\to\infty} x_t = -\infty$.

Proof. The process $\{|x_t|\}$ has bounded and finite increments and by hypothesis satisfies the condition of Theorem 4.1. Thus, $\lim_{t\to\infty} |x_t| = +\infty$ almost surely.

Assume that on a positive measure of trajectories it is $\liminf_{t\to\infty} x_t = -\infty$ and $\limsup_{t\to\infty} x_t = +\infty$. Consider a $M > B$. Then on those trajectories, for any t for which $x_t > M$ there exists a t' for which $x_{t'} < -M$. Since $M > B$, this implies that there is a t'' such that $t < t'' < t'$ and $x_{t''} \in [-M, M]$, that is $|x_{t''}| \in [0, M]$. Hence the set $[0, M]$ would be recurrent for the process $\{|x_t|\}$, but this is absurd given the previous result. The statement follows directly. □

According to the previous theorem, if one defines two sets $\Sigma_{-\infty}$ and $\Sigma_{+\infty}$ of trajectories converging, respectively, to minus and plus infinity, it is $P(\Sigma_1 \cup \Sigma_2) = 1$.

All the previous results can also be applied to diverging processes, e.g. by removing an unconditional drift $\bar{\mu}$. In this case, it is the sign of the relative drift $\mu_t(x) - \bar{\mu}$ for large positive and negative x that can be used to decide whether the trajectories of the process visit with probability one a neighborhood of $\bar{\mu}\,t$ or accumulate far away from it.

5 Applications and Examples

In this section, inspired by problems in the economic discipline, we briefly discuss simple examples that clarify the domain of application of the previous results. We shall consider the problem of characterizing long-run populations shares in an evolutionary game with stochastic payoffs, long-run relative consumption in a dynamic stochastic exchange economy with heterogeneous agents, and long-run posterior beliefs of a Bayesian learner who has a misspecified prior support and undereacts to information.

In all examples we assume that in each period t a state of nature $s_t \in \{1, \ldots, s, \ldots, S\}$ can occur. $\sigma_t \in \Sigma^t$ is the history of states till t and σ is a path $\in \Sigma$. $\Im$ is the σ-algebra generated by the partial histories. The measure P on $(\Sigma, \Im)$ is kept general or derived from an i.i.d. Bernoulli process with $\pi = (\pi_1, \pi_2, \ldots, \pi_S) \gg 0$, depending on the example.

5.1 Discrete-time stochastic replicator dynamics

We are concerned with a stochastic version of the discrete-time replicator dynamics (Taylor and Jonker, 1978; Bishop and Cannings, 1978; Weibull, 1997) describing the competition of strategies in repeated games or, equivalently, of species in a varied fitness landscape. Assume to have N strategies (species) and let $\boldsymbol{p}_t = (p_t^1, \ldots, p_t^N)$ be the fraction of each strategy in the whole population at time t. Then the fraction of strategy h at time $t+1$ is

$$p_{t+1}^h = p_t^h \, \frac{f^h(\boldsymbol{p}_t)}{\sum_{i=1}^N p_t^i f^i(\boldsymbol{p}_t)} \,, \quad h = 1, \ldots, N \,, \tag{7}$$

where $f^h(\boldsymbol{p}) > 0$ is the fitness (payoff) of strategy h when $\boldsymbol{p}$ is the vector of population shares.

First, we shall make the simplifying assumption that the fitness f^h does not depend on the relative abundance of strategy $\boldsymbol{p}$ but instead on some external factor described by the i.i.d. process $\{s_t\}$ with probability $\pi(s) > 0$.[5] Then $f^h(s) > 0$ is the fitness of strategy h if the external condition s is realized. On a realization σ of the process, the evolution of the population shares reads

$$p_{t+1}^h = p_t^h \, \frac{f^h(s_{t+1})}{\sum_{i=1}^N p_t^i f^i(s_{t+1})} \,, \quad h = 1, \ldots, N \,. \tag{8}$$

The assumption that f^h does not depend on $\boldsymbol{p}$ makes the deterministic version of model in (7) rather trivial: if h is the strategy with the highest fitness, it is $\lim_{t\to\infty} p_t^h = 1$. For the stochastic version (8), one has the following

Theorem 5.1. *If strategy h is such that the geometric average of its fitness is greater that the geometric average of the fitness of any other strategy,*

$$\prod_{s=1}^S f^h(s)^{\pi_s} > \prod_{s=1}^S f^i(s)^{\pi_s} \quad \forall i \neq h \,,$$

then a.s. $\lim_{t\to\infty} p_t^h = 1$.

