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Social Choice among Complex Objects: Mathematical Tools

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SOCIAL CHOICE AMONG COMPLEX OBJECTS:MATHEMATICAL TOOLS

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Abstract. Here the reader can find some basic definitions and notations in order to better understand the model for social choise described by L. Marengo and S. Settepanella in their paper: *Social choice among complex objects*.
The interested reader can refer to [Bou68], [Massey] and [OT92] to go into more depth.

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1. Arrangements

In geometry and combinatorics, an **arrangement of hyperplanes** is a finite set $\mathcal{A}$ of hyperplanes in a linear, affine, or projective space S. Questions about a hyperplane arrangement $\mathcal{A}$ generally concern geometrical, topological, or other properties of the complement, $M(\mathcal{A})$, which is the set that remains when the hyperplanes are removed from the whole space. One may ask how these properties are related to the arrangement and its intersection semilattice.

The intersection semilattice of $\mathcal{A}$, written $L(\mathcal{A})$, is the set of all subspaces that are obtained by intersecting some of the hyperplanes; among these subspaces are S itself, all the individual hyperplanes, all intersections of pairs of hyperplanes, etc. (excluding, in the affine case, the empty set). These subspaces are called the **flats** of $\mathcal{A}$. $L(\mathcal{A})$ is partially ordered by reverse inclusion.

If the whole space S is 2-dimensional, the hyperplanes are lines; such an arrangement is often called an arrangement of lines. Historically, real arrangements of lines were the first arrangements investigated. If S is 3-dimensional one has an arrangement of planes.

More precisely, let $\mathbb{K}$ be a field and let $V_{\mathbb{K}}$ be a vector space of dimension n. A **hyperplane** H in $V_{\mathbb{K}}$ is an affine subspace of dimension (n-1). A hyperplane arrangement

$$\mathcal{A}_{\mathbb{K}} = (\mathcal{A}_{\mathbb{K}}, V_{\mathbb{K}})$$

is a finite set of hyperplanes in $V_{\mathbb{K}}$.

We are interesting in the real and complex cases, then from now on $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $V = \mathbb{R}^n, \mathbb{C}^n$. Then choosen the canonical basis $\{e_1, \ldots, e_n\}$ in V, each hyperplane $H \in \mathcal{A}$ is the kernel of a polynomial $\alpha_H \in \mathbb{K}[x_1, \ldots, x_n]$ of degree 1 defined up to a costant. The product:

$$\mathcal{Q}(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$$

is called **defining polynomial** of $\mathcal{A}$.

The **cardinality** $\mid \mathcal{A} \mid$ of the arrangement $\mathcal{A}$ is the number of hyperplanes in $\mathcal{A}$.

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If $\mathcal{B} \subset \mathcal{A}$ is a subset, then it is called a **subarrangement** of $\mathcal{A}$. We define the set of all nonempty intersections of elements of $\mathcal{A}$ as:

$$L(\mathcal{A}) = \{\cap_{H\in\mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}\}$$

Given an element $X \in L(\mathcal{A})$, we define a subarrangement $\mathcal{A}_X$ of $\mathcal{A}$ by:

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\},$$

and an arrangement in X by:

$$\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X and X \cap H \neq \emptyset\}.$$

$\mathcal{A}^X$ is called the **restriction** of $\mathcal{A}$ to X.

Finally we define the **complement** of $\mathcal{A}$ by

$$M(\mathcal{A}) = V \setminus \cup_{H\in\mathcal{A}} H.$$

The complement of an arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is clearly disconnected: it is made up of separate pieces called **chambers** or **regions**, each of which is either a bounded region or an unbounded region which goes off to infinity.

Each flat of $\mathcal{A}$ is also divided into pieces by the hyperplanes that do not contain the flat; these pieces are called the **faces** of $\mathcal{A}$. The regions are faces because the whole space is a flat. The faces of codimension 1 may be called the **facets** of A.

The **face semilattice** of an arrangement is the set of all faces, ordered by inclusion. Adding an extra top element to the face semilattice gives the face lattice.

Example 1. *Let us give examples:*

- *if the arrangement consists of three parallel lines, the intersection semilattice consists of the plane and the three lines, but not the empty set. There are four regions, none of them bounded.*
- *If we add a line crossing the three parallels, then the intersection semilattice consists of the plane, the four lines, and the three points of intersection. There are eight regions, still none of them bounded.*
- *If we add one more line, parallel to the last, then there are 12 regions, of which two are bounded parallelograms.*

Every arrangement $(\mathcal{A}_{\mathbb{R}}, \mathbb{R}^n)$ gives rise to an arrangement over $\mathbb{C}$. Let $(\mathcal{A}_{\mathbb{R}}, \mathbb{R}^n)$ be an arrangement with defining polynomial $\mathcal{Q}(\mathcal{A}_{\mathbb{R}})$. The $\mathbb{C}$-extended arrangement is in $\mathbb{C}^n$. It consists of the hyperplanes which are the kernel of the polynomial α_H in $\mathbb{C}^n$ instead of $\mathbb{R}^n$.

