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Social choice on complex objects: A geometric approach

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Abstract

In [MP08] L. Marengo and C. Pasquali present a model of object construction in majority voting and show that, in general, by appropriate changes of such bundles, different social outcomes may be obtained. In this paper we extend and generalize this approach by providing a geometric model of individual preferences and social aggregation based on hyperplanes and their arrangements. As an application of this model we give a necessary condition for existence of a local social optimum.

Moreover we address the question if a social decision rule depends also upon the number of voting agents. More precisely: are there social decision rules that can be obtained by an odd (even) number of voting agent which cannot be obtained by only three (two) voting agent? The answer is negative. Indeed three (or two) voting agent can produce all possible social decision rules.

Keywords: Social choice; object construction power; agenda power; intransitive cycles; arrangements; graph theory .

JEL classification: D71, D72

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1 Introduction

The baseline of every argument on individual and social choice is that agents choose among exogenously given and uni-dimensional objects according to their preferences.
Social choice models usually assume that choice is among pre-defined, unidimensional and simple objects. Very often, on the contrary, choice is among multifeatured and complex objects: a candidate in an election stands for an electoral programme which is a complex bundle of many interdependent political positions on a wide variety of issues. Also in committees and organizations of various sorts collective choices are most often made among policy bundles and authorities can act upon the pre-choice stage of construction of such bundles. This pre-choice power of alternatives construction may grant authorities

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a highly effective device to influence the outcome of social choice even when the latter is totally free and democratic.
In [MP08] authors propose a model which investigates within a simple majority vote framework the role of the object construction power, an analogous to the agenda power. Even when object construction is simply defined as the possibility of assembling and dis-assembling a fixed set of choice components into bundles, they show that, under rather general condition, it can radically change the outcome of the majority voting process. In particular they show that any set of bundles (that they call *choice modules*) is associated to a set of possible social outcomes which can be attained depending upon the initial conditions. Moreover they shows that also Condorcet-Arrow cycles can appear or disappear depending upon which set of modules is chosen.
More precisely, in [MP08] authors focus on and question about both the exogenously given and the uni-dimensional hypotheses. In particular, they study the case in which objects, far from being unstructured points in an abstract choice space, are composed of different parts, traits and features that can be variously instantiated and combined with one another. Under this perspective, objects can be conceived of as largely under-determined labels that stand for specific compositions of the underlying set of features and dimensions they are composed of. At the same time, they ask: where do alternatives come from? In answering this question, they try and model situations in which alternatives are endogenously constructed by a social actor that has an alternatives generation power which is fulfilled by structuring and instantiating objects features sets.
Three points are at stake and define the subject of their paper. First, as one's preferences might vary as long as the same object receives different instantiations, the power of defining an object by concretely coupling and instantiating its features set might have a significant relevance with respect to driving and constraining individual choice. Second, there is a wide room for interesting trade-offs to emerge as long as non separabilities (interdependencies) and non-monotonicities exist between different features of the same object. Third, there is an extent to which object construction can lead to specific social outcomes through the selection and categorization of appropriate traits/features sets. Broadly speaking, their results are about choice as taking place within an institutionally framed scenario which, at a minimum, constructs a set of alternatives.
They show that the very construction process is far from being neutral neither with respect to individual choice nor to the selection of social outcomes. In particular, they define some precise tools to investigate the relation between the possibility of aggregating individual preferences, their structure and the existence of some centralized form of power. Broadly speaking, their results suggest that the possibility of constructing aggregate states is to some extent founded upon the categorization performed by an underlying pre-choice institution.

Yet L. Marendo e C. Pasquali give only a computational model. The main idea of this paper is to give a precise mathematical description of the structures defined by authors in [MP08]. Indeed we believe that arrangement theory and related topics are a very pawerfull mathematical objects in order to give a mathematization of social choise theory. This language is usefull not only to explain the already known facts, but also to go farther and to find new and interesting results.

Example of relation between arrangement and social decision theory is already in [Te07] where the author generalizes the Arrow's impossibility theorem to all class of arrangements.

The first and biggest part of the paper is devoted to introduce classical mathematical objects which are necessary in order to better understand our construction.

In the second part authors rewrite the main part of [MP08] using a new mathematical description which allows to better understand results in [MP08].

In the third part authors give a first interesting application of this new mathematical model. They answer to the question if a social decision rule depends or not from the number of voting people. More precisely if it exists a social decision rule which can be generated by $2k+1$ (or $2k$) agents but cannot be generated by $2h+1$ (or $2h$) for $h \neq k$.
What if only two or three people are enough to generate all possible decision rule?
This would mean that a social decision rule is indipendet from the number of voting people, the only difference beeing between even and odd numbers. The authors prove exactly this assertion.

If $\Delta = \{(x,x) \in X \times X\}$ is the *diagonal* of the cartesian product $X \times X$, then, given a social decision rule $\succeq_{\mathcal{R}}$ or, simply, $\mathcal{R}$, they define a subset

$$Y_{\mathcal{R}} \subset X \times X \setminus \Delta \tag{1}$$

as follow: a couple (x_i, x_j) is in $Y_{\mathcal{R}}$ if and only if $x_i \succeq_{\mathcal{R}} x_j$; both (x_i, x_j) and (x_j, x_i) are in $Y_{\mathcal{R}}$ iff $x_i \succeq_{\mathcal{R}} x_j$ and $x_j \succeq_{\mathcal{R}} x_i$, While if both (x_i, x_j) and (x_j, x_i) aren't in $Y_{\mathcal{R}}$, then x_i and x_j are *indifferents*.
With these notations they prove the following

Theorem 1 *Given a subset $Y \subset X \times X \setminus \Delta$, there always exists a social decision rule $\mathcal{R}$ such that $Y = Y_{\mathcal{R}}$. Moreover any social decision rule $\mathcal{R}$ can be obtained by at most three voting agents.*

In the forth part authors use the new mathematical tools in order to prove that for all social decision rules $\mathcal{R}$, given a configuration z, it is possible to build a modules sheme A_z in the sense of [MP08] such that z is a local optimum for A_z under given conditions on z.
Moreover they notice that their result is a generalization of the Arrow's impossibility theorem [Arr51].

2 Mathematical tools: arrangements and Salvetti's complex

In order to construct our model for social choise, we need to introduce some mathematical tools. We will start giving some basic definitions and notations. For more detailed information the authors chose three books which they consider very good references for the following topics. The Orlik and Terao's book on arrangement [OT92] (subsection 2.1), the Bourbaki's book in General Topology [Bou66] (2.2), the Massey's book in Algebrai Topology [Ma91] (subsections 2.3, 2.4 and 2.5).

2.1 Arrangements

In geometry and combinatorics, an **arrangement of hyperplanes** is a finite set $\mathcal{A}$ of hyperplanes in a linear, affine, or projective space S. Questions about a hyperplane arrangement $\mathcal{A}$ generally concern geometrical, topological, or other properties of the complement, $M(\mathcal{A})$, which is the set that remains when the hyperplanes are removed from the whole space. One may ask how these properties are related to the arrangement and its intersection semilattice.

The intersection semilattice of $\mathcal{A}$, written $L(\mathcal{A})$, is the set of all subspaces that are obtained by intersecting some of the hyperplanes; among these subspaces are S itself, all the individual hyperplanes, all intersections of pairs of hyperplanes, etc. (excluding, in the affine case, the empty set). These subspaces are called the **flats** of $\mathcal{A}$. $L(\mathcal{A})$ is partially ordered by reverse inclusion.

If the whole space S is 2-dimensional, the hyperplanes are lines; such an arrangement is often called an arrangement of lines. Historically, real arrangements of lines were the first arrangements investigated. If S is 3-dimensional one has an arrangement of planes.

More precisely, let $\mathbb{K}$ be a field and let $V_{\mathbb{K}}$ be a vector space of dimension n. A **hyperplane** H in $V_{\mathbb{K}}$ is an affine subspace of dimension (n-1). A hyperplane arrangement

$$\mathcal{A}_{\mathbb{K}} = (\mathcal{A}_{\mathbb{K}}, V_{\mathbb{K}})$$

is a finite set of hyperplanes in $V_{\mathbb{K}}$.

We are interesting in the real and complex cases, then from now on $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $V = \mathbb{R}^n, \mathbb{C}^n$. Then choosen the canonical basis $\{e_1, \ldots, e_n\}$ in V, each hyperplane $H \in \mathcal{A}$ is the kernel of a polynomial $\alpha_H \in \mathbb{K}[x_1, \ldots, x_n]$ of degree 1 defined up to a costant. The product:

$$\mathcal{Q}(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$$

is called **defining polynomial** of $\mathcal{A}$.

The **cardinality** $\mid \mathcal{A} \mid$ of the arrangement $\mathcal{A}$ is the number of hyperplanes in $\mathcal{A}$.

If $\mathcal{B} \subset \mathcal{A}$ is a subset, then it is called a **subarrangement** of $\mathcal{A}$. We define the set of all nonempty intersections of elements of $\mathcal{A}$ as:

$$L(\mathcal{A}) = \{\cap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}\}$$

Given an element $X \in L(\mathcal{A})$, we define a subarrangement $\mathcal{A}_X$ of $\mathcal{A}$ by:

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\},$$

and an arrangement in X by:

$$\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X and X \cap H \neq \emptyset\}.$$

$\mathcal{A}^X$ is called the **restriction** of $\mathcal{A}$ to X.

