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LEM
Working Paper Series

How to Construct Alternatives
A Computational Voting Model

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2005/23 November 2005

ISSN (online) 2284-0400

How to construct alternatives

A computational voting model*

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Abstract

Social choice models usually assume that choice is among pre-defined, uni-dimensional and "simple" objects. Very often, on the contrary, choice is among multi-featured and "complex" objects: a candidate in an election stands for an electoral programme which is a complex bundle of many interdependent political positions on a wide variety of issues. Also in committees and organizations of various sorts collective choices are most often made among policy "bundles" and authorities can act upon the pre-choice stage of construction of such bundles. This pre-choice power of alternatives construction may grant authorities a highly effective device to influence the outcome of social choice even when the latter is totally free and democratic.
In this paper we propose a model which investigates within a simple majority vote framework the role of the object construction power, an analogous to the agenda power. Even when object construction is simply defined as the possibility of assembling and dis-assembling a fixed set of choice components into bundles, we show that, under rather general condition, it can radically change the outcome of the majority voting process. In particular we show that any set of bundles (that we call "choice modules") is associated to a set of possible social outcomes which can be attained depending upon the initial conditions. Moreover we shows that also Condorcet-Arrow cycles can appear or disappear depending upon which set of modules is chosen.

*This paper was originally prepared for the 46th Annual Meeting of the Italian Economic Association, Naples, 21-22 October 2005. We thank the Conference participants, Giovanni Dosi, Marco Faillo and Matteo Ploner for useful comments and suggestions. The usual caveats apply.

1 Introduction

In the 60^{th} paragraph of his Philosophical Investigations, Ludwig Wittgenstein considers the following problem:

> When I say: “My broom is in the corner”, is this really a statement about the broomstick and the brush? [...] Well, if the broom is there, that surely means that the stick and brush must be there, and in a particular relation to one another; [...] Then does someone who says that the broom is in the corner really mean: the broomstick is there, and so is the brush, and the broomstick is fixed in the brush? [...] Suppose that, instead of saying “Bring me the broom”, you said “Bring me the broomstick and the brush which is fitted on to it.” Isn’t the answer: “Do you want the broom? Why do you put it so oddly?”

Wittgenstein expresses - among other things - that there is a level of analysis of the world and of objects within the world, upon which people talk, live, discuss and choose: going any further in analyzing objects in their constituent parts amounts to “putting things oddly”.

Thus, according to this passage - even at a probably very superficial reading of it - a distinction can be recognized between an “odd” level of analysis and one that does effectively serve mutual understanding, between an atomistic logical level of analysis and one that might be further refined.

But what is the social relevance of distinguishing between parts of things and things as wholes? Who does perform the distinction? What is the effect of adopting a specific level of analysis with respect to individual and social choice? What is the effect of different compositions/decompositions of objects in their constituent parts with respect to the expression of individual and social preferences?

An apparently trivial fact is that a wide range of social activities such as communication and choice only take place once objects have been defined e.g. in terms of their component parts. That is, once someone has declared: “this is not a broomstick plus a brush: this is a broom and we treat it as such in talking and acting upon it: as a non furtherly separable object.” Brooms thus become objects of choice as wholes. In this sense, one can only express his preferences on brooms as opposed to, say, hammers but not on broomsticks and brushes as separate objects. It would thus follow that preferences on possible features of the brush or of the broomstick (e.g. color, shape, material, etc.) cannot play any role in the choice process even if an agent might have a preference structure defined on them.

Our point here is that every act of choice takes place once a choice set and its members are defined. That is: objects must be constructed and presented to an agent before he can actually choose among them. The key point we will address is that, far from being unstructured points, objects of choice are very much like Wittgenstein's broom: they are composed of different parts and traits that can be variously combined with one another.