2. Basic notions in Topology

Topological spaces are mathematical structures that allow the formal definition of concepts such as convergence, connectedness, and continuity. They appear in virtually every branch of modern mathematics and are a central unifying notion. The branch of mathematics that studies topological spaces in their own right is called *Topology*.

A **topological space** is a set X together with T, a collection of subsets of X, satisfying the following axioms:

1. *the empty set and X are in T;*
2. *the union of any collection of sets in T is also in T;*
3. *the intersection of any finite collection of sets in T is also in T.*

The collection T is called a **topology** on X.

The elements of X are usually called **points**, though they can be any mathematical objects. A topological space in which the points are functions is called a function space.

The sets in T are the **open sets**, and their complements in X are called **closed sets**. A set may be neither closed nor open, either closed or open, or both.

Given an open set $U \in T$, the smaller closed set which contain it is called **closure** of U and it is usually indicated by $\overline{U}$.

Example 2. *Let us give some elementary example of topological spaces:*

(1) $X = \{1, 2, 3, 4\}$ *and collection* $T = \{\{\}, \{1, 2, 3, 4\}\}$ *of two subsets of* X *form a trivial topology.*
(2) $X = \{1, 2, 3, 4\}$ *and collection* $T = \{\{\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$ *of six subsets of* X *form another topology.*
(3) $X = \mathbb{Z}$*, the set of integers and collection* T *equal to all finite subsets of the integers plus* $\mathbb{Z}$ *itself is not a topology, because (for example) the union over all finite sets not containing zero is infinite but is not* $\mathbb{Z}$*, and so is not in* T*.*

A variety of topologies can be placed on a set to form a topological space. When every set in a topology T_1 is also in a topology T_2, we say that T_2 is *finer* than T_1, and T_1 is *coarser* than T_2.

A **neighbourhood** of a point x is any set that contains an open set containing x. The neighbourhood system at x consists of all neighbourhoods of x. A topology can be determined by a set of axioms concerning all neighbourhood systems.

As usual in mathematics we define relations between topological spaces. These relations are **continuous functions**.

A function between topological spaces is said to be continuous if the inverse image of every open set is open. In mathematical terms, given two spaces X and Y and topologies T_X and T_Y on X and Y respectively, then a function

$$f : X \to Y$$

is continuous if and only if $f^{-1}(U) \in T_X$ for all $U \in T_Y$.

This is an attempt to capture the intuition that there are no *breaks* or *separations* in the function.

A **homeomorphism** is a bijection that is continuous and whose inverse is also continuous. Two spaces are said to be homeomorphic if there exists a homeomorphism between them.

From the standpoint of topology, homeomorphic spaces are essentially identical.

The category of topological spaces with topological spaces as objects and continuous functions as morphisms is one of the fundamental categories in mathematics. The attempt to classify the objects of this category (up to homeomorphism) by invariants has motivated and generated entire areas of research, such as homotopy theory, homology theory, and K-theory, to name just a few.

Let see some other important and useful example of topologies on a given set X, remarking that if a set is given a different topology, it is viewed as a different topological space.

Any set can be given the **discrete topology** in which every subset is open. Also, any set can be given the **trivial topology** (also called the indiscrete topology), in which only the empty set and the whole space are open.

However, oftentimes topological spaces are required to be **Hausdorff spaces**: let x and y be points in a topological space X. We say that x and y can be *separated by neighbourhoods* if there exists a neighbourhood U of x and a neighbourhood V of y such that U and V are disjoint ($U \cap V = \emptyset$). X is a Hausdorff space if any two distinct points of X can be separated by neighborhoods.

There are many ways of defining a topology on $\mathbb{R}$, the set of real numbers. The **standard topology** on $\mathbb{R}$ is generated by the open intervals. The set of all open intervals forms a base or basis for the topology, meaning that every open set is a union of some collection of sets from the base. In particular, this means that a set is open if there exists an open interval of non zero radius about every point in the set.

More generally, the Euclidean spaces $\mathbb{R}^n$ can be given a topology. In the usual topology on $\mathbb{R}^n$ the basic open sets are the open balls, meaning n-dimensional balls centered in a point $x \in \mathbb{R}^n$ with radius $\varepsilon > 0$.