Finally we define the **complement** of $\mathcal{A}$ by

$$M(\mathcal{A}) = V \setminus \cup_{H\in\mathcal{A}} H.$$

The complement of an arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is clearly disconnected: it is made up of separate pieces called **chambers** or **regions**, each of which is either a bounded region or an unbounded region which goes off to infinity.

Each flat of $\mathcal{A}$ is also divided into pieces by the hyperplanes that do not contain the flat; these pieces are called the **faces** of $\mathcal{A}$. The regions are faces because the whole space is a flat. The faces of codimension 1 may be called the **facets** of A.

The **face semilattice** of an arrangement is the set of all faces, ordered by inclusion. Adding an extra top element to the face semilattice gives the face lattice.

Example 1 *Let us give examples:*

- *if the arrangement consists of three parallel lines, the intersection semilattice consists of the plane and the three lines, but not the empty set. There are four regions, none of them bounded.*
- *If we add a line crossing the three parallels, then the intersection semilattice consists of the plane, the four lines, and the three points of intersection. There are eight regions, still none of them bounded.*
- *If we add one more line, parallel to the last, then there are 12 regions, of which two are bounded parallelograms.*

Every arrangement $(\mathcal{A}_{\mathbb{R}}, \mathbb{R}^n)$ gives rise to an arrangement over $\mathbb{C}$. Let $(\mathcal{A}_{\mathbb{R}}, \mathbb{R}^n)$ be an arrangement with defining polynomial $\mathcal{Q}(\mathcal{A}_{\mathbb{R}})$. The $\mathbb{C}$-extended arrangement is in $\mathbb{C}^n$. It consists of the hyperplanes which are the kernel of the polynomial α_H in $\mathbb{C}^n$ instead of $\mathbb{R}^n$.

For more information on arrengement theory see [OT92].

2.2 Basic notions in General Topology

Topological spaces are mathematical structures that allow the formal definition of concepts such as convergence, connectedness, and continuity. They appear in virtually every branch of modern mathematics and are a central unifying notion. The branch of mathematics that studies topological spaces in their own right is called *Topology*.

A **topological space** is a set X together with T, a collection of subsets of X, satisfying the following axioms:

1. *the empty set and X are in T;*
2. *the union of any collection of sets in T is also in T;*
3. *the intersection of any finite collection of sets in T is also in T.*

The collection T is called a **topology** on X.

The elements of X are usually called **points**, though they can be any mathematical objects. A topological space in which the points are functions is called a function space.

The sets in T are the **open sets**, and their complements in X are called **closed sets**. A set may be neither closed nor open, either closed or open, or both.

Given an open set $U \in T$, the smaller closed set which contain it is called **closure** of U and it is usually indicated by $\overline{U}$.

Example 2 *Let us give some elementary example of topological spaces:*

1. $X = \{1, 2, 3, 4\}$ *and collection* $T = \{\{\}, \{1, 2, 3, 4\}\}$ *of two subsets of* X *form a trivial topology.*
2. $X = \{1, 2, 3, 4\}$ *and collection* $T = \{\{\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$ *of six subsets of* X *form another topology.*
3. $X = \mathbb{Z}$*, the set of integers and collection* T *equal to all finite subsets of the integers plus* $\mathbb{Z}$ *itself is not a topology, because (for example) the union over all finite sets not containing zero is infinite but is not* $\mathbb{Z}$*, and so is not in* T*.*

A variety of topologies can be placed on a set to form a topological space. When every set in a topology T_1 is also in a topology T_2, we say that T_2 is *finer* than T_1, and T_1 is *coarser* than T_2.

A **neighbourhood** of a point x is any set that contains an open set containing x. The neighbourhood system at x consists of all neighbourhoods of x. A topology can be determined by a set of axioms concerning all neighbourhood systems.

As usual in mathematics we define relations between topological spaces. These relations are **continuous functions**.

A function between topological spaces is said to be continuous if the inverse image of every open set is open. In mathematical terms, given two spaces X and Y and topologies T_X and T_Y on X and Y respectively, then a function

$$f : X \to Y$$

is continuous if and only if $f^{-1}(U) \in T_X$ for all $U \in T_Y$.

This is an attempt to capture the intuition that there are no *breaks* or *separations* in the function.

A **homeomorphism** is a bijection that is continuous and whose inverse is also continuous. Two spaces are said to be homeomorphic if there exists a homeomorphism between them.

From the standpoint of topology, homeomorphic spaces are essentially identical.

The category of topological spaces with topological spaces as objects and continuous functions as morphisms is one of the fundamental categories in mathematics. The attempt to classify the objects of this category (up to homeomorphism) by invariants has motivated and generated entire areas of research, such as homotopy theory, homology theory, and K-theory, to name just a few.

Let see some other important and useful example of topologies on a given set X, remarking that if a set is given a different topology, it is viewed as a different topological space.

Any set can be given the **discrete topology** in which every subset is open. Also, any set can be given the **trivial topology** (also called the indiscrete topology), in which only the empty set and the whole space are open.

However, oftentimes topological spaces are required to be **Hausdorff spaces**: let x and y be points in a topological space X. We say that x and y can be *separated by neighbourhoods* if there exists a neighbourhood U of x and a neighbourhood V of y such that U and V are disjoint ($U \cap V = \emptyset$). X is a Hausdorff space if any two distinct points of X can be separated by neighborhoods.

There are many ways of defining a topology on $\mathbb{R}$, the set of real numbers. The **standard topology** on $\mathbb{R}$ is generated by the open intervals. The set of all open intervals forms a base or basis for the topology, meaning that every open set is a union of some collection of sets from the base. In particular, this means that a set is open if there exists an open interval of non zero radius about every point in the set.

More generally, the Euclidean spaces $\mathbb{R}^n$ can be given a topology. In the usual topology on $\mathbb{R}^n$ the basic open sets are the open balls, meaning n-dimensional balls centered in a point $x \in \mathbb{R}^n$ with radius $\varepsilon > 0$.

Similarly, $\mathbb{C}$ and $\mathbb{C}^n$ have a standard topology in which the basic open sets are open balls.

Every metric space can be given a metric topology, in which the basic open sets are open balls defined by the metric. This is the *standard topology* on any normed vector space. On a finite-dimensional vector space this topology is the same for all norms.

Many sets of operators in functional analysis are endowed with topologies that are defined by specifying when a particular sequence of functions converges to the zero function.

Any local field has a topology native to it, and this can be extended to vector spaces over that field.

Every manifold has a natural topology since it is locally Euclidean.

The real line can also be given the lower limit topology. Here, the basic open sets are the half open intervals $[a, b)$. This topology on $\mathbb{R}$ is strictly finer than the Euclidean topology defined above. This example shows that a set may have many distinct topologies defined on it.

A very important notion in mathematics is the notion of *substructure*. Every subset $Y \subset X$ of a topological space X can be given the **subspace topology** or **induced topology** in which the open sets are the intersections of the open sets of X with the subset Y, i.e.

$$T_Y = \{W \cap X \mid U \in T_X\}.$$

For any indexed family of topological spaces $\{X_i\}_{i\in I}$, the product $\prod_{i\in I} X_i$ can be given the **product topology**, which is generated by the inverse images of open sets of the factors under the projection mappings

$$p_j : \prod_{i\in I} X_i \to X_j.$$

For example, in finite products, a basis for the product topology consists of all products of open sets. For infinite products, there is the additional requirement that in a basic open set, all but finitely many of its projections are the entire space.

A quotient space is defined as follows: if X is a topological space and Y is a set, and if

$$f : X \longrightarrow Y$$

is a surjective function, then the **quotient topology** on Y is the collection

$$T_{q,Y} = \{U \subset Y \mid f^{-1}(U) \in T_X\}$$

of subsets of Y that have open inverse images under f. In other words, the quotient topology is the finest topology on Y for which f is continuous. A common example of a quotient topology is when an equivalence relation is defined on the topological space X. The map f is then the natural projection onto the set of equivalence classes.

Topological spaces can be broadly classified, up to homeomorphism, by their topological properties. A **topological property** is a property of spaces that is invariant under homeomorphisms. To prove that two spaces are not homeomorphic it is sufficient to find a topological property which is not shared by them. Examples of such properties include connectedness, compactness, and various separation axioms.

See [Bou66] on references for more details and examples.

2.3 Simplicial and CW complexes

In mathematics, a **simplicial complex** is a topological space of a particular kind, constructed by "gluing together" points, line segments, triangles, and their n-dimensional counterparts. In order to define a simplicial complex we need to define **simplexes**.

Let us define the **convex hull** for a set of points $X = \{x_1, \ldots, x_m\}$ in a real vector space V as follow:

$$H(X) = \{\sum_{i=1}^{k} \alpha_i x_i \mid x_i \in X, \alpha_i \in \mathbb{R}, \alpha_i \geq 0, \sum_{i=1}^{k} \alpha_i = 1, k = 1, 2, \ldots\}.$$

In geometry, a **simplex** (plural simplexes or simplices) or n-simplex is an n-dimensional analogue of a triangle. Specifically, a simplex is the convex hull of a set of (n + 1) affinely independent points in some Euclidean space of dimension n or higher (i.e., a set of points such that no m-plane contains more than (m + 1) of them; such points are said to be in *general position*).