According to classical choice theory, objects of choice are described as one-dimensional entities whose most direct representation is a set of points which can be arranged on a line from most to the least preferred. However, unlike what it is usually assumed, the alternatives among which a society has to choose are often not "simple" one-dimensional objects: they are rather formed by multiple dimensions that possibly generate complex trade-offs. According to this perspective, we shall try to model how objects populating choice sets are constructed, how the construction of objects interacts with choices and how choosers' preferences interact with objects' constructions.

The main issue at hand is that interesting dynamics can emerge from possible clashes between different ways of clustering sets of traits into wholes and more or less separable preferences on the part of agents. For instance, in the "broom" example agents are not allowed to separately express their preferences on the brush or the stick as these can only be jointly considered under the composite category or label "broom".

We believe that a fundamental though neglected part of social choice is the process of building alternatives' sets and thus of categorizing objects in the world. These two fundamental activities, as we will try to show, grant the power to influence and to some extent direct the outcomes of individual and social choice and they appear to be at least as important as the power of setting the agenda according to which alternatives are compared.

In other words, we take the following stance: agents might indeed have well defined preferences on the objects of their choice *once objects are given* but still how objects are defined and constructed is a crucial as much as a neglected point. Very often, framing a choice set in terms of one-dimensional points have very little meaning as alternatives within the set might be broad and grossly underdefined labels for wide classes of very different (and differently attractive) alternatives. One might, for instance, imagine thousands of different brushes (as to color, shape, material, weight...) and broom-sticks (as to length, weight...) and, on the other hand, agents endowed with complete preference structures on each of these possibilities. Notwithstanding this, given that agents are compelled to choose on a given object-like categorization of sets of features, (i.e. they are called upon to make their choices on brooms as non separable objects), none of his single-feature preferences will be reflected in his choice as such. Given this, it follows that it might have very little sense to talk about the choice between a broom and a hammer as this might be grossly underdefined labels for wide classes of very different (and differently attractive) alternatives.

Our approach here focuses on the way that object construction works as an institution with respect to selecting subsets of feasible outcomes. In particular, we view an institution as essentially characterized by a set of categories. Under this respect, our fundamental question deals with the extent to which category construction can lead to specific social outcomes through the selection and categorization of appropriate features sets. Our main focus is on the relations between objects structures and individual preference structures and our main question is about the measure in which specific objects' constructions drive and constrain individual preferences with respect to their satisfaction.

As a matter of fact, there appears to be a category-formation analogue of “agenda power” which is not just given by the power of setting the order in which alternatives are voted but the power of constructing the alternatives when the latter are bundles of different traits or features. As to this point, we ask what is the extent to which, by appropriately forming such bundles, one can influence the social outcome. Further, if alternatives are categorized bundles of features, individual preferences might not be separable in each component and interdependencies might show up. Or, on the contrary, agents may have some areas of indifference on some features. Thus, by exploiting interdependencies and indifference, institutions with the power of categorization can influence a social choice by selecting a specific outcome out of a multitude of possible ones. As to this point, we ask to what extent categorization is able to select social outcomes.

In what follows we develop a simple model of majority voting whereby a plurality of individual agents possess heterogeneous individual orderings which have to be aggregated in a collective outcome. A well established literature shows that the aggregation of elements into a collective choice is not always straightforward. Arrow (1951) shows that no universal voting procedure exists that aggregates individual preferences into social orderings that satisfy a set of minimal conditions. McKelvey (1979) has proved that under majority rule the stake of agenda manipulation can encompass the entire range of feasible outcomes however individual preferences are defined. Far from being seen as simple sums of components, aggregation processes do have the potential for unstable, arbitrary, intransitive and chaotic behavior.

Our approach here focuses on the way that object construction - rather than agenda manipulation - works as an institution with respect to selecting subsets of feasible outcomes. In our model, an institution proposes instances (i.e. choice configurations) to agents based on its category set. Agents vote according to their preferences and following the majority rule. In doing so, a topological space is generated which we call a *social decision surface*: the voting procedure determines a walk on such a surface, whose outcome depends – generally speaking – on the starting point, on the sequence through which alternatives are presented, and, especially, on how components are aggregated into what we call *decision modules*.