[5]Examples of external factors are aggregate demand or technological shocks in a model where firms compete for market shares.

Proof. Consider $k \neq h$ and let $x_t = \log p_t^h / p_t^k$, so that $x_t - x_{t-1} = \log f^h(s_t)/f^k(s_t)$. By hypothesis, for at least one s, it is $\log f^h(s_t)/f^k(s_t) > 0$ so that the process $\{x_t\}$ has finite positive increments (c.f. Definition 4.1). For each strategy i, define the lowest and greatest fitness $\underline{f}^i = \min_s f^i(s)$ and $\overline{f}^i = \max_s f^i(s)$. Then $\log \underline{f}^h / \overline{f}^k \leq x_t - x_{t-1} \leq \log \overline{f}^h / \underline{f}^k$ so that the process $\{x_t\}$ has bounded increments (c.f. Definition 2.2). Notice that, by hypothesis, $E[x_t - x_{t-1}] = \sum_{s=1}^S \log f^h(s)/f^k(s) > 0$. Thus, according to Theorem 4.1, it is $\lim_{t\to\infty} p_t^k = 0$ almost surely. Since this is true for any $k \neq h$, the statement follows. □

The stochastic version is qualitatively similar to the deterministic one. The appropriate quantity to measure the dominance of strategies is the geometric average of their stochastic fitness.

Consider now the case in which there are two strategies and the fitness of strategy $h = 1, 2$ depends on random external factors as before but also, linearly, on the fractions of the two strategies composing the population $f^h(\boldsymbol{p}, s) = \sum_{i=1}^2 a_i^h(s)\, p^i$, with $a_j^i(s) > 0$ for any $i, j = 1, 2$ and any s. Thus on a realization σ, the evolution of the population shares reads

$$p_{t+1}^h = p_t^h \, \frac{\sum_{j=1}^2 a_j^h(s_{t+1}) p_t^j}{\sum_{i=1}^2 \sum_{j=1}^2 p_t^i \, a_j^i(s_{t+1})\, p_t^j}, \quad h = 1, 2 \,. \tag{9}$$

Let $\bar{a}_j^i = \prod_{s=1}^S a_j^i(s)^{\pi_s}$. Then we have the following

Theorem 5.2. *If* $\bar{a}_2^1 > \bar{a}_2^2$, *then a.s.* $\liminf_{t\to\infty} p_t^1 > 0$. *If* $\bar{a}_1^1 < \bar{a}_1^2$, *then a.s.* $\liminf_{t\to\infty} p_t^2 > 0$.

Proof. We prove the first part of the statement. Let $x_t = \log p_t^1/p_t^2$ and define the lowest and greatest fitness $\underline{a}^1 = \min_{s,i} a_i^1(s)$ and $\overline{a}^1 = \max_{s,i} a_i^1(s)$. Then $\log \underline{a}^1/\overline{a}^1 \leq x_t - x_{t-1} \leq \log \overline{a}^1/\underline{a}^1$ so that the process $\{z_t\}$ has bounded increments (c.f. Definition 2.2). Define $\mu_t(x) = \mathrm{E}[x_{t+1} | x_t = x] - x$ and notice that $\lim_{x\to-\infty} \mu_t(x) = \log \bar{a}_2^1 - \log \bar{a}_2^2$. Since the latter is positive by hypothesis, the statement follows from Theorem 3.1. The second part of the statement is proved analogously. □

The case in which both conditions of the previous theorem are satisfied is that of heterophilic interactions between strategies. The result is consistent with the shares of the two strategies reaching an equilibrium but also with persistent, possibly huge, fluctuations in the composition of the population.