For example, a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and a 4-simplex is a pentachoron (in each case with interior). The convex hull of any nonempty subset of the n+1 points that define an n-simplex is called a **face** of the simplex. Faces are simplices themselves.

In particular, the convex hull of a subset of size m+1 (of the n+1 defining points) is an m-simplex, called an m-face of the n-simplex.
The 0-faces (i.e., the defining points themselves as sets of size 1) are called the **vertices** (singular: vertex), the 1-faces are called the **edges**, the (n-1)-faces are called the **facets**, and the sole n-face is the whole n-simplex itself.
In general, the number of m-faces is equal to the binomial coefficient C (n + 1 , m + 1).

A **simplicial complex** X is a set of simplices that satisfies the following conditions:

1. Any face of a simplex from X is also in X.
2. The intersection of any two simplices $\sigma_1, \sigma_2 \in X$ is a face of both σ_1 and σ_2.

Note that the empty set is a face of every simplex.

Let us remark that every $n-$simplex and every $n-$simplicial complex inherits a natural topology from $\mathbb{R}^n$.

A **simplicial k-complex** X is a simplicial complex where the largest dimension of any simplex in X equals k. For instance, a simplicial 2-complex must contain at least one triangle, and must not contain any tetrahedra or higher-dimension simplices.

A **pure** or **homogeneous** simplicial k-complex is a simplicial complex where every simplex of dimension less than k is the face of some simplex of dimension exactly k. Informally, a pure 1-complex "looks" like it's made of a bunch of lines, a 2-complex "looks" like it's made of a bunch of triangles, etc. An example of a non-homogeneous complex is a triangle with a line segment attached to one of its vertices.

A **facet** is any simplex in a complex that is not the face of any larger simplex. (Note the difference from the "facet" of a simplex.)

A pure simplicial complex can be thought of as a complex where all facets have the same dimension.

Sometimes the term face is used to refer to a simplex of a complex, not to be confused with the face of a simplex. For a simplicial complex embedded in a k-dimensional space, the k-faces are sometimes referred to as its **k-cells**. The term cell is sometimes used in a broader sense to denote a set homeomorphic to a simplex, leading to the definition of cell complex.

A **cell complex** or simply complex in $\mathbb{R}^n$ is a set X in $\mathbb{R}^n$ of convex polyhedra, i.e. the set of solutions to a finite system of linear inequalities, satisfying two conditions:

1. Every face of a cell is a cell (i.e. it is in X).

2. If σ_1 and σ_2 are cells, then their intersection is a common face of both.

Each convex polyhedron is called **cell**.

A simplicial complex is a cell complex whose cells are all simplices.

We are interested in a particula cell complex, the CW-complex. In order to define it we need to introduce the following:

A **finite graph** is a pair consisting of a Hausdorff space X and a subspace X_0 (called the *set of vertices of* X) such that the following conditions hold:

1. X_0 *is a finite discrete (i.e. it inherits the discrete topology from* X*), closed subspace of* X*. Points of* X_0 *are called* **vertices**;
2. $X \setminus X_0$ *is the finite disjoint union of open subsets* e_i*, where each* e_i *is homeomorphic to an open interval of the real line. The set* e_i *are called* **edges**;
3. *for each edge* e_i*, if* $\overline{e_i}$ *is the closure of* e_i*, then its* **boundary** $\overline{e_i} \setminus e_i$ *is a subset of* X_0 *consisting of one or two points. If* $\overline{e_i} \setminus e_i$ *consists of two points, then the pair* $(\overline{e_i}, e_i)$ *is homeomorphic to the pair* $([0,1],(0,1))$*,if* $\overline{e_i} \setminus e_i$ *consists of one point, then the pair* $(\overline{e_i}, e_i)$ *is homeomorphic to the pair* $(S_1, S_1 \setminus \{1\})$*, where* S_1 *is the unit circle in the plane.*

It is a simple remark to notice that the definition of graph is a generalization of a 1-dimensional simplicial complex.

Moreover it is possible to give an orientation to a graph, simply ordering its vertices. So an edge will be oriented going from the lesser to the bigger vertices.

Now, we can give the definition of **CW-complex**.

In topology, a CW complex is a type of topological space introduced by J. H. C. Whitehead to meet the needs of homotopy theory. The idea was to have a class of spaces that was broader than simplicial complexes, but still retained a combinatorial nature, so that computational considerations were not ignored. For these purposes a closed cell is a topological space homeomorphic to a simplex, or equally a ball (of which a sphere is the boundary) or cube in n dimensions.

First of all we need to introduce the notion of *adjoining* cells to a space (for a more precise definition see also [Ma91]).

A n-dimensional cell is attached to a space X by gluing a closed n-dimensional ball D_n to the **(n-1)-skeleton** X_{n-1} of X, i.e., the union of all cells of dimension lower than n in X.

The gluing is specified by a continuous function

$$f : S_{n-1} \to X_{n-1}$$

from the $n-1$-dimensional unit sphere $S_{n-1} = \partial D_n$ to X_{n-1}.

The points on the new space are exactly the equivalence classes of points in the disjoint union $X \dot{\cup} D_n$ of the old space and the closed cell D_n, the equivalence relation being the transitive closure of $x \equiv f(x)$ (i.e., the smallest transitive relation that contains $\equiv$).

The function f plays an essential role in determining the nature of the newly enlarged complex. For example, if the 2−dimensional ball D_2 is glued onto the circle S_1 in the usual way, i.e. with the funcion f given by the identity map, we get D_2 itself; if f has winding number 2, we get the real projective plane instead.

The process of adjoining cells to a space leads naturally to the notion of CW-complex. Roughly speaking, a CW-complex is a space X which can be built up as follows:

Start with a graph X_1 and adjoin a collection of 2-cells as described above to obtain a space X_2. Next adjoin a collection of 3-cells and so on. Then

$$X = \cup_{n=1}^{\infty} X_n$$

is a CW-complex. Moreover if all attaching maps are homeomorphisms, the structure is called a **regular CW-complex**.

More precisely, a CW-complex is defined on an Housdorff space X by the prescription of an ascending sequence

$$X_0 \subset X_1 \subset X_2 \subset \dots$$

of closed subspaces of X which satisfies the following conditions:

1. *X_0 is a discrete space;*
2. *for $n \geq 0$, X_n is obtained from $X_{(n-1)}$ by attachment of a collection of n-cells as described above.*
3. $$X = \cup_{n=1}^{\infty} X_n$$
4. **(weak topology)***X and each of X_n have the weak topology; i.e, a subset A of X (or X_n) is closed if and only if the intersection $A \cup \overline{e_q}$ of A with the closure of a q-cell e_q is closed for each e_q.*

The subspace X_n is called the **n-skeleton** of X.

If there are no cells of dimension greater than n, X is called **finite dimensional**.

Let us notice that a graph is a 1-dimensional CW-complex.

Note that there normally are many possible choices of a filtration by skeleta for a given CW-complex. A particular choice of skeleta and attaching maps for the cells is called a **CW-structure** on the space.

A subspace Y of a CW-complex X is called a **subcomplex** if Y is a union of cells of X, and if for any q-cell e_q, if $e_q \subset Y$ then $\overline{e_q} \subset Y$. If this is the case, we define the n-skeleton Y_n by

$$Y_n = X_n \cap Y.$$

It can be shown that Y is also a CW-complex, and it is a closed subset of X.

Associated to a cell complex there is the **Euler characteristic** which is defined as the alternating sum

$$\chi = k_0 - k_1 + k_2 - k_3 + \ldots,$$

where k_n denotes the number of cells of dimension n in the complex.

For more detail see [Ma91]

2.4 Some notion of homotopy theory: the fundamental group

In mathematics, the fundamental group is one of the basic concepts of algebraic topology. Associated with every point of a topological space there is a **fundamental group** that conveys information about the 1-dimensional structure of the portion of the space surrounding the given point. The fundamental group is the first homotopy group.

Before giving a precise definition of the fundamental group, we try to describe the general idea in non-mathematical terms. Take some space, and some point in it, and consider all the loops both starting and ending at this point: paths which start at this point, wander around as much as they like and eventually return to the starting point.

Two loops can be combined together in an obvious way: travel along the first loop, then along the second. The set of all the loops with this method of combining them is the fundamental group, except that for technical reasons it is necessary to consider two loops to be the same if one can be deformed into the other without breaking.

For the precise definition, let X be a topological space, and let $x_0 \in X$ be a point of X. We are interested in the set of continuous functions

$$f : [0,1] \longrightarrow X$$

with the property that $f(0) = x_0 = f(1)$. These functions are called **loops** with base point x_0. Any two such loops, say f and g, are considered equivalent if there is a continuous function

$$h : [0,1] \times [0,1] \longrightarrow X$$

with the property that, for all $0 \leq t \leq 1$, $h(t,0) = f(t)$, $h(t,1) = g(t)$ and $h(0,t) = x_0 = h(1,t)$. Such an h is called a **homotopy** from f to g, and the corresponding equivalence classes are called **homotopy classes**.