Similarly, $\mathbb{C}$ and $\mathbb{C}^n$ have a standard topology in which the basic open sets are open balls.

Every metric space can be given a metric topology, in which the basic open sets are open balls defined by the metric. This is the *standard topology* on any normed vector space. On a finite-dimensional vector space this topology is the same for all norms.

Many sets of operators in functional analysis are endowed with topologies that are defined by specifying when a particular sequence of functions converges to the zero function.

Any local field has a topology native to it, and this can be extended to vector spaces over that field.

Every manifold has a natural topology since it is locally Euclidean.

The real line can also be given the lower limit topology. Here, the basic open sets are the half open intervals $[a, b)$. This topology on $\mathbb{R}$ is strictly finer than the Euclidean topology defined above. This example shows that a set may have many distinct topologies defined on it.

A very important notion in mathematics is the notion of *substructure*. Every subset $Y \subset X$ of a topological space X can be given the **subspace topology** or **induced topology** in which the open sets are the intersections of the open sets of X with the subset Y, i.e.

$$T_Y = \{W \cap X \mid U \in T_X\}.$$

For any indexed family of topological spaces $\{X_i\}_{i \in I}$, the product $\prod_{i \in I} X_i$ can be given the **product topology**, which is generated by the inverse images of open sets of the factors under the projection mappings

$$p_j : \prod_{i \in I} X_i \to X_j.$$

For example, in finite products, a basis for the product topology consists of all products of open sets. For infinite products, there is the additional requirement that in a basic open set, all but finitely many of its projections are the entire space.

A quotient space is defined as follows: if X is a topological space and Y is a set, and if

$$f : X \longrightarrow Y$$

is a surjective function, then the **quotient topology** on Y is the collection

$$T_{q,Y} = \{U \subset Y \mid f^{-1}(U) \in T_X\}$$

of subsets of Y that have open inverse images under f. In other words, the quotient topology is the finest topology on Y for which f is continuous. A common example of a quotient topology is when an equivalence relation is defined on the topological space X. The map f is then the natural projection onto the set of equivalence classes.

Topological spaces can be broadly classified, up to homeomorphism, by their topological properties. A **topological property** is a property of spaces that is invariant under homeomorphisms. To prove that two spaces are not homeomorphic it is sufficient to find a topological property which is not shared by them. Examples of such properties include connectedness, compactness, and various separation axioms. See books on references for more details and examples.

3. Simplicial and CW complexes

In mathematics, a **simplicial complex** is a topological space of a particular kind, constructed by "gluing together" points, line segments, triangles, and their n-dimensional counterparts. In order to define a simplicial complex we need to define **simplexes**.

Let us define the **convex hull** for a set of points $X = \{x_1, \ldots, x_m\}$ in a real vector space V as follow:

$$H(X) = \{\sum_{i=1}^{k} \alpha_i x_i \mid x_i \in X, \alpha_i \in \mathbb{R}, \alpha_i \geq 0, \sum_{i=1}^{k} \alpha_i = 1, k = 1, 2, \ldots\}.$$

In geometry, a **simplex** (plural simplexes or simplices) or n-simplex is an n-dimensional analogue of a triangle. Specifically, a simplex is the convex hull of a set of (n + 1) affinely independent points in some Euclidean space of dimension n or higher (i.e., a set of points such that no m-plane contains more than (m + 1) of them; such points are said to be in *general position*).

For example, a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and a 4-simplex is a pentachoron (in each case with interior). The convex hull of any nonempty subset of the n+1 points that define an n-simplex is called a **face** of the simplex. Faces are simplices themselves.

In particular, the convex hull of a subset of size m+1 (of the n+1 defining points) is an m-simplex, called an m-face of the n-simplex.
The 0-faces (i.e., the defining points themselves as sets of size 1) are called the **vertices** (singular: vertex), the 1-faces are called the **edges**, the (n-1)-faces are called the **facets**, and the sole n-face is the whole n-simplex itself.
In general, the number of m-faces is equal to the binomial coefficient C (n + 1 , m + 1).

A **simplicial complex** X is a set of simplices that satisfies the following conditions:

(1) Any face of a simplex from X is also in X.
(2) The intersection of any two simplices $\sigma_1, \sigma_2 \in X$ is a face of both σ_1 and σ_2.

Note that the empty set is a face of every simplex.

Let us remark that every $n-$simplex and every $n-$simplicial complex inherits a natural topology from $\mathbb{R}^n$.

A **simplicial k-complex** X is a simplicial complex where the largest dimension of any simplex in X equals k. For instance, a simplicial 2-complex must contain at least one triangle, and must not contain any tetrahedra or higher-dimension simplices.