We ask: given the total surface, does it present a single global optimum, many local optima or cycles? We show that under general conditions (notably if preferences are not fully separable) the answers to the previous questions are entirely dependent upon the decision modules. We show algorithmically that – given a set of individual preferences – by appropriate modifications of the decision modules we can obtain all the three outcomes, i.e. a single global optimum, multiple local optima or cycles. Thus the chosen modules determine the dynamics and outcome of the voting procedure. In the case of many local optima, by appropriately selecting the decision modules and the starting point any of the local optima can be obtained. Finally we show that cycles *à la* Condorcet-Arrow (de Caritat Marquis de Condorcet 1785, Arrow 1951)[1], may also appear and disappear by appropriately modifying the decision modules.

Finally, note that we show these results in a setting in which there is a given and finite set of components and in which the set of decision modules always covers such a set entirely. Different decision modules are simply different decompositions (not necessarily partitions) of such a set, and the results we obtain show that different decompositions can generate vastly different outcomes. Thus what we show has nothing to do with the correct but trivial observation that there exist an obvious “issue raising” power, i.e. that in world in which there exist potentially infinite choices to be made, a

[1] This is a well known result for which even in the presence of transitive individual preferences, social preferences expressed through some voting rule may be cyclical.

primary and fundamental power is exerted in focussing the social attention on some issues and neglecting others. In our finite setting all possible issues are always decided upon.

We believe that categorization and framing are important parts of social choice and that building alternatives based on particular categories confer - to some extent - the power to determine, influence and direct the selection of specific social outcomes. This point seems to be very consonant in spirit with some recent work of George Lakoff on the use of frames and metaphors in politics. According to Lakoff (2004):

> Frames are mental structures that shape the way we see the world. As a result, they shape the goals we seek, the plans we make, the way we act and what counts as a good or a bad outcome of our actions. In politics our frames shape our social policies and the institutions we form to carry out policies. To change our frames is to change all of this. Reframing *is* social change.

The paper is organized as follows. In the following section 2 we illustrate our main points by means of a simple example. Section 3 presents our formalism, which we use in order to present, in section 4 presents some "possibility" examples, i.e. we provide examples in which social outcomes depend upon categorization both in that the number and location of social optima depend upon categories and in that the presence of cycles depend upon them. In section 5 instead we discuss the likelihood of such phenomena in randomly generated social decision problems. In particular we show that in general cycles are very likely and their likelihood can be reduced by refining categories, but this increases the number of different social outcomes which can be achieved depending upon categories. Decidability seems therefore to be linked to the alternative setting power and the agenda power. Finally, in section 6 we draw some conclusions.

2 An example

Let us imagine and describe two possible objects of choice such as spending an evening out and staying home. We shall call these objects *going out* and *staying home*. As such - i.e. with no further specification - both objects are largely underdetermined objects that can be instantiated in a variety of ways depending on how different traits relative to the object itself are defined. In this sense, the category "evening out" derives its attractiveness from how the traits within it are actually instantiated. We might imagine, as a first approximation, that *going out* is defined by a place to go (e.g. restaurant, disco, pub, rave) and a set of people to spend the evening with. While the object *staying home* is defined by what we will do (e.g. talking, watching a movie, working) and who we will do it with.