The **product** $f * g$ of two loops f and g is defined by setting

$$(f * g)(t) := \begin{cases} f(2t), & 0 \leq t \leq 1/2 \\ g(2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

Thus the loop $f * g$ first follows the loop f with *twice the speed* and then follows g with twice the speed. The product of two homotopy classes of loops $[f]$ and $[g]$ is then defined as $[f * g]$, and it can be shown that this product does not depend on the choice of representatives.

With the above product, the set of all homotopy classes of loops with base point x_0 forms the **fundamental group** of X at the point x_0 and is denoted:

$$\pi_1(X, x_0),$$

or simply $\pi(X, x_0)$. The *identity element* is the constant map at the basepoint, and the *inverse* of a loop f is the loop g defined by $g(t) = f(1-t)$. That is, g follows f backwards.

Although the fundamental group in general depends on the choice of base point, it turns out that, up to isomorphism, this choice makes no difference if the space X is *path-connected*, i.e. for all two points x_1, x_2 in X there is a **path** which joins x_1 and x_2 in X, i.e. there is a continuous function

$$f : [0,1] \longrightarrow X$$

such that $f(0) = x_1$ and $f(1) = x_2$.

For path-connected spaces, therefore, we can write $\pi(X)$ instead of $\pi(X, x_0)$ without ambiguity whenever we care about the isomorphism class only.

In many spaces, such as $\mathbb{R}^n$, there is only one homotopy class of loops, and the fundamental group is therefore trivial, i.e. $\pi(\mathbb{R}^n) = (0, +)$.

A path-connected space with a trivial fundamental group is said to be **simply connected**.

A more interesting example is provided by the circle S_1. It turns out that each homotopy class consists of all loops which wind around the circle a given number of times (which can be positive or negative, depending on the direction of winding). The product of a loop which winds around m times and another that winds around n times is a loop which winds around $m+n$ times. So the fundamental group $\pi(S_1)$ of the circle S_1 is isomorphic to $(\mathbb{Z}, +)$, the additive group of integers.

Since the fundamental group is a homotopy invariant, the theory of the winding number for the complex plane minus one point is the same as for the circle.

The fundamental group of a graph G is a free group. Here the rank of the free group is equal to $1-\chi(G)$: one minus the Euler characteristic of G, when G is connected, i.e., there is a path from any point to any other point in the graph.

Given two topologicxal spaces X and Y, if $f : X \longrightarrow Y$ is a continuous map, $x_0 \in X$ and $y_0 \in Y$ with $f(x_0) = y_0$, then every loop in X with base point x_0 can be composed with f to yield a loop in Y with base point y_0.

This operation is compatible with the homotopy equivalence relation and with composition of loops. The resulting group homomorphism, called the **induced homomorphism**, is written as $\pi(f)$ or, more commonly,

$$f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0).$$

If $f, g : X \rightarrow Y$ are continuous maps with $f(x_0) = g(x_0) = y_0$, f and g are homotopic if and only if exists a continuous function

$$H : X \times [0,1] \rightarrow Y$$

from the product of the space X with the unit interval $[0, 1]$ to Y such that, for all points x in X, $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

If f and g are homotopic relative to $\{x_0\}$, then $f_* = g_*$.

Given two spaces X and Y, we say they are **homotopy equivalent** or of the same homotopy type if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map id_X and $f \circ g$ is homotopic to id_Y.

The maps f and g are called homotopy equivalences in this case.

Clearly, every homeomorphism is a homotopy equivalence, but the converse is not true: for example, a solid disk is not homeomorphic to a single point, although the disk and the point are homotopy equivalent.

Intuitively, two spaces X and Y are homotopy equivalent if they can be transformed into one another by bending, shrinking and expanding operations. For example, a solid disk or solid ball is homotopy equivalent to a point, and $\mathbb{R}^2 \setminus \{(0, 0)\}$ is homotopy equivalent to the unit circle S_1.

Spaces that are homotopy equivalent to a point are called **contractible**.

A function f is said to be **null-homotopic** if it is homotopic to a constant function. The homotopy from f to a constant function is then sometimes called a null-homotopy. For example, it is simple to show that a map from the circle S_1 is null-homotopic precisely when it can be extended to a map of the disc D_2.

It follows from these definitions that a space X is contractible if and only if the identity map from X to itself, which is always a homotopy equivalence, is null-homotopic.

From the above definitions follows that two homotopy equivalent path-connected spaces have isomorphic fundamental groups:

$$X \simeq Y \Rightarrow \pi_1(X, x_0) \cong \pi_1(Y, y_0).$$

Moreover, if X and Y are path connected, then

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$$

and

$$\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y).$$

In the latter formula, $\vee$ denotes the wedge sum of topological spaces, and $*$ the free product of groups. Both formulas generalize to arbitrary products.

For more detail see [Ma91]

2.5 Some notions on singular homology theory

In algebraic topology singular homology refers to the study of a certain set of topological invariants of a topological space X, the so-called homology groups $H_n(X)$. Singular homology is a particular example of a homology theory, which has now grown to be a rather broad collection of theories. Of the various theories, it is perhaps one of the simpler ones to understand, being built on fairly concrete constructions.

In brief, singular homology is constructed by taking maps of the standard n-simplex to a topological space, and composing them into formal sums, called singular chains. The boundary operation on a simplex (see below) induces a singular chain complex. The singular homology is then the homology of the chain complex.

The resulting homology groups are the same for all homotopically equivalent spaces, which is the reason for their study.

These constructions can be applied to all topological spaces. These ideas are developed in greater detail below.

Start defining **singular simplices**. A **singular n-simplex** is a continuous mapping from the standard n-simplex Δ^n to a topological space X.

Notationally, one writes

$$\sigma_n : \Delta^n \to X$$

. This mapping need not be injective, and there can be non-equivalent singular simplices with the same image in X.

The boundary of σ_n, denoted as $\partial_n \sigma_n$, is defined to be the formal sum of the singular $(n-1)$-simplices represented by the restriction of σ_n to the faces of the standard n-simplex, with an alternating sign to take orientation into account.

That is, if

$$\sigma_n = [p_0, p_1, \cdots, p_n] = \sigma_n([e_0, e_1, \cdots, e_n])$$

are the corners of the n-simplex corresponding to the vertices e_k of the standard n-simplex Δ^n, then

$$\partial_n \sigma_n = \sum_{k=0}^{n} (-1)^k [p_0, \cdots, p_{k-1}, p_{k+1}, \cdots p_n]$$

is the formal sum of the (oriented) faces of the simplex.

Thus, for example, the boundary of a 1-simplex $\sigma = [p_0, p_1]$ is the formal difference $\sigma_1 - \sigma_0 = [p_1] - [p_0]$.

Consider first the set $\sigma_n(X)$ of all possible singular n-simplices on a topological space X. Then we can consider all finite formal sums of singular simplices with integer coefficients. All these sums give rise to an abelian group which is very large, usually infinite, frequently uncountable, as there are many ways of mapping a simplex into a typical topological space. This group is commonly denoted as $C_n(X)$. Elements of $C_n(X)$, i.e. a finite formal sum of simplices, are called **singular n-chains**.

The boundary ∂ is readily extended to act on singular n-chains. The extension, called the **boundary operator**, written as

$$\partial_n : C_n \to C_{n-1},$$

is a homomorphism of groups. The boundary operator, together with the C_n, form a **chain complex** of abelian groups, called the **singular complex**. It is often denoted as $(C_\bullet(X), \partial_\bullet)$ or more simply $C_\bullet(X)$.

The kernel of the boundary operator is $Z_n(X) = \ker(\partial_n)$, and is called the group of **singular n-cycles**. The image of the boundary operator is $B_n(X) = \mathrm{im}(\partial_{n+1})$, and is called the group of **singular n-boundaries**.

Clearly, one has $\partial_n \circ \partial_{n+1} = 0$. The **$n$-th homology group** of X is then defined as the factor group

$$H_n(X) = Z_n(X)/B_n(X).$$

The elements of $H_n(X)$ are called **homology classes**.

A very important property of homology is the homotopy invariance.
If X and Y are two topological spaces with the same homotopy type, then

$$H_n(X) = H_n(Y),$$

for all $n \geq 0$. This means homology groups are topological invariants.
In particular, if X is a contractible space, then all its homology groups are 0, except $H_0(X) = \mathbb{Z}$.

Given any unital ring R, the set of singular n-simplices on a topological space can be taken to be the generators of a free R-module. That is, rather than performing the above constructions considering all the finite formal sums with integer coefficients, one instead uses as coefficients the elements of R. All of the constructions go through with little or no change. The result of this is

$$H_n(X, R)$$

which is now an R-module. Of course, it is usually not a free module. The usual homology group is regained by noting that

$$H_n(X, \mathbb{Z}) = H_n(X)$$

when one takes the ring to be the ring of integers. The notation $H_n(X, R)$ should not be confused with the nearly identical notation $H_n(X, A)$, which denotes the relative homology (below).