The analysis can be extended to two groups $g = 1, 2$ of a finite populations of strategy $h = 1, \ldots, N$. Without loss of generality assume that the first M_1 strategies are in group $g = 1$, and the other $M_2 = N - M_1$ agents are in group $g = 2$. We name $p_{g,t}$ the aggregate size population of group g so that

$$p_{1,t} = \sum_{h=1}^{M_1} p_t^h \quad \text{and} \quad p_{2,t} = \sum_{h=M_1+1}^{N} p_t^h .$$

Let $\phi_{g,t}^h \in \Delta^{M_g}$ be the share of population h in group g at date t, so that $p_t^h = \phi_{g,t}^h p_{g,t}$. From eq. (9), the group population dynamics is

$$p_{g,t+1} = p_{g,t} \frac{\sum_{f=1}^2 l_f^g(\phi_{1,t}, \phi_{2,t}, s_{t+1}) p_{f,t}}{\sum_{e=1}^2 \sum_{f=1}^2 p_{e,t}\, l_f^e(\phi_{1,t}, \phi_{2,t}, s_{t+1})\, p_{f,t}}, \quad g = 1, 2 .$$

where

$$l_f^g(\phi_{1,t}, \phi_{2,t}, s_{t+1}) = \sum_{h \in M_g} \sum_{j \in M_f} a_j^h(s_{t+1}) \phi_{g,t}^h \phi_{f,t}^j .$$

Note that the process $x_t = \log(p_{1,t}/p_{2,t})$ is real, adapted, and, given an initial state z_0 its value in (t, σ_t) depends on the whole history σ_t. That is, in general, the process is not Markov. Nevertheless, if the drift conditions are valid for all possible realizations, or even more, for all possible composition of the fractions, then we can still use them to characterize long-run survival. Let $\bar{l}_f^g(\phi_1, \phi_2) = \prod_{s=1}^S l_f^g(\phi_1, \phi_2, s)^{\pi_s}$. Then we have the following

Theorem 5.3. *If* $\min_{\phi_1 \in \Delta^{M_1}, \phi_2 \in \Delta^{M_2}} \left\{ \bar{l}_2^1 / \bar{l}_2^2 \right\} > 1$, *then a.s.* $\liminf_{t \to \infty} p_{1,t} > 0$. *If* $\min_{\phi_1 \in \Delta^{M_1}, \phi_2 \in \Delta^{M_2}} \left\{ \bar{l}_1^1 / \bar{l}_1^2 \right\} > 1$, *then a.s.* $\liminf_{t \to \infty} p_{2,t} > 0$.

Proof. Along the lines of the proof of Theorem 5.2, define $\mu_t(x) = \mathrm{E}[x_{t+1} | x_t = x] - x$ and notice that $\lim_{z \to -\infty} \mu_t(x) > min_{\phi_1 \in \Delta^{M_1}, \phi_2 \in \Delta^{M_2}} \left\{ \log \bar{l}_2^1(\phi_1, \phi_2) - \log \bar{l}_2^2(\phi_1, \phi_2) \right\}$. Since the latter is positive by hypothesis, the statement follows from Theorem 3.1. The second part of the statement is proved analogously. □

5.2 Market selection in growing CRRA economies

In what follows, we consider an endowment economy populated by two agents with heterogeneous beliefs and, possibly, preferences. The latter are represented by a time separable subjective expected utility with constant relative risk aversion. The question is to determine the long-run consumption distribution when agents can engage in speculative trades. Sandroni (2000) and Blume and Easley (2006) provides results for bounded economies with complete markets, we shall instead look at possibly unbounded economies.

The measure P on paths is kept general and all states occur with positive probability. We denote with $\pi_{t,s}(\sigma_t) = P_{t+1}(\sigma_t, s)/P_t(\sigma_t)$ the probability of the realization of state (σ_t, s) conditional upon the history σ_t. Each agent $i = 1, 2$ observes the realization of the process $\{s_t\}$ and assigns to partial histories σ_t a marginal probability P^i_t. We assume that for each i and each t, P^i_t and P_t are absolutely continuous with respect to each other.

There is a unique good after each history. Agent i receives an endowment $\{e^i_t\}$ and is allowed to trade in a complete market. We assume that the expected aggregate endowment $e_t = e^1_t + e^2_t$ is growing (or shrinking) at a (possibly stochastic) rate $g_{t+1} = \log e_{t+1}/e_t$ and that the process has bounded increments. We denote $\bar{g}_t$ the expected log growth rate, $\bar{g}_t(\sigma_t) = \sum_{s=1}^{S} \pi_{t,s}(\sigma_t) \log\left(e_{t+1}(\sigma_t, s)/e_t(\sigma_t)\right)$. Under date 0 trading, naming $q(\sigma_t)$ the price of consumption in node σ_t, each agent $i = 1, 2$ solves