A **pure** or **homogeneous** simplicial k-complex is a simplicial complex where every simplex of dimension less than k is the face of some simplex of dimension exactly k. Informally, a pure 1-complex "looks" like it's made of a bunch of lines, a 2-complex "looks" like it's made of a bunch of triangles, etc. An example of a non-homogeneous complex is a triangle with a line segment attached to one of its vertices.

A **facet** is any simplex in a complex that is not the face of any larger simplex. (Note the difference from the "facet" of a simplex.)

A pure simplicial complex can be thought of as a complex where all facets have the same dimension.

Sometimes the term face is used to refer to a simplex of a complex, not to be confused with the face of a simplex. For a simplicial complex embedded in a k-dimensional space, the k-faces are sometimes referred to as its **k-cells**. The term cell is sometimes used in a broader sense to denote a set homeomorphic to a simplex, leading to the definition of cell complex.

A **cell complex** or simply complex in $\mathbb{R}^n$ is a set X in $\mathbb{R}^n$ of convex polyhedra, i.e. the set of solutions to a finite system of linear inequalities, satisfying two conditions:

(1) Every face of a cell is a cell (i.e. it is in X).
(2) If σ_1 and σ_2 are cells, then their intersection is a common face of both.

Each convex polyhedron is called **cell**.

A simplicial complex is a cell complex whose cells are all simplices.

We are interested in a particula cell complex, the CW-complex. In order to define it we need to introduce the following:

A **finite graph** is a pair consisting of a Hausdorff space X and a subspace X_0 (called the *set of vertices of* X) such that the following conditions hold:

(1) *X_0 is a finite discrete (i.e. it inherits the discrete topology from X), closed subspace of X. Points of X_0 are called* **vertices**;
(2) *$X \setminus X_0$ is the finite disjoint union of open subsets e_i, where each e_i is homeomorphic to an open interval of the real line. The set e_i are called* **edges**;
(3) *for each edge e_i, if $\overline{e_i}$ is the closure of e_i, then its* **boundary** *$\overline{e_i} \setminus e_i$ is a subset of X_0 consisting of one or two points. If $\overline{e_i} \setminus e_i$ consists of two points, then the pair $(\overline{e_i}, e_i)$ is homeomorphic to the pair $([0,1],(0,1))$,if $\overline{e_i} \setminus e_i$ consists of one point, then the pair $(\overline{e_i}, e_i)$ is homeomorphic to the pair $(S_1, S_1 \setminus \{1\})$, where S_1 is the unit circle in the plane.*

It is a simple remark to notice that the definition of graph is a generalization of a 1-dimensional simplicial complex.

Moreover it is possible to give an orientation to a graph, simply ordering its vertices. So an edge will be oriented going from the lesser to the bigger vertices.

Now, we can give the definition of **CW-complex**.
In topology, a CW complex is a type of topological space introduced by J. H. C. Whitehead to meet the needs of homotopy theory. The idea was to have a class of spaces that was broader than simplicial complexes, but still retained a combinatorial nature, so that computational considerations were not ignored. For these purposes a closed cell is a topological space homeomorphic to a simplex, or equally a ball (of which a sphere is the boundary) or cube in n dimensions.

First of all we need to introduce the notion of *adjoining* cells to a space (for a more precise definition see also [Massey]).

A n-dimensional cell is attached to a space X by gluing a closed n-dimensional ball D_n to the **(n-1)-skeleton** X_{n-1} of X, i.e., the union of all cells of dimension lower than n in X.

The gluing is specified by a continuous function

$$f : S_{n-1} \rightarrow X_{n-1}$$

from the $n-1$-dimensional unit sphere $S_{n-1} = \partial D_n$ to X_{n-1}.

The points on the new space are exactly the equivalence classes of points in the disjoint union $X \dot{\cup} D_n$ of the old space and the closed cell D_n, the equivalence relation being the transitive closure of $x \equiv f(x)$ (i.e., the smallest transitive relation that contains $\equiv$).

The function f plays an essential role in determining the nature of the newly enlarged complex. For example, if the $2-$dimensional ball D_2 is glued onto the circle S_1 in the usual way, i.e. with the funcion f given by the identity map, we get D_2 itself; if f has winding number 2, we get the real projective plane instead.

The process of adjoining cells to a space leads naturally to the notion of CW-complex. Roughly speaking, a CW-complex is a space X which can be built up as follows:

Start with a graph X_1 and adjoin a collection of 2-cells as described above to obtain a space X_2. Next adjoin a collection of 3-cells and so on. Then

$$X = \cup_{n=1}^{\infty} X_n$$

is a CW-complex. Moreover if all attaching maps are homeomorphisms, the structure is called a **regular CW-complex**.