For a subspace $A \subset X$, the relative homology $H_n(X, A)$ is understood to be the homology of the quotient of the chain complexes, that is,

$$H_n(X, A) = H_n(C_\bullet(X)/C_\bullet(A))$$

where the quotient of chain complexes is given by the short exact sequence

$$0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X)/C_\bullet(A) \to 0,$$

where a sequence of maps $\varphi_1, \varphi_2, \ldots$ is **exact** if and only if $\mathrm{Im}\ \varphi_i \subset \mathrm{Ker}\ \varphi_{i+1}$.

There is an important relationship between homology and foundamental group.
The fundamental groups of a topological space X are related to its first singular homology group, because a loop is also a singular 1-cycle. Mapping the homotopy class of each loop at a base point x_0 to the homology class of the loop gives a homomorphism

$$\varphi : \pi(X, x_0) \to H_1(X)$$

from the fundamental group $\pi(X, x_0)$ to the homology group $H_1(X)$.

If X is path-connected, then this homomorphism is surjective, then H_1 is isomorphic to $\pi(X, x_0)/$ Ker φ.

Moreover, for whom who knows theory of abelian group, Ker φ is the commutator subgroup of $\pi(X, x_0)$, and $H_1(X)$ is therefore isomorphic to the abelianization of $\pi(X, x_0)$. This is a special case of the Hurewicz theorem of algebraic topology.

Clearly the singular homology theory can be generalized to simplicial homology, i.e. the case which involve simplicial (or CW) complex instead of singular ones.

Let S be a simplicial complex. A **simplicial k-chain** is a formal sum of k-simplices

$$\sum_{i=1}^{N} c_i \sigma^i \,.$$

where c_i are integers (or element in a ring R).The group of k-chains on S, the free abelian group defined on the set of k-simplices in S, is denoted C_k.

Consider a basis element of C_k, a k-simplex,

$$\sigma = \left\langle v^0, v^1, ..., v^k \right\rangle .$$

The boundary operator

$$\partial_k : C_k \to C_{k-1}$$

is a homomorphism defined by:

$$\partial_k(\sigma) = \sum_{i=0}^{K} (-1)^i \left\langle v^0, ..., \hat{v}^i, ..., v^k \right\rangle ,$$

where the simplex

$$\left\langle v^0, ..., \hat{v}^i, ..., v^k \right\rangle$$

is the i-th face of σ obtained by deleting its ith vertex.

In C_k, elements of the subgroup

$$Z_k = \ker \partial_k$$

are referred to as **cycles**, and the subgroup

$$B_k = \operatorname{im} \partial_{k+1}$$

is said to consist of **boundaries**.

Direct computation shows that B_k lies in Z_k. The boundary of a boundary must be a cycle. In other words,

$$(C_k, \partial_k)$$

form a simplicial chain complex.

The k-th homology group H_k of S is defined to be the quotient

$$H_k(S) = Z_k/B_k.$$

A homology group H_k is not trivial if the complex at hand contains k-cycles which are not boundaries. This indicates that there are k-dimensional holes in the complex.

For example consider the complex obtained by glueing two triangles (with no interior) along one egde. The edges of each triangle form a cycle. These two cycles are by construction not boundaries (there are no 2-chains). Therefore one has two *1-holes*.

Holes can be of different dimensions. The rank of the homology groups, the numbers

$$\beta_k = \text{rank}(H_k(S))$$

are referred to as the **Betti numbers** of the space S, and gives a measure of the number of k-dimensional holes in S.

The same construction applies if we consider CW-complexes instead of simplicial ones.

for more detail see [Ma91]

2.6 Salvetti's complex

The Salvetti's complex is a CW-complex associated to an hyperplane arrangement $\mathcal{A}$ which is the complexification of a real one.

This complex is very important in the arrangement theory because it is homotopically equivalent to the complement $M(\mathcal{A})$ of the arrangement $\mathcal{A}$.

This complex is interesting also for our studies. It will be useful in order to describe our model for objects in social choise.

Let $\mathcal{A} = \{H\}$ be a finite affine hyperplane arrangement in $\mathbb{R}^n$. Assume $\mathcal{A}$ essential, i.e. the minimal dimensional non-empty intersections of hyperplanes are points (which we call *vertices* of the arrangement). Equivalently, the maximal elements of the associated *intersection lattice* $L(\mathcal{A})$ (see above) have rank n.

Let

$$M(\mathcal{A}) \ = \mathbb{C}^n \setminus \bigcup_{H \in A} H_{\mathbb{C}}$$

be the complement to the complexified arrangement. We can construct (see [Sal87]) the regular CW-complex $\mathbf{S} = \mathbf{S}(\mathcal{A})$ which is a deformation retract of $M(\mathcal{A})$ as follow:

Let

$$\mathcal{S} := \{F^k\}$$

be the stratification of $\mathbb{R}^n$ into faces F^k which is induced by the arrangement (see above), where exponent k stands for codimension (i.e. the F^1 are the facets and $F^0 = C$ are the chambers of the complement $M(\mathcal{A})$). Then $\mathcal{S}$ has standard partial ordering

$$F^i \ \prec F^j \quad \text{iff} \quad clos(F^i) \supset F^j$$

where $clos(F^i)$ is the topological closure of the open F^i (see section 2.1).

The k-cells of the Salvetti complex $\mathbf{S}$ bijectively correspond to pairs

$$[C \prec F^k]$$

where $C = F^0$ is a chamber of $\mathcal{S}$.

Let $|F|$ be the affine subspace spanned by F, i.e. the minimal subspace which contains F, and let us consider the subarrangement

$$\mathcal{A}_F \;=\; \{H \in \mathcal{A} \;:\; F \subset H\}.$$

A cell $[C \prec F^k]$ is in the boundary of $[D \prec G^j]$ $(k < j)$ iff

i) $F^k \prec G^j$
ii) the chambers C and D are contained in the same chamber of $\mathcal{A}_{F^k}$.

Previous conditions are equivalent to say that C is the chamber of $\mathcal{A}$ which is "closest" to D among those which contain F^k in their closure.

Then the boundary $\partial[C \prec F^k]$ of a given k-cell on **S** is defined as an alternanting formal sum of the $(k-1)$-cells in its boundary.

Notation 2.1 *i) We denote the chamber D which appear in the boundary cell $[D \prec G^j]$ of a cell $[C \prec F^k]$ by $C.G^j$.*
ii) More generally, given a chamber C and a facet F, we denote by $C.F$ the unique chamber containing F and lying in the same chamber as C in $\mathcal{A}_{F^k}$. Given two facets F, G we will use also for $(C.F).G$ the notation (without brackets) $C.F.G$.

It is possible to realize **S** inside $\mathbb{C}^n$ with explicitly given attaching maps of the cells (see [Sal87]).

3 Social decision surfaces

We assume that choices are made over a set of n elements or *features* $F = \{f_1, \ldots, f_n\}$ taking a value out of a finite set of $m+1$ possibilities.
For simplicity, without loss of generality, we label these possibilities respectively $0, 1, 2, \cdots, m$; therefore $f_i \in \{0, 1, 2, \ldots, m\}$. Then the space of possibilities is given by $(m+1)^n$ possible *configurations* $X = \{x_1, \ldots, x_{(m+1)^n}\}$.
This corresponds to choose in $\mathbb{R}^n$ an hyperplane arrangement

$$\mathcal{A}_{n,m} = \{H_{i,j}\}_{\substack{1 \le i \le n \\ 0 \le j \le m-1}}, \tag{2}$$

where $H_{i,j}$ is the hyperplane of equation $y_i = j$.
Then each configuration x_i corresponds to a chamber C_i of the arrangement $\mathcal{A}_{n,m}$.
In particular there is a standard correspondence between configurations and chambers given by assigning to the configuration $x_i = i_1 \cdots i_n$ the chamber which contains the open set

$$\{(y_1, \ldots, y_n) \in \mathbb{R}^n \mid i_j - 1 < y_j < i_j j = 1, \ldots, n\}. \tag{3}$$

Moreover given a social decision rule $\mathcal{R}$, we define a subset

$$Y_{\mathcal{R}} \subset X \times X \setminus \Delta \tag{4}$$

as follow: a couple (x_i, x_j) is in $Y_{\mathcal{R}}$ if and only if $x_i \succ_{\mathcal{R}} x_j$; both (x_i, x_j) and (x_j, x_i) are in $Y_{\mathcal{R}}$ iff $x_i \succsim_{\mathcal{R}} x_j$ and $x_j \succsim_{\mathcal{R}} x_i$, while if both (x_i, x_j) and (x_j, x_i) aren't in $Y_{\mathcal{R}}$, then x_i and x_j are indifferents.
The subset $\mathcal{C}_{\mathcal{R}} \subset \mathcal{C}$ of the set $\mathcal{C}$ of chambers in $\mathcal{A}_{n,m}$ is defined as:

$$\mathcal{C}_{\mathcal{R}} = \{C_i \in \mathcal{C} \mid \exists x_j : (x_i, x_j) \in Y_{\mathcal{R}} \text{ or } (x_j, x_i) \in Y_{\mathcal{R}}\}. \tag{5}$$

A first interesting result is the following

Theorem 2 *Given a subset $Y \subset X \times X \setminus \Delta$, there always exists a social decision rule $\mathcal{R}$ such that $Y = Y_{\mathcal{R}}$. Moreover any social decision rule $\mathcal{R}$ can be obtained by at most three voting agents.*

In order to prove this statement we need more notations and lemmas.