$$\max_{\{c^i_t\}_{(t,\sigma)}} \sum_{t\geq 0, \sigma_t \in \Sigma^t} (\beta^i)^t P^i(\sigma_t) u^i(c^i_t(\sigma_t))$$

$$\text{such that } c^i_0 + \sum_{t>0, \sigma_t\in\Sigma^t} q(\sigma_t) c^i_t(\sigma_t) \leq e^i_0 + \sum_{t>0,\sigma_t\in\Sigma^t} q(\sigma_t) e^i_t(\sigma_t),$$

where $u^i(c) = c^{1-\gamma^i}$ when $\gamma^i > 0$ and $\gamma^i \neq 1$, and $u^i(c) = \log c$ when $\gamma^i = 1$. Using F.O.C. and the expression of the Bernoulli utility $u^i(c)$ one gets

$$c^i_t(\sigma_t) = \left(\frac{(1-\gamma^i)(\beta^i)^t P^i_t(\sigma_t)}{\lambda^i q(\sigma_t)}\right)^{1/\gamma^i},$$

where λ^i is the Lagrange multiplier of the individual intertemporal budget constraint. Prevailing prices $q(\sigma_t)$ and the Lagrange multipliers can be computed using budget constraints and the market clearing condition $e_t = c^1_t + c^2_t$. In general, they are only implicitly defined. The relative consumption dynamics can be derived from Euler equations and reads

$$c^i_{t+1}(\sigma_{t+1}) = c^i_t(\sigma_t)\left(\beta^i \pi^i_{t,s_{t+1}}(\sigma_t)\frac{q(\sigma_{t+1})}{q(\sigma_t)}\right)^{1/\gamma^i}, \tag{10}$$

where $\pi^i_{t,s}(\sigma_t) = P^i_{t+1}(\sigma_t, s)/P^i_t(\sigma_t)$ is the probability assigned in t to the realization of state (σ_t, s) conditional upon the history σ_t. For each agent $i = 1, 2$ define a survival index

$$I^i(\sigma_t) = \log\beta_i - \gamma^i\,\bar{g}_t(\sigma_t) - \Pi^i_t(\sigma_t) \tag{11}$$

where $\Pi_t^i(\sigma_t) = \sum_{s=1}^S \pi_{t,s}(\sigma_t)\log(\pi_{t,s}(\sigma_t)/\pi_{t,s}^i(\sigma_t))$ is the relative entropy of the individual conditional probabilities with respect to the truth. Thus the following applies

Theorem 5.4. *Consider a bounded increments process $\{e_t\}$. If for all (t,σ) $I^1(\sigma_t) > I^2(\sigma_t)$, then a.s. $\lim_{t\to\infty} c_t^1(\sigma_t)/c_t^2(\sigma_t) = 0$.*

Proof. Consider the process $x_t(\sigma_t) = \log c_t^1(\sigma_t)/c_t^2(\sigma_t)$. From (10)

$$x_{t+1}(\sigma_t, s) - x_t(\sigma_t) = \frac{\beta^1}{\gamma^1} - \frac{\beta^2}{\gamma^2} + \frac{\pi_s^1(\sigma_t)}{\gamma^1} - \frac{\pi_s^2(\sigma_t)}{\gamma^2} + \left(\frac{1}{\gamma^2} - \frac{1}{\gamma^1}\right)\log\frac{q(\sigma_t,s)}{q(\sigma_t)} \ . \quad (12)$$

If $c^1 \to 0$, that is $x \to -\infty$, agent 2 consumption becomes equal to the whole endowment so that by (10) it is

$$\frac{q(\sigma_t,s)}{q(\sigma_t)} = \beta^2\pi_s^2(\sigma_t)\left(\frac{e_t(\sigma_t)}{e_{t+1}(\sigma_t,s)}\right)^{\gamma^2} \ .$$

By direct substitution in (12) and taking expectation, the drift of the process x_t at $-\infty$ becomes

$$\mu(-\infty) = \frac{1}{\gamma^1}\left(I^1(\sigma_t) - I^2(\sigma_t)\right) \ .$$

By the same argument, when $c^2 \to 0$ one can write the drift of the process x_t at $+\infty$ to obtain

$$\mu(+\infty) = \frac{1}{\gamma^2}\left(I^1(\sigma_t) - I^2(\sigma_t)\right) \ .$$

Thus, if we can prove that the process $x(\sigma_t)$ has bounded and finite positive increments, we can apply Theorem 4.1 and conclude that a.s. $\lim_{t\to\infty} x(\sigma_t) = +\infty$, hence the assertion.