Any agent called to express his preferences and choose one of the two objects, will possibly have preferences defined on each single dimension of the object and preferences defined on their possible combination/categorization. Consider the following "narrative": shall we go to a restaurant? Shall we choose an Arab restaurant or an Italian one? Well, we would really like to head for an Arab if the restaurant is "Shawarma Station" and if we are going there at eight sharp with Françoise and Mara but we would really prefer an Italian place if the restaurant is "Pommidoro" and we are having dinner at ten with Giovanni and Gabriele. Are we rather going to the cinema? That sounds great if our movie will be the latest Lars Von Trier's "Dogma" masterpiece and the theater is close to our place so that we might invite Matteo and Cecilia. However we would prefer to invite some people at our place for a drink if it is too late for the movies and parking is too dark a nightmare after nine in the city center. Shall we invite Giovanni to join us? He is a real friend and a nice guy but he will almost certainly come along with Paola whom we can hardly stand and, besides, she doesn't like the company of Claudio and Stefano which we really like......

If we view *going out tonight* as an object, we can imagine it as being constructed by a set of traits whose union results in a different construction and, in turn, in a different object. Let us imagine *going out tonight* as constructed by the bundle of three traits: {*where, who-is-coming, when*}. If we suppose that each of these traits may assume different values (i.e. the *where* might be {*cinema, restaurant, pub, ...*}, the *who* might be {*Giovanni, Marco, Paola, ...* }) then a whole set of different instances of *going out tonight* based on the specific value assumed by each of its traits. Some possible instances of *going out tonight* might be: {*restaurant, Giovanni and Paola, 8 sharp*}, {*cinema, Françoise, 10 pm*}, {*pub, Cecilia and Gabriele, 9 pm*} and so on.

Let us now suppose that another object is introduced and call it *stay at home*. In turn, this object will be constructed as a bundle of different traits each with its own specific value out of a class of possible ones. Let us imagine that *stay at home* = {*who-is-coming, when, to-do-what*} and that each of these traits might assume different values. Possible instances of *stay at home* might be: {*Nicole Kidman, 8 pm, see what happens*}, {*Naomi Klein, 6 am, talk about globalization*}, {*Matteo, 10 pm, have a beer*}.

Were one supposed to choose what to do tonight facing the two alternatives *going out* and *staying at home*, he will certainly choose according to his preferences. It is however reasonable that individual preferences and the choice resulting from them be largely dependent on how the two alternatives are constructed i.e. on how the objects populating them are constructed. It might well be possible that one is enthusiastic about Françoise as a table-companion (she's a brilliant talker) but totally dislikes her as a movie fellow (she can't stop laughing during love scenes) or that one prefers Nicole Kidman if he is supposed to talk about globalization issues but values Naomi Klein far more if the *to-do-what* issue is *see what happens*. On the other hand it might happen that one wishes to enjoy Giovanni's company no matter what we are supposed to do and that I will choose any situation whatsoever provided that the *who-is-coming* issue contains Giovanni.

3 Social decision surfaces

We assume that choices are made over a set of N elements or **features** $F = \{f_1, f_2, \ldots, f_N\}$, each of which taking a value out of a finite set possibilities. For simplicity and without loss of generality we make the assumption that such a set is the same for all elements and contains two values that we label respectively 0 and 1: $f_i \in \{0, 1\}$. Thus the space of possibilities is given by 2^N possible **choice configurations**: $X = \{x^1, x^2, \ldots, x^{2^N}\}$.

In the "what-to-do-tonight" example f_i can designate a yes or no choice on each single item at stake and X is the set of virtually possible decisions.

Let us assume now that there exist h individual agents $A = \{a_1, a_2, \ldots, a_h\}$, each characterized by a (weak) ordering on the set of choice configurations: given any two configurations x_i and x_j agent a_k can always state whether $x_i \succ_k x_j$ or $x_j \succ_k x_i$ or $x_i \approx_k x_j$. For the time being we make no further assumption on agents' preferences: any ranking is allowed. We will obtain most of our results for random orderings and make some restrictions when appropriate.[2] We call this ranking agent k's *individual decision surface* Ω_k.

[2] In the appendix we will develop a characterization of preferences which can account for all such restrictions.