Notations. Given $Y \subset X \times X \setminus \Delta$, let us define the subset $Y_0 \subset X$ of *interesting* configurations as follow:

$$Y_0 = \{x_i \in X \mid \exists x_j : (x_i, x_j) \in Y \text{ or } (x_j, x_i) \in Y\}. \tag{6}$$

Then we can represent the space Y_0 as the set of points, or vertices, in an oriented graph $\overline{Y}$ where two points x_i and x_j are connected by an edge if and only if $(x_i, x_j) \in Y$ or $(x_j, x_i) \in Y$, while the orientation is from x_i to x_j in the first case and from x_j to x_i in the latter.
Without loss of generality we will still denote by x_i the vertices of $\overline{Y}$ and (x_i, x_j) its edges.
If, given two subgraph Y_1, Y_2 in $\overline{Y}$, there aren't edges between any two vertices $x_i \in Y_1$ and $x_j \in Y_2$, then Y_1 and Y_2 are *disconnected* subgraph of $\overline{Y}$.
Let us remark that if the graph $\overline{Y_{\mathcal{R}}}$ associated to a decision rule $\mathcal{R}$ contains disconnected subgraph, then this is equivalent to the indifference between the two groups of choices. As a consequence we can consider two disjoint subgraph as the result of two different voting processes.
Then from now on we will consider only *connected* graphs.
Given an oriented graph $\overline{Y}$ a *cycle of lenght n* is a subgraph γ_I of $\overline{Y}$ corresponding to a subset of Y of the form $\{(x_{i_1}, x_{i_2}), (x_{i_2}, x_{i_3}), \ldots, (x_{i_n}, x_{i_1})\}$. It is a simple remark that if $Y = Y_{\mathcal{R}}$ then a cycle in the graph $\overline{Y}$ corresponds to a cycle in the social decision rule $\mathcal{R}$.
Given the cycle γ_I above, we will say that $\gamma_I^{i_j, i_{j+1}}$ is obtained *breaking γ_I in* $(x_{i_j}, x_{i_{j+1}})$ if it is generated by the set $\{(x_{i_1}, x_{i_2}), \ldots, (x_{i_{j+1}}, x_{i_j}), \ldots, (x_{i_{n-1}}, x_n)\}$ instead of the old one. Clearly the new subgraph is no longer a cycle.
Let p be a finite set of edges in $\overline{Y}$, we will say that a new graph $\overline{Y}_p$ is obtained from $\overline{Y}$ going along the *breacking path p* if it is obtained from $\overline{Y}$ breacking the edges in p. We have the following:

Lemma 3.1 *Let G be an oriented graph with m cycles, x_i a vertex and $(x_i, x_{i_1}), (x_i, x_{j_1})$ two edges of a cycle in G. Then there are at list two breacking path p_1 and p_2 of the form:*

$$\begin{array}{l} p_1 = \{(x_i, x_{i_1}), (x_i, x_{i_2}), \ldots, (x_i, x_{i_h})\} \\ p_2 = \{(x_i, x_{j_1}), (x_i, x_{j_2}), \ldots, (x_i, x_{j_k})\} \end{array} \tag{7}$$

such that

1. $p_1 \cap p_2 = \emptyset$

2. G_{p_1} and G_{p_2} contain $m-1$ cycles.

Moreover if G contains k cycles through x_i then we can iterate the process breacking G along pairwise disjoint paths $p_1^1, p_2^1, \ldots, p_1^k, p_2^k$ in order to obtain a new graph with $m-k$ cycles none containing x_i.

Proof. The proof of the first part arises by costruction remarking that in a cycle two edges through the same vertex have to have opposite orientations.
Then if (x_i, x_{i_1}) come out from x_i, (x_i, x_{j_1}) has to go in x_i. Then we can start breacking the edge (x_i, x_{i_1}) obtaining an edge going in x_i; if the next edge (x_i, x_{i_2}) go in x_i then we don't have any more cycle and we have done. Otherwise we broke it and we go ahead until we meet an edge which go in x_i; such an edge exists because at list (x_i, x_{j_1}) go in x_i.
The same applies starting from (x_i, x_{j_1}).
The proof of the second part of the statement is by induction on the number k of the cycle through x_i. The base of induction is proved above.
Let us assume the statement true for $k-1$ cycles through x_i and let us add a new cycle through x_i, then the proof arises from an argument similar to the one used above.

Lemma 3.2 *Given an oriented finite graph G with m cycles it is alwais possible to obtain from G a new oriented graph with $m-1$ cycles breacking a finite number of edges. Moreover starting from three different edges of a cycle it is possible to obtain three different new graphs in a way such that all the edges involved are always different.*

Proof. The proof is by double induction on the number n of vertices and the number m of cycles.
Let be $G_{3,1}$ a graph with 3 vertices and 1 cycle. Then the proof is a simple remark: we need just to breack one edge at a time.
Let us assume the statement true for $G_{n,1}$ and let us add to $G_{n,1}$ a vertex x_{n+1} and edges from x_{n+1} such that the new graph $G_{n+1,1}$ has still one cycle.
Then if we breack the cycle in an edge (x_i, x_j) there are two possibilities: the new graph doesn't have any more cycles and then we have done or the new graph has a new cycle. Then, by construction, the subgraph given by the edges of the old cycle and the new one without (x_i, x_j) is a cycle too. This is not possible by hypothesis, in fact $G_{n+1,1}$ is a graph with only one cycle. Then the statement comes breacking one edge of the cycle at a time.
Let us assume the statement true for a graph $G_{n,m}$ with n vertices and m cycles. Then there are three breacking path p_1, p_2 and p_3 with empty intersection such that the new three graph $G_{n,m-1,p_1}$, $G_{n,m-1,p_2}$ and $G_{n,m-1,p_3}$ obtained breacking $G_{n,m}$ along, respectively, p_1, p_2 and p_3 contain $m-1$ cycles.
Let us add to $G_{n,m}$ a vertex x_{n+1} and new edges with vertex in x_{n+1} such that the new graph $G_{n+1,m+1}$ has $m+1$ cycles.
If none of the three path p_1, p_2 and p_3 give rise to new cycles through x_{n+1} then the proof is done: p_1, p_2 and p_3 are the path we are looking for.
Let assume that the paths p_1, p_2 and p_3 give rise to new cycles through x_{n+1}. If this new cycles involve three different edges with vertex in x_{n+1} then the proof follows from Lemma 3.1.
Otherwise if they involve the same two edges with vertex in x_{n+1} then let us

consider the new paths p'_1 and p'_2 which come from p_1 and p_2 applying the lemma 3.1.
Then there is at list one more edge in the cycle araising from p_3 which is not in $p'_1 \cup p'_2$. If we breack this edge one of the following situation occurs:

1. there aren't any more cycles or there is a new cycle with an edge already in p_3 and then we have done;
2. there is a new cycle with an edge in p_1 (or in p_2). Then we can swap p_3 with p_1 (or p_2) considering p'_3 instead of p'_1 (or p'_3 instead of p'_2) and we finish in (1) ;
3. there is a new cycle with an edge in $p'_1 \setminus p_1$ or $p'_2 \setminus p_2$. Then there is an edge with vertex in x_{n+1} which is not involved in any path and we can apply lemma 3.1;
4. there is a new cycle with an edge in $p'_1 \setminus p_1$ and one in $p'_2 \setminus p_2$. Then we are in the situation above and we can iterate.

This conclude the proof, in fact the graph is finite and iterating we have to end up with one of the situation (1), (2) or (3) .

Proof of the Theorem 2. The proof is by double induction on the number n of vertices of the graph $\overline{Y}$ and the number k of cycles in $\overline{Y}$.
If there aren't cycle in $\overline{Y}$ then the statement follows immediately. Indeed Y will be equal to the set $Y_{\mathcal{R}}$ where $\mathcal{R}$ is the social decision rule given by only one voting agent who states his preferences going along the oriented path in $\overline{Y}$.
Let assume that $\overline{Y}$ contains only one cycle of lenght m. Then, in a simple way, the social decision rule $\mathcal{R}$ such that $\overline{Y} = \overline{Y_{\mathcal{R}}}$ is obtained by three agents who state their preferences along the three oriented path obtained by $\overline{Y}$ breaking the cycle in three different edges as in Lemma 3.2. Indeed is a simple remark that in this way any edge (x_i, x_j) of $\overline{Y}$ is the preferred choice of at list two agents, i.e. at list two agents prefer x_i to x_j.
By induction we can assume that the statement is true for a graph $\overline{Y_{n,k}}$ with k cicles and n vertices for any n.
Then there are three agents a_1, a_2 and a_3 with preferences v_1, v_2 and v_3 which give rise to the social decision rule $\mathcal{R}_{n,k}$ such that $Y_{n,k} = Y_{\mathcal{R}_{n,k}}$, where $Y_{n,k}$ is the set associated to the graph $\overline{Y_{n,k}}$.
Let us assume to join a new vertex $n+1$ and new edges connecting $n+1$ to the other edges of $\overline{Y_{n,k}}$, such that the new graph $\overline{Y_{n+1,k+1}}$ has $k+1$ cicles.
If preferences v_1, v_2 and v_3 already break the new cycle γ_I then we have done. Otherwise we can broke it in three edges as in Lemma 3.2. Then $\overline{Y_{n+1,k+1}}$ is the result of a voting proces of three agents whose preferences are obtained by v_1, v_2 and v_3 changing the order between preferences involved in the path obtained breacking γ_I.
Now, by induction, let us assume that the statement is true if adding x_{n+1} we add $h-1$ cycles to $\overline{Y_{n,k}}$.
If adding $n+1$ we add h cycles then we can consider the graph $\overline{Y_{n+1,k+h-1}}$ obtained by our graph $\overline{Y_{n+1,k+h}}$ removing opportunely an edge with vertex in x_{n+1}. Then, again, the proof follows by inductive hypothesis and by Lemma

3.2 applied to the removed cycle.