Having, by assumption, $I^1(\sigma_t) > I^2(\sigma_t)$ for all (t,σ), implies that for all (t,σ) there exists an s such that

$$\beta^1\pi_{t,s}^1(\sigma_t)\left(\frac{e_{t+1}(\sigma_t)}{e_t(\sigma_t,s)}\right)^{\gamma^1} > \beta^2\pi_{t,s}^2(\sigma_t)\left(\frac{e_{t+1}(\sigma_t)}{e_t(\sigma_t,s)}\right)^{\gamma^2} \ . \quad (13)$$

Now notice that, due to market clearing conditions, it is

$$\left(\beta^i\pi_{t,s}^i(\sigma_t)\frac{q(\sigma_t)}{q(\sigma_t,s)}\right)^{1/\gamma^i} \le \frac{e_{t+1}(\sigma_t,s)}{e_t(\sigma_t)} \le \left(\beta^j\pi_{t,s}^j(\sigma_t)\frac{q(\sigma_t)}{q(\sigma_t,s)}\right)^{1/\gamma^j} \quad (14)$$

for $i \neq j$ with $i,j = 1,2$. But according to (13) it must be $i = 2$, so that $c_{t+1}^1(\sigma_t,s)/c_t^1(\sigma_t) > e_{t+1}(\sigma_t,s)/e_t(\sigma_t)$ and $c_{t+1}^2(\sigma_t,s)/c_t^2(\sigma_t) < e_{t+1}(\sigma_t,s)/e_t(\sigma_t)$, which in turn implies $x_{t+1}(\sigma_t, s) > x_t(\sigma_t)$. Hence, the process x_t has finite positive increments. Moreover for (14) for all (t, σ) it is

$$\frac{q(\sigma_t, s)}{q(\sigma_t)} \leq \max_{i=1,2} \left\{ \beta^i \pi^i_{t,s}(\sigma_t) \left(\frac{e_t(\sigma_t)}{e_{t+1}(\sigma_t, s)} \right)^{\gamma^i} \right\} .$$

Since by assumption the process $\{e_t\}$ has bounded increments, then the process $\{x_t\}$ has bounded increments too. □

The agent with the lowest survival index asymptotically consumes a negligible fraction of the aggregate endowment and disappears from the economy. When $\bar{g}_t$ is positive, a large intertemporal elasticity of consumption (low γ) denotes a high propensity to transfer consumption to future dates and it is thus advantageous for survival, in line with results of the incumbent literature (see e.g. Yan, 2008). If Bernoulli risk aversions are equal, $\gamma^1 = \gamma^2$, the aggregate growth rate of the economy does not matter and the agent with the highest value of $\log \beta^i - \Pi(\pi^i)$ dominates, asymptotically consuming the whole aggregate endowment. If intertemporal discount factors are also equal, $\beta^1 = \beta^2$, it is the agent with the most accurate beliefs, in relative entropy terms, who dominates. This is the only case in which one is certain that prices reflect wholly, in the long run, the most accurate model initially available in the market.

5.3 Bayesian learning with under-reaction and miss-pecified models

Assume that states $s_t \in \{1, 2, \ldots, S\}$ are generated by Bernoulli process with probability $\pi = (\pi_1, \ldots, \pi_S) \gg 0$. The measure P on paths is derived accordingly.

Consider a Bayesian learner with a prior $w_0 = (w^1_0, w^2_0)$ on the two Bernoulli models $\{\pi^1, \pi^2\}$ with $\pi^1 \neq \pi^2$ and $\pi^i \gg 0$, $i = 1, 2$. We shall also assume that neither model coincides with the truth: $\pi^i \neq \pi$, $i = 1, 2$. For simplicity we consider only two models and each model is a Bernoulli process. Note however that we could generalize the analysis and each model could be quite complicated, as in Bayesian model averaging.