More precisely, a CW-complex is defined on an Housdorff space X by the prescription of an ascending sequence

$$X_0 \subset X_1 \subset X_2 \subset \dots$$

of closed subspaces of X which satisfies the following conditions:

(1) *X_0 is a discrete space;*
(2) *for $n \geq 0$, X_n is obtained from $X_{(n-1)}$ by attachment of a collection of n-cells as described above.*
(3)
$$X = \cup_{n=1}^{\infty} X_n$$
(4) **(weak topology)***X and each of X_n have the weak topology; i.e, a subset A of X (or X_n) is closed if and only if the intersection $A \cup \overline{e_q}$ of A with the closure of a q-cell e_q is closed for each e_q.*

The subspace X_n is called the **n-skeleton** of X.

If there are no cells of dimension greater than n, X is called **finite dimensional**.

Let us notice that a graph is a 1-dimensional CW-complex.

Note that there normally are many possible choices of a filtration by skeleta for a given CW-complex. A particular choice of skeleta and attaching maps for the cells is called a **CW-structure** on the space.

A subspace Y of a CW-complex X is called a **subcomplex** if Y is a union of cells of X, and if for any q-cell e_q, if $e_q \subset Y$ then $\overline{e_q} \subset Y$. If this is the case, we define the n-skeleton Y_n by

$$Y_n = X_n \cap Y.$$

It can be shown that Y is also a CW-complex, and it is a closed subset of X.

Associated to a cell complex there is the **Euler characteristic** which is defined as the alternating sum

$$\chi = k_0 - k_1 + k_2 - k_3 + \dots,$$

where k_n denotes the number of cells of dimension n in the complex.

4. Some notion of homotopy theory: the fundamental group

In mathematics, the fundamental group is one of the basic concepts of algebraic topology. Associated with every point of a topological space there is a **fundamental group** that conveys information about the 1-dimensional structure of the portion of the space surrounding the given point. The fundamental group is the first homotopy group.

Before giving a precise definition of the fundamental group, we try to describe the general idea in non-mathematical terms. Take some space, and some point in it, and consider all the loops both starting and ending at this point: paths which start at this point, wander around as much as they like and eventually return to the starting point.

Two loops can be combined together in an obvious way: travel along the first loop, then along the second. The set of all the loops with this method of combining them is the fundamental group, except that for technical reasons it is necessary to consider two loops to be the same if one can be deformed into the other without breaking.

For the precise definition, let X be a topological space, and let $x_0 \in X$ be a point of X. We are interested in the set of continuous functions

$$f : [0,1] \longrightarrow X$$

with the property that $f(0) = x_0 = f(1)$. These functions are called **loops** with base point x_0. Any two such loops, say f and g, are considered equivalent if there is a continuous function

$$h : [0,1] \times [0,1] \longrightarrow X$$

with the property that, for all $0 \leq t \leq 1$, $h(t,0) = f(t)$, $h(t,1) = g(t)$ and $h(0,t) = x_0 = h(1,t)$. Such an h is called a **homotopy** from f to g, and the corresponding equivalence classes are called **homotopy classes**.

The **product** $f * g$ of two loops f and g is defined by setting

$$(f * g)(t) := \begin{cases} f(2t), & 0 \leq t \leq 1/2 \\ g(2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

Thus the loop $f * g$ first follows the loop f with *twice the speed* and then follows g with twice the speed. The product of two homotopy classes of loops $[f]$ and $[g]$ is then defined as $[f * g]$, and it can be shown that this product does not depend on the choice of representatives.

With the above product, the set of all homotopy classes of loops with base point x_0 forms the **fundamental group** of X at the point x_0 and is denoted:

$$\pi_1(X, x_0),$$

or simply $\pi(X, x_0)$. The *identity element* is the constant map at the basepoint, and the *inverse* of a loop f is the loop g defined by $g(t) = f(1-t)$. That is, g follows f backwards.

Although the fundamental group in general depends on the choice of base point, it turns out that, up to isomorphism, this choice makes no difference if the space X is *path-connected*, i.e. for all two points x_1, x_2 in X there is a **path** which joins x_1 and x_2 in X, i.e. there is a continuous function

$$f : [0,1] \longrightarrow X$$

such that $f(0) = x_1$ and $f(1) = x_2$.

For path-connected spaces, therefore, we can write $\pi(X)$ instead of $\pi(X, x_0)$ without ambiguity whenever we care about the isomorphism class only.

In many spaces, such as $\mathbb{R}^n$, there is only one homotopy class of loops, and the fundamental group is therefore trivial, i.e. $\pi(\mathbb{R}^n) = (0, +)$.