Let us now introduce a social decision rule $\Re : (\Omega_1, \Omega_2, \ldots, \Omega_h) \mapsto \mathbf{\Omega}$. In this paper we consider only a simple majority voting: given a status quo x_i and an alternative x_j agents truthfully vote according to their preferences. Agent k votes for x_i if $x_i \succ_k x_j$, votes for x_j if $x_j \succ_i x_i$ and abstains if $x_i \approx_i x_j$. The alternative which receives more votes is chosen as the new status quo, in case of a draw the current status quo is kept. We make the hypothesis that this process continues until no new alternative wins against the status quo.

Given an initial configuration and a social decision rule $\Re$ this process defines a *walk on the social decision surface* which can either end up on a social optimum or cycle forever among a subset of alternatives. If voting processes end up on the same social optimum for all possible initial conditions we call them ergodic or path-independent, otherwise voting is non-ergodic or path-dependent. If voting processes are non-ergodic the social outcome depends upon the initial condition and upon the agenda, i.e. the sequence in which choice configurations undergo examination and voting. A further problem may arise from the combinatorial nature of the set of alternatives: the cardinality of the set X is exponential in the number N of features and even for relatively small values of N the number of alternatives may be so large that no real life voting process can possibly examine all of them. A fundamental part of the social decision is the pre-voting generative mechanism[3] which is led by framing and categorization through which alternatives are generated within the pre-defined categories. The influence of generative mechanisms upon social outcomes is twofold: on the one hand they define the sequence of voting, on the other hand they define which subset of alternatives undergoes examination. As we shall show, different sets of categories may generate different social outcomes because of these two phenomena: social optima do – in general – change when categories are different both because the subset of generated alternative is different (and some social optima may not belong to many of these subsets) and because the agenda is different (and this may determine different outcomes). Framing power appears therefore as a more general phenomenon than agenda power.

[3] The reader might recognize a similarity with evolutionary thinking, in which evolution is normally considered as the outcome of the interaction between selection and variation, whereby the latter provides the variety upon which selection operates and – in many cases – may heavily limit the power of selection. In our approach social outcomes are the product of the interaction between social selection, here represented by majority voting, and alternative generation.

3.1 Walking on social decision surfaces

We assume that voting occurs among bundles of features, whereby one bundle may contain any number of features between 1 and N, that we call decision modules. Decision modules are kinds of categories or templates through which decision alternatives are generated and compared. Thus they can strongly influence the dynamics and outcomes of the voting process.

Let $I = \{1, 2, \ldots, N\}$ be the set of indexes and let a **decision module** $C_i \subseteq I$ be a non-empty subset of it, we call the **size of module** C_i , its cardinality $|C_i|$. We define a **a modules scheme** as a set of modules:

$$C = \{C_1, C_2, \ldots, C_k\}$$

such that $\bigcup_{i=1}^{k} C_i = I$

Note that a decomposition scheme does not have necessarily to be a partition as modules can have non-empty intersections.

Given a choice configuration x^i and a module C_j, we call module-configuration $x^i(C_j)$ the substring of length $|C_j|$ containing the features of configuration x^i belonging to module C_j:

$$x^i(C_j) = f^i_{j_1} f^i_{j_2} \ldots f^i_{j_{|C_j|}}$$

for all $j_h \in C_j$.

We also use the notation $x^i(C_{-j})$ to indicate the sub-configuration of length $N - |C_j|$ containing the components of configuration x^i not belonging to module C_j.

Two module-configurations can be united into a larger module-configuration by means of the $\vee$ operator so defined:

$$x(C_j) \vee y(C_h) = z(C_j \cup C_h) \text{ where } z_\nu = \begin{cases} x_\nu \text{ if } \nu \in C_j \\ y_\nu \text{ otherwise} \end{cases}$$

We can therefore write $x^i = x^i(C_j) \vee x^i(C_{-j})$ for any C_j.

We define the **size of a decomposition scheme** as the size of its largest defining module:

$$|C| = \max\{|C_1|, |C_2|, \ldots |C_k|\}$$

An **agenda** $\alpha = C_{\alpha_1} C_{\alpha_2} \ldots C_{\alpha_k}$ over the module set C is a permutation of the set of modules which states the order according to which modules are examined.