4 Walking on social decision surfaces

Let $\mathcal{R}$ be a social decision rule and $\mathcal{A}_{n,m}$ be the hyperplane arrangement associated to n features and m possibilities. Then given a subset $I \subset \{1, \ldots, n\}$, a *decision module* $\mathcal{A}_I$ is a non empty subset of the arrangement $\mathcal{A}_{n,m}$ of the form

$$\mathcal{A}_I = \{H_{i,j}\}_{\substack{i \in I \\ 0 \leq j \leq m-1}}; \tag{8}$$

where cardinality of $\mathcal{A}_I$ is called *size*.
We will also denote with $\mathcal{A}_{I^c} = \mathcal{A}_{n,m} \setminus \mathcal{A}_I$ the complement of the arrangement $\mathcal{A}_I$ in $\mathcal{A}_{n,m}$. A set of decision modules $A = \{\mathcal{A}_{I_1}, \ldots, \mathcal{A}_{I_k}\}$ such that $\cup_{j=1}^k I_j = \{1, \ldots, n\}$ is a *modules scheme*.

Let x_j be a configuration in X, i.e. a chamber of $\mathcal{A}_{n,m}$, then the *module-configuration* $x_j(\mathcal{A}_I)$ is the chamber of the arrangement $\mathcal{A}_I$ which contains the chamber corresponding to x_j.

We can also define an operator between module-configurations as follow:

$$x_i(\mathcal{A}_I) \vee x_j(\mathcal{A}_I) = z \tag{9}$$

where z is the chamber of the arrangement $\mathcal{A}_I \cup \mathcal{A}_I$ obtained as intersection between $x_i(\mathcal{A}_I) \cap x_j(\mathcal{A}_I \setminus \mathcal{A}_I)$.

Moreover the *size* of a module scheme is the size of its largest defining module:

$$\mid A \mid = \max\{\mid \mathcal{A}_{I_1} \mid, \ldots, \mid \mathcal{A}_{I_k} \mid\}. \tag{10}$$

Given a module scheme $A = \{\mathcal{A}_{I_1}, \ldots, \mathcal{A}_{I_k}\}$, we say that a configuration x_i is a *preferred neighbor* of a configuration x_j with respect to a module $\mathcal{A}_{I_h} \in A$ if the following conditions hold:

1. $(x_i, x_j) \in Y_{\mathcal{R}}$,
2. $x_i(\mathcal{A}_{I_h^c}) = x_j(\mathcal{A}_{I_h^c})$, i.e. x_i and x_j correspond to the same chamber of the arrangement $\mathcal{A}_{I_h^c}$,
3. $x_i(\mathcal{A}_{I_h}) \neq x_j(\mathcal{A}_{I_h})$, i.e. x_i and x_j correspond to different chamber of the arrangement $\mathcal{A}_{I_h}$.

Let us define:

$$H(x_i, \mathcal{A}_{I_h}) = \{x_j \mid x_j \text{is a preferred neighbor of } x_i \text{with respect to } \mathcal{A}_{I_h}\} \tag{11}$$

and $H(x_i, A) = \cup_{j=1}^k H(x_i, \mathcal{A}_{I_j})$. We call $P(x_i, x_j, A)$ *a path through* A*, starting from* x_i *and ending in* x_j a succession of prefereed neighbors with respect to modules in A, i.e. a succession:

$$x_i = x_{i_0}, x_{i_1}, \ldots, x_{i_{s+1}} = x_j \tag{12}$$

such that there exist modules $\mathcal{A}_{I_{h_0}}, \ldots \mathcal{A}_{I_{h_s}} \in C$ wiht $x_{i_{t+1}} \in H(x_{i_t}, \mathcal{A}_{I_{h_t}})$ for all $0 \leq t \leq s$.
A configuration x_j is *reachable* from x_i with respect to a module scheme A if

and only if it exists a path $P(x_i, x_j, A)$.
Let us consider $P(x_i, A)$ the set of all path starting from x_i with respect to A; then a path can end up either in a social (local) optimum, i.e. a configuration which doesn't have any preferred neighbor or in a limit cycle, i.e. a cycle among a set of configurations which are preferred neighbors to each other. The latter is the well known case of intransitive social preferences.
The set of *best neighbors* $B(x_i, \mathcal{A}_{I_h}) \subset H(x_i, \mathcal{A}_{I_h})$ is defined as:

$$B(x_i, \mathcal{A}_{I_h}) = \{x_j \in H(x_i, \mathcal{A}_{I_h})\text{such that } (x_j, x_k) \in Y_{\mathcal{R}}\text{for all } x_k \in H(x_i, \mathcal{A}_{I_h})\}. \tag{13}$$

A configuration x_i is a *local optimum* for A if and only if for all $x_j \in H(x_i, A)$, $(x_j, x_i) \notin Y_{\mathcal{R}}$. Clearly if $\mathcal{R}$ is strong then the local optimum condition is equivalent to $H(x_i, A) = \emptyset$.
For a given local optimum x_i with respect to C, the *basin of attraction* of x_i is the set

$$\Psi(x_i, A) = \{x_j \mid P(x_j, x_i, A) \neq \emptyset\}. \tag{14}$$

Moreover an *agenda* α over a module scheme $A = \{\mathcal{A}_{I_1}, \dots, \mathcal{A}_{I_k}\}$ is a permutation $\alpha = \{\alpha_1, \dots, \alpha_k\}$ which states the order of the modules $\mathcal{A}_{I_i}$.

5 Cycles in social preferences

Let $\mathcal{A}_{n,m}$ in $\mathbb{R}^n$ be an arrangement of configurations and $\mathcal{R}$ be a social decision rule. Then it is easy to verify that the local optimum strictly depends on the choise of the module scheme A. Let us consider the following example.
Consider the case of three agents a_1, a_2 and a_3 and three objects x, y and z with individual preferences expressed by:

$$a_1 : x, y, z \qquad a_2 : y, z, x \qquad a_3 : z, x, y \tag{15}$$

where x, y and z are encoded according to the following map:

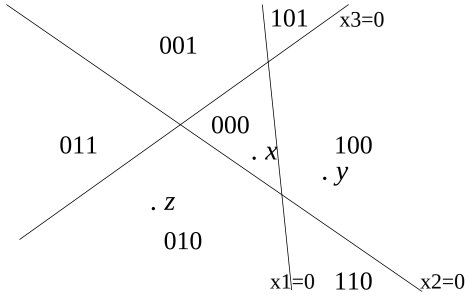
$$x \mapsto 000 \qquad y \mapsto 100 \qquad z \mapsto 010. \tag{16}$$

These individual preferences give rise to the decision rule $\mathcal{R}$ expressed by the cycle $x \succ_{\mathcal{R}} y$, $y \succ_{\mathcal{R}} z$ and $z \succ_{\mathcal{R}} x$.
This is equivalent to consider the arrangement $A_{3,1}$ of the coordinate hyperplanes in $\mathbb{R}^3$ and chambers $\{C_x, C_y, C_z\}$ corresponding to the three open sets:

$$\begin{array}{l} C_x = \{(x_i, x_j, x_h) \mid x_i < 0, x_j < 0, x_h < 0\}; \\ C_y = \{(x_i, x_j, x_h) \mid x_i > 0, x_j < 0, x_h < 0\}; \\ C_z = \{(x_i, x_j, x_h) \mid x_i < 0, x_j > 0, x_h < 0\}. \end{array} \tag{17}$$

Let consider a general 2-dimensional section of the arrangement through C_x, C_y and C_z as in figure.

Looking at the figure ..., it is easy to observe that if we consider the module schemes $A = \{\{H_{1,0}\}, \{H_{2,0}, H_{3,0}\}\}$ and $A = \{\{H_{1,0}, H_{3,0}\}, \{H_{2,0}\}\}$,then C_z is the only chamber which verifies conditions 4. While if $A = \{\mathcal{A}_{3,0}\}$ or $A = \{\{H_{1,0}\}, \{H_{2,0}\}, \{H_{3,0}\}\}$ then the voting process ends up in the limit cycle among x, y and z.