A Bayesian learner under-react to the news, giving a weight $\alpha \in [0, 1)$ to the old posterior as in Epstein et al. (2008), if the map from prior to posterior is

$$w^i_{t+1}(\sigma_{t+1}) = (1 - \alpha) \frac{\pi^i_{s_{t+1}} w^i_t(\sigma_t)}{\pi^1_{s_{t+1}} w^1_t + \pi^2_{s_{t+1}} w^2_t(\sigma_t)} + \alpha\, w^i_t(\sigma_t) \quad i = 1, 2 . \tag{15}$$

Define the stochastic process $x_t = \log w^1_t / w^2_t$. One has the following

Lemma 5.1. *The process $\{x_t\}$ has bounded and finite positive and negative increments.*

Proof. Having $\pi^1 \neq \pi^2$ implies that there exists an s such that $\pi^1_s > \pi^2_s$ and an s' such that $\pi^2_{s'} > \pi^1_{s'}$. It is immediate to verify that if $s_{t+1} = s$ then $x_{t+1} > x_t$ and if $s_{t+1} = s'$ then $x_{t+1} < x_t$. Thus the process has finite positive and negative increments. Let $\pi^+ = \max_{i,s}\{\pi^i_s\}$ and $\pi^- = \min_{i,s}\{\pi^i_s\}$. It is immediate to verify that

$$\log \frac{\pi^-}{\pi^+} \leq x_{t+1} - x_t \leq \log \frac{\pi^+}{\pi^-}$$

so that the increments of the process are bounded. □

Define the relative entropy of the model π^i with respect to the truth[6]

$$I_\pi(\pi^i) = -\sum_{s=1}^{S} \pi_s \log \frac{\pi_s}{\pi^i_s} \ .$$

Based on the previous Lemma one can easily prove the following

Theorem 5.5. *If $I_\pi(\pi^1) < I_\pi(\pi^2)$ then a.s. $\lim_{t\to+\infty} x_t > -\infty$.*

Proof. Consider the asymptotic conditional drift

$$\mu(-\infty) = \lim_{x\to-\infty} \mathrm{E}[x_{t+1} - x_t | x_t = x] = I_\pi(\pi^2) - I_\pi((1-\alpha)\pi^1 + \alpha\pi^2) \ .$$

By the concavity of the log function

$$I_\pi((1-\alpha)\pi^1 + \alpha\pi^2) < (1-\alpha)\, I_\pi(\pi^1) + \alpha\, I_\pi(\pi^2)$$

so that, substituting in the expression of the asymptotic drift, one gets

$$\mu(-\infty) > (1-\alpha)\left(I_\pi(\pi^2) - I_\pi(\pi^1)\right) > 0$$

and the statement follows from Theorem 3.1. □

The meaning of the previous theorem is that, also with under-reaction updating, the Bayesian learner always assigns a positive weight to the strategy that is better in terms of entropic divergence. But one can say more.

Theorem 5.6. *Assume $I_\pi(\pi^1) < I_\pi(\pi^2)$. If, furthermore, $I_\pi(\pi^1) < I_\pi(\alpha\,\pi^1 + (1-\alpha)\,\pi^2)$, then the process $\{x_t\}$ is transient and a.s. $\lim_{t\to+\infty} x_t = +\infty$. If instead $I_\pi(\pi^1) > I_\pi(\alpha\,\pi^1 + (1-\alpha)\,\pi^2)$ then the process $\{x_t\}$ is persistent.*

[6]Using the notation of the previous example, the relative entropy of π^i is Π^i.

Proof. First note that having assumed $I_\pi(\pi^1) < I_\pi(\pi^2)$, Theorem 5.5 applies. Consider the asymptotic conditional drift

$$\mu(+\infty) = \lim_{x \to +\infty} \mathrm{E}[x_{t+1} - x_t | x_t = x] = I_\pi(\alpha\, \pi^1 + (1-\alpha)\, \pi^2) - I_\pi(\pi^1)\ .$$

If the first hypothesis is true, then $\mu(+\infty) > 0$ and, according to Cor. 4.1, the process is transient. If the second hypothesis is true, then $\mu(+\infty) < 0$ and, according to Theorem 3.2, the process is persistent. □

The previous theorem provides sufficient conditions for the learner to converge to the best model: the relative entropy of the best model should be lower than the entropy of an appropriate mixtures of the two. As expected, in the case of pure Bayesian learning, $\alpha = 0$, the best model gets all the weight in the long run. More generally, the case in which the learner does not converge to a specific model but rather keeps on updating her model indefinitely results generic.

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