Movements on the social decision surface are driven by the social decision rule $\Re$, which, more concretely, we suppose works as follows. We suppose that an initial choice configuration is (randomly) given[4] then the first module of the agenda is considered and all module configurations are generated. At every step agents vote the status quo against a new configuration in which the features of the module under consideration are replaced by the current module configuration, whereas all other modules are kept constant and equal to the initial condition. Every time the configuration which obtains the majority becomes the (new) status quo.

When all modules configurations have been examined for the first module in the agenda, the same procedure is repeated for the second, third, ..., $k-th$ module in the agenda. As to the stopping rule we can consider two possibilities:

1. modules which have already been settled cannot be re-examined
2. modules which have already been settled can be re-examined and if new social improvements have become possible

Normally we will use the latter stopping rule, as it is more general and limits the role of the agenda, though – as we shall see – does not eliminate agenda power. In fact even is modules can be re-examined over and over again until some social improvement keeps being possible, widespread path dependency generally implies that the order in which modules are examined is often relevant in determining the social outcome.

More precisely, we will use the following algorithmic implementation of majority voting:

1. repeat for all initial conditions $x = x^1, x^2, \ldots, x^{2^N}$
2. repeat for all modules $C_{\alpha_i} = C_{\alpha_1}, C_{\alpha_2}, \ldots, C_{\alpha_k}$ until a cycle or a local optimum is found;
3. repeat for j=1 to $2^{|C_{\alpha_i}|}$
    - generate a module-configuration $C^j_{\alpha_i}$ of module C_{α_i}
    - vote between x and $x' = C^j_{\alpha_i} \vee x(C_{-\alpha_i})$
    - if $x' \succeq^{\Re} x$ then x' becomes the new current configuration

[4] In what follows we actually find properties for all possible initial choice configurations.

A module scheme is therefore a sort of template which determines how new choice configurations are generated and can therefore undergo selection by the social rule $\Re$.

Given a module scheme $C = \{C_1, C_2, \ldots, C_k\}$, we say that a configuration x^i is a **preferred neighbor** of configuration x^j with respect to a module $C_h \in C$ if the following three conditions hold:

1. $x^i \succeq^{\Re} x^j$
2. $x^i_\nu = x^j_\nu \; \forall \nu \notin C_h$
3. $x^i \neq x^j$

Conditions 2 and 3 require that the two configurations differ only by components belonging to module C_h. According to the definition, a neighbor can be reached from a given configuration through voting on a single module.

We call $H_i(x, C_i)$ the set of neighbors of a configuration x for module C_i.

A **path** $P(x^i, C)$ from a configuration x^i and for a module scheme C is a sequence, starting from x^i, of preferred neighbors:

$$P(x^i, C) = x^i, x^{i+1}, x^{i+2}, \ldots \text{ with } x^{i+m+1} \in H(x^{i+m}, C)$$

A configuration x^j is **reachable** from another configuration x^i and for decomposition C if there exist a path $P(x^i, C)$ such that $x^j \in P(x^i, C)$.

A path can end up either on a social (local) optimum, i.e. a configuration which does not have any preferred neighbor, or in a limit cycle, i.e. a cycle among a set of configurations which are preferred neighbors to each other. The latter is the well known case of intransitive social preferences.

The set of **best neighbors** $B_i(x, C_i) \subseteq H_i(x, C_i)$ of a configuration x for block C_i is the set of the socially most preferred configurations in the set of neighbors:

$$B_i(x, C_i) = \{y \in H_i(x, C_i) \text{ such that } y \succeq^{\Re} z \; \forall z \in H_i(x, C_i)\}$$

By extension from a single module to the entire module scheme, we can give the following definition of the set of neighbors for a module scheme as:

$$H(x, C) = \bigcup_{i=1}^{k} H_i(x, C_i)$$

A configuration x is a **local optimum** for the decomposition scheme C if there does not exist a configuration y such that $y \in H(x, C)$ and $y \succ^{\Re} x$.