A path-connected space with a trivial fundamental group is said to be **simply connected**.

A more interesting example is provided by the circle S_1. It turns out that each homotopy class consists of all loops which wind around the circle a given number of times (which can be positive or negative, depending on the direction of winding). The product of a loop which winds around m times and another that winds around n times is a loop which winds around $m+n$ times. So the fundamental group $\pi(S_1)$ of the circle S_1 is isomorphic to $(\mathbb{Z}, +)$, the additive group of integers.

Since the fundamental group is a homotopy invariant, the theory of the winding number for the complex plane minus one point is the same as for the circle.

The fundamental group of a graph G is a free group. Here the rank of the free group is equal to $1 - \chi(G)$: one minus the Euler characteristic of G, when G is connected, i.e., there is a path from any point to any other point in the graph.

Given two topologicxal spaces X and Y, if $f : X \longrightarrow Y$ is a continuous map, $x_0 \in X$ and $y_0 \in Y$ with $f(x_0) = y_0$, then every loop in X with base point x_0 can be composed with f to yield a loop in Y with base point y_0.

This operation is compatible with the homotopy equivalence relation and with composition of loops. The resulting group homomorphism, called the **induced homomorphism**, is written as $\pi(f)$ or, more commonly,

$$f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

If $f, g : X \to Y$ are continuous maps with $f(x_0) = g(x_0) = y_0$, f and g are homotopic if and only if exists a continuous function

$$H : X \times [0,1] \to Y$$

from the product of the space X with the unit interval $[0,1]$ to Y such that, for all points x in X, $H(x,0) = f(x)$ and $H(x,1) = g(x)$.

If f and g are homotopic relative to $\{x_0\}$, then $f_* = g_*$.

Given two spaces X and Y, we say they are **homotopy equivalent** or of the same homotopy type if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map id_X and $f \circ g$ is homotopic to id_Y.

The maps f and g are called homotopy equivalences in this case.

Clearly, every homeomorphism is a homotopy equivalence, but the converse is not true: for example, a solid disk is not homeomorphic to a single point, although the disk and the point are homotopy equivalent.

Intuitively, two spaces X and Y are homotopy equivalent if they can be transformed into one another by bending, shrinking and expanding operations. For example, a solid disk or solid ball is homotopy equivalent to a point, and $\mathbb{R}^2 \setminus \{(0,0)\}$ is homotopy equivalent to the unit circle S_1.

Spaces that are homotopy equivalent to a point are called **contractible**.

A function f is said to be **null-homotopic** if it is homotopic to a constant function. The homotopy from f to a constant function is then sometimes called a null-homotopy. For example, it is simple to show that a map from the circle S_1 is null-homotopic precisely when it can be extended to a map of the disc D_2.

It follows from these definitions that a space X is contractible if and only if the identity map from X to itself, which is always a homotopy equivalence, is null-homotopic.

From the above definitions follows that two homotopy equivalent path-connected spaces have isomorphic fundamental groups:

$$X \simeq Y \Rightarrow \pi_1(X, x_0) \cong \pi_1(Y, y_0).$$

Moreover, if X and Y are path connected, then

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$$

and

$$\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y).$$

In the latter formula, $\vee$ denotes the wedge sum of topological spaces, and $*$ the free product of groups. Both formulas generalize to arbitrary products.

5. Some notions on singular homology theory

In algebraic topology singular homology refers to the study of a certain set of topological invariants of a topological space X, the so-called homology groups $H_n(X)$. Singular homology is a particular example of a homology theory, which has now grown to be a rather broad collection of theories. Of the various theories, it is perhaps one of the simpler ones to understand, being built on fairly concrete constructions.

In brief, singular homology is constructed by taking maps of the standard n-simplex to a topological space, and composing them into formal sums, called singular chains. The boundary operation on a simplex (see below) induces a singular chain complex. The singular homology is then the homology of the chain complex.

The resulting homology groups are the same for all homotopically equivalent spaces, which is the reason for their study.

These constructions can be applied to all topological spaces. These ideas are developed in greater detail below.

Start defining **singular simplices**. A **singular n-simplex** is a continuous mapping from the standard n-simplex Δ^n to a topological space X.

Notationally, one writes

$$\sigma_n : \Delta^n \to X$$

. This mapping need not be injective, and there can be non-equivalent singular simplices with the same image in X.

The boundary of σ_n, denoted as $\partial_n \sigma_n$, is defined to be the formal sum of the singular $(n-1)$-simplices represented by the restriction of σ_n to the faces of the standard n-simplex, with an alternating sign to take orientation into account.