6 A condition for existence of a local optimum

Let $\mathcal{R}$ be a social decision rule and $\mathcal{A}_{n,m}$ be the hyperplane arrangement associated to n features and m possibilities.

A first interesting question is when, given a configuration z, there exists a modules sheme $A = \{\mathcal{A}_{I_1}, \ldots, \mathcal{A}_{I_k}\}$ such that z is a local optimum for A.

Let z and x be two configurations, we will say that z and x are *separate* by an hyperplane $H \in \mathcal{A}_{n,m}$, $z \mid H \mid x$, if H separates the chambers C_z and C_x. Moreover we will say that z and x are *prominently separate* if there exists two hyperplanes $H_{i_1,j_1}, H_{i_2,j_2} \in \mathcal{A}_{n,m}$ with $i_1 \neq i_2$ and $z \mid H_{i_1,j_1} \mid x$, $z \mid H_{i_2,j_2} \mid x$.

We define the *distance* between z and x as:

$$d(z,x) = \min\sharp\{H \in \mathcal{A}_{n,m} \text{ such that } z \mid H \mid x\}. \tag{18}$$

While the *prominent distance* $d_p(z,x)$, will be the minimum number of hyperplanes which prominently separate z and x. If $z \neq x$, we will say that $d_p(x,z) = 1$ if z and x aren't prominently separate.

Let us remark that, by definition of decision module $\mathcal{A}$, if $H_{i,j_1} \in \mathcal{A}$ then $H_{i,j} \in \mathcal{A}$ for all $0 \leq j \leq m-1$. As consequence if $d_p(x,z) = 1$ and $d(x,z) > 1$ then all hyperplanes which separate z and x have to be in the same decision module $\mathcal{A}$.

Now we can give the main result of this section:

Theorem 3 *Let $\mathcal{R}$ be a social decision rule and $\mathcal{A}_{n,m}$ be the hyperplane arrangement associated. Given a configuration z, we can build a modules sheme A_z such that z is a local optimum for A_z if and only if for any configuration x such that $x \succ_{\mathcal{R}} z$ then $d_p(x,z) > 1$. Moreover the construction is independent from the order of the decision modules inside the chosen modules scheme.*

Proof. Given a configuration z, let $x_{i_1}, \ldots, x_{i_k}$ be all configurations such that $x_{i_j} \succ_{\mathcal{R}} z$. By hypothesis $d_p(x_{i_j}, z) > 1$, then x_{i_j} and z are prominently separate at list by two hyperplanes $H_1^{i_j}, H_2^{i_j}$.

Let us consider a module scheme A_z such that for any x_{i_j} there exist at list two decision modules $\mathcal{A}_1^{i_j}, \mathcal{A}_2^{i_j}$ in A_z whit $H_1^{i_j} \in \mathcal{A}_1^{i_j}$ and $H_2^{i_j} \in \mathcal{A}_2^{i_j}$.

It is a simple remark to see that such a module scheme exists. Moreover z is a local optimum for A_z. Indeed for all $x_{i_j} \succ_{\mathcal{R}} z$ and for all $\mathcal{A} \in A_z$ the chambers $C_{i_j}(\mathcal{A}^c)$ and $C_z(\mathcal{A}^c)$ are always separate by $H_1^{i_j}$ or $H_2^{i_j}$. That is $x_{i_j}(\mathcal{A}^c) \neq z(\mathcal{A}^c)$ and then $H(z,\mathcal{A}) = \emptyset$ for all $\mathcal{A} \in A_z$. Clearly this construction is independent from the order of the decision modules inside the chosen module scheme.

On the other hand if x is a configuration $x \succ_{\mathcal{R}} z$ such that $d_p(x, z) = 1$ then for any module scheme A there is at list one decision module $\mathcal{A}$ such that all hyperplanes $x \mid H \mid z$ which separate x and z are $H \in \mathcal{A}$. Then, by definition, $x \in H(z, \mathcal{A}) \neq \emptyset$. This concludes the proof.

Remark 6.1 *Let us remark that the indipendence of our construction from the* 1*-dimensional distance, i.e. the distance along the* 1*-dimensional family of hyperplanes* $\{H_{i,j}\}_{0 \leq j \leq m-1}$ *for a fixed* i *is a consequence of the indipendence of the choice from the order in which a* 1*-dimensional list of objects is given.*

An interesting question is to study how close we can go to our chosen configuration z when it cannot be a local optimum for any module scheme.

We will say that a configuration z is *free with respect to a decision rule* $\mathcal{R}$ if and only if for any configuration x such that $x \succ_{\mathcal{R}} z$, $d_p(x, z) > 1$. Then, by the theorem 3, z is the local optimum for a module scheme A_z if and only if z is free.

Moreover a configuration $\overline{z}$ is of *minimal distance from* z *with respect to* $\mathcal{R}$ if and only if $\overline{z}$ is free with respect to a decision rule $\mathcal{R}$ and $d(z, \overline{z})$ =min$\{d(z, x) \mid$ x is free $\}$. If z is free then it coincides with its configuration of minimal distance.

Then as a direct consequence of the theorem 3 we have the following:

Corollary 6.2 *Given a decision rule* $\mathcal{R}$ *and a configuration* z *it is always possibile to build a module scheme* $A_{\overline{z}}$ *such that the configuration* $\overline{z}$ *of minimal distance from* z *with respect to* $\mathcal{R}$ *is a local optimum.*

Remark 6.3 *If we consider the classical* 1*-dimensional problem, then the prominent distance between two configurations* x *and* z *is always* $d_p(x, z) = 1$*. It follows:*

- *the configuration* z *is free if and only if* z *is an optimum, i.e. for any configuration* x*,* $z \succ_{\mathcal{R}} x$*;*
- *if the configuration of minimal distance from* z *with respect to* $\mathcal{R}$ *exists then it is the only optimum;*
- *if the configuration of minimal distance from* z *with respect to* $\mathcal{R}$ *doesn't exist, then our theorem simply recover the Arrow's impossibility theorem.*

Salvetti's Complex The theorem 2 is an interesting application of the graph theory to the social decision theory. While thorem 3 give an application of the arrangement theory to the social decision one.
Then it seems that these two mathematical tools are very usefull and powerfull in order to investigate the social decision problem.
The natural consequence is to use a tool which put togheter these two different mathematical objects: the Salvetti's complex.

Indeed given the arrangement $\mathcal{A}_{n,m}$, the correspondence between chambers and configurations is the above one. While if $\mathbf{S}(\mathcal{A}_{\mathbf{n},\mathbf{m}})$ is the Salvetti's complex associated to the complexification of $\mathcal{A}_{n,m}$, each chamber C of $\mathcal{A}_{n,m}$ corresponds to the 0-cell $[C \prec C]$.

Moreover, let x_i and x_j two configurations which correspond to chambers C_i and C_j. Then we can define a *path from C_i to C_j* as follow:

$$([C_{i_1} \prec F^1_{j_1}], [C_{i_2} \prec F^1_{j_2}], \ldots, [C_{i_k} \prec F^1_{j_k}]) \tag{19}$$

such that C_{i_h}, $C_{i_{h+1}}$ are adjacent chambers separated by $F^1_{j_h}$, $C_{i_1} = C_i$ and $F^1_{j_k}$ is the 1-codimensional face which separates C_{i_k} from C_j. The number k is the *lenght* of the path. Obviously, given two chambers C_i and C_j there are paths between them of minimal lenght.

Let $([C_{i_1} \prec F^1_{j_1}], [C_{i_2} \prec F^1_{j_2}], \ldots, [C_{i_k} \prec F^1_{j_k}])$ be a path of minimal lenght between C_i and C_j. Then the oriented edge associated to (x_i, x_j) is simply the chain given by the sum

$$\sum_{h=1}^{k} [C_{i_h} \prec F_{j_h}]. \tag{20}$$

This definition deserve farther studies. We will come back on this matter in an upcoming work.

Other interesting questions arise. In particolar two of them deserve farther studies:

1. Which is the basin of attraction of a configuration z?

2. which are conditions such that not only a configuration z is a local optimum for a modules scheme A_z, but it is the only local optimum?

References

[Arr51] K. Arrow, *Social Choice and individual values*, Wiley, New York, (1951).

[Bou66] N. Bourbaki, *Elements of Mathematics: General Topology*, vol. Chapters I-IV, Addison-Wesley (1966).

[Bou68] N. Bourbaki, *Groupes et algebrès de Lie*, vol. Chapters IV-VI, Hermann, (1968).

[MP08] L. Marengo, C. Pasquali *The construction of choice. A computational voting model*, LEM Working paper (2008/24)

[Ma91] W.S. Massey, *A Basic Course in Algebraic topology*, Springer Verlag, New York, (1991).

[OT92] P. Orlik, M. Terao, *Arrangements of hyperplanes*, Springer-Verlag **300** (1992)

[Sal87] M. Salvetti, *Topology of the complement of real hyperplanes in* $\mathbb{C}^n$, Inv. Math., **88** (1987), no.3, 603–618.

[Te07] H. Terao *Chambers of Arrangements of Hyperplanesand Arrow's Impossibility Theorem*, Advances in Math., **214** (2007), 366-378.