That is, if

$$\sigma_n = [p_0, p_1, \cdots, p_n] = \sigma_n([e_0, e_1, \cdots, e_n])$$

are the corners of the n-simplex corresponding to the vertices e_k of the standard n-simplex Δ^n, then

$$\partial_n \sigma_n = \sum_{k=0}^{n} (-1)^k [p_0, \cdots, p_{k-1}, p_{k+1}, \cdots p_n]$$

is the formal sum of the (oriented) faces of the simplex.

Thus, for example, the boundary of a 1-simplex $\sigma = [p_0, p_1]$ is the formal difference $\sigma_1 - \sigma_0 = [p_1] - [p_0]$.

Consider first the set $\sigma_n(X)$ of all possible singular n-simplices on a topological space X. Then we can consider all finite formal sums of singular simplices with integer coefficients. All these sums give rise to an abelian group which is very large, usually infinite, frequently uncountable, as there are many ways of mapping a simplex into a typical topological space. This group is commonly denoted as $C_n(X)$. Elements of $C_n(X)$, i.e. a finite formal sum of simplices, are called **singular n-chains**.

The boundary ∂ is readily extended to act on singular n-chains. The extension, called the **boundary operator**, written as

$$\partial_n : C_n \to C_{n-1},$$

is a homomorphism of groups. The boundary operator, together with the C_n, form a **chain complex** of abelian groups, called the **singular complex**. It is often denoted as $(C_\bullet(X), \partial_\bullet)$ or more simply $C_\bullet(X)$.

The kernel of the boundary operator is $Z_n(X) = \ker(\partial_n)$, and is called the group of **singular n-cycles**. The image of the boundary operator is $B_n(X) = \mathrm{im}(\partial_{n+1})$, and is called the group of **singular n-boundaries**.

Clearly, one has $\partial_n \circ \partial_{n+1} = 0$. The **$n$-th homology group** of X is then defined as the factor group

$$H_n(X) = Z_n(X)/B_n(X).$$

The elements of $H_n(X)$ are called **homology classes**.

A very important property of homology is the homotopy invariance.

If X and Y are two topological spaces with the same homotopy type, then

$$H_n(X) = H_n(Y),$$

for all $n \geq 0$. This means homology groups are topological invariants.

In particular, if X is a contractible space, then all its homology groups are 0, except $H_0(X) = \mathbb{Z}$.

Given any unital ring R, the set of singular n-simplices on a topological space can be taken to be the generators of a free R-module. That is, rather than performing the above constructions considering all the finite formal sums with integer coefficients, one instead uses as coefficients the elements of R. All of the constructions go through with little or no change. The result of this is

$$H_n(X, R)$$

which is now an R-module. Of course, it is usually not a free module. The usual homology group is regained by noting that

$$H_n(X, \mathbb{Z}) = H_n(X)$$

when one takes the ring to be the ring of integers. The notation $H_n(X, R)$ should not be confused with the nearly identical notation $H_n(X, A)$, which denotes the relative homology (below).

For a subspace $A \subset X$, the relative homology $H_n(X, A)$ is understood to be the homology of the quotient of the chain complexes, that is,

$$H_n(X, A) = H_n(C_\bullet(X)/C_\bullet(A))$$

where the quotient of chain complexes is given by the short exact sequence

$$0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X)/C_\bullet(A) \to 0,$$

where a sequence of maps $\varphi_1, \varphi_2, \ldots$ is **exact** if and only if Im $\varphi_i \subset$ Ker φ_{i+1}.

There is an important relationship between homology and foundamental group.

The fundamental groups of a topological space X are related to its first singular homology group, because a loop is also a singular 1-cycle. Mapping the homotopy class of each loop at a base point x_0 to the homology class of the loop gives a homomorphism

$$\varphi : \pi(X, x_0) \to H_1(X)$$

from the fundamental group $\pi(X, x_0)$ to the homology group $H_1(X)$.

If X is path-connected, then this homomorphism is surjective, then H_1 is isomorphic to $\pi(X, x_0)/$ Ker φ.

Moreover, for whom who knows theory of abelian group, Ker φ is the commutator subgroup of $\pi(X, x_0)$, and $H_1(X)$ is therefore isomorphic to the abelianization of $\pi(X, x_0)$. This is a special case of the Hurewicz theorem of algebraic topology.

Clearly the singular homology theory can be generalized to simplicial homology, i.e. the case which involve simplicial (or CW) complex instead of singular ones.

Let S be a simplicial complex. A **simplicial k-chain** is a formal sum of k-simplices

$$\sum_{i=1}^{N} c_i \sigma^i .$$

where c_i are integers (or element in a ring R).The group of k-chains on S, the free abelian group defined on the set of k-simplices in S, is denoted C_k.

Consider a basis element of C_k, a k-simplex,

$$\sigma = \left\langle v^0, v^1, ..., v^k \right\rangle .$$

The boundary operator

$$\partial_k : C_k \to C_{k-1}$$

is a homomorphism defined by:

$$\partial_k(\sigma) = \sum_{i=0}^{K} (-1)^i \left\langle v^0, ..., \hat{v}^i, ..., v^k \right\rangle ,$$

where the simplex

$$\left\langle v^0, ..., \hat{v}^i, ..., v^k \right\rangle$$

is the i-th face of σ obtained by deleting its ith vertex.

In C_k, elements of the subgroup

$$Z_k = \ker \partial_k$$

are referred to as **cycles**, and the subgroup

$$B_k = \operatorname{im} \partial_{k+1}$$

is said to consist of **boundaries**.

Direct computation shows that B_k lies in Z_k. The boundary of a boundary must be a cycle. In other words,

$$(C_k, \partial_k)$$

form a simplicial chain complex.

The k-th homology group H_k of S is defined to be the quotient

$$H_k(S) = Z_k/B_k.$$

A homology group H_k is not trivial if the complex at hand contains k-cycles which are not boundaries. This indicates that there are k-dimensional holes in the complex.

For example consider the complex obtained by glueing two triangles (with no interior) along one egde. The edges of each triangle form a cycle. These two cycles are by construction not boundaries (there are no 2-chains). Therefore one has two *1-holes*.

Holes can be of different dimensions. The rank of the homology groups, the numbers

$$\beta_k = \operatorname{rank}(H_k(S))$$

are referred to as the **Betti numbers** of the space S, and gives a measure of the number of k-dimensional holes in S.

The same construction applies if we consider CW-complexes instead of simplicial ones.

6. Salvetti's complex

The Salvetti's complex is a CW-complex associated to an hyperplane arrangement $\mathcal{A}$ which is the complexification of a real one.

This complex is very important in the arrangement theory because it is homotopically equivalent to the complement $M(\mathcal{A})$ of the arrangement $\mathcal{A}$.

This complex is interesting also for our studies. It will be useful in order to describe our model for objects in social choise.

Let $\mathcal{A} = \{H\}$ be a finite affine hyperplane arrangement in $\mathbb{R}^n$. Assume $\mathcal{A}$ essential, i.e. the minimal dimensional non-empty intersections of hyperplanes are points (which we call *vertices* of the arrangement). Equivalently, the maximal elements of the associated *intersection lattice* $L(\mathcal{A})$ (see above) have rank n.

Let

$$M(\mathcal{A}) \ = \mathbb{C}^n \setminus \bigcup_{H \in A} H_{\mathbb{C}}$$

be the complement to the complexified arrangement. We can construct (see [Sal87]) the regular CW-complex $\mathbf{S} = \mathbf{S}(\mathcal{A})$ which is a deformation retract of $M(\mathcal{A})$ as follow:

Let

$$\mathcal{S} := \{F^k\}$$

be the stratification of $\mathbb{R}^n$ into faces F^k which is induced by the arrangement (see above), where exponent k stands for codimension (i.e. the F^1 are the facets and $F^0 = C$ are the chambers of the complement $M(\mathcal{A})$). Then $\mathcal{S}$ has standard partial ordering

$$F^i \prec F^j \quad \text{iff} \quad clos(F^i) \supset F^j$$

where $clos(F^i)$ is the topological closure of the open F^i (see section 1).

The k-cells of the Salvetti complex $\mathbf{S}$ bijectively correspond to pairs

$$[C \prec F^k]$$

where $C = F^0$ is a chamber of $\mathcal{S}$.

Let $|F|$ be the affine subspace spanned by F, i.e. the minimal subspace which contains F, and let us consider the subarrangement

$$\mathcal{A}_F \ = \ \{H \in \mathcal{A} \ : \ F \subset H\}.$$

A cell $[C \prec F^k]$ is in the boundary of $[D \prec G^j]$ $(k < j)$ iff

i) $F^k \prec G^j$
ii) the chambers C and D are contained in the same chamber of $\mathcal{A}_{F^k}$.

Previous conditions are equivalent to say that C is the chamber of $\mathcal{A}$ which is "closest" to D among those which contain F^k in their closure.

Then the boundary $\partial[C \prec F^k]$ of a given k-cell on $\mathbf{S}$ is defined as an alternanting formal sum of the $(k-1)$-cells in its boundary.

It is possible to realize $\mathbf{S}$ inside $\mathbb{C}^n$ with explicitly given attaching maps of the cells (see [Sal87]).

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