Suppose configuration x^j is a local optimum for decomposition C, we call basin of attraction of x^j for decomposition C the set of all configurations from which x^j is reachable:

$$\Psi(x^j, C) = \{y, \text{ such that } \exists P(y, C) \text{ with } x^j \in P(y, C)\}$$

A **limit cycle** is a set $X_0 = \{x_0^1, x_0^2, \ldots, x_0^j\}$ of configurations such that $x_0^1 \succ^{\Re} x_0^2 \succ^{\Re} \ldots \succ^{\Re} x_0^j \succ^{\Re} x_0^1$ and that for all $x \in X_0$, if x has a preferred neighbor $y \in H(x, C)$ then necessarily $y \in X_0$. In other words if we have a cycling set but at least a reachable configuration which is outside this set and is a preferred neighbors of its element, sooner or later the voting process will exit the cycle.

4 Local optima and cycles on social decision surfaces

Having defined the basic characteristics of the walks on social decision surfaces which are generated by voting processes, we are ready to discuss their fundamental properties and, in particular, the social outcomes they may determine. Our algorithmic approach allows to trace all the possible paths on a social decision surface and characterize all possible outcomes for every initial condition. We elaborate on previous work: Marengo and Dosi (2005) and Marengo, Pasquali and Valente (2005) provide a methodology for mapping every modules scheme into possible outcomes in the case in which all modules can be re-examined endlessly until no further improvements can be made, while Page (1996) offers similar results in the case in which once decided a module cannot be re-examined even if improvement become later possible.[5] As already mentioned, in this paper we will discuss only the more general case in which all modules can be always re-examined until no further social improvement whatsoever becomes possible.

In this section we show that, in general, social outcomes depend upon the adopted modules scheme and that by appropriately modifying it one can obtain different social outcomes or even the appearance or disappearance of intransitive limit cycles. In this section we basically provide "possibility"

[5] Because of interdependencies it may in fact happen that a module for which no further improvement is presently possible given the configuration of the other modules can become improvable once other modules have been subsequently modified.

results, i.e. we show examples of occurrences of such phenomena, in the next section we will attempt a discussion of their generality and likelihood.

We first show that different modules schemes can produce different social outcomes.

Proposition 1: Social outcome are, in general, dependent upon the modules scheme.

Consider first a very simple example in which 5 agents have a common most preferred choice. The following table presents such an example of individual preferences, ranked from the most to the least preferred for each agent:

Order	**Agent1**	**Agent2**	**Agent3**	**Agent4**	**Agent5**
1st	011	011	011	011	011
2nd	111	000	010	101	111
3rd	000	001	001	111	000
4th	010	110	101	110	010
5th	100	010	000	100	001
6th	110	111	110	001	101
7th	101	101	111	010	110
8th	001	100	100	000	100

Modules and social outcomes

It is easy to show that if voting is based upon the modules scheme $C = \{\{f_1, f_2, f_3\}\}$ the only local optimum is the global one 011 whose basin of attraction is the entire set X.

If instead voting is based upon the modules scheme $C = \{\{f_1\}, \{f_2\}, \{f_3\}\}$ we have the appearance of multiple local optima and agenda-dependence. If for instance the agenda is the sequence $\{f_1\}, \{f_2\}, \{f_3\}$ then 000 is the local optimum whose basin of attraction contains half the possible initial configurations. For instance, if we start from 110, three out of five agents will vote for changing the first feature into a 0: 010 is in fact the best neighbor of 110 for module $\{f_1\}$. Then module $\{f_2\}$ is considered and again the majority (3 out of 5) decide to move to 000. Then no other change can get the majority. If instead the agenda is the sequence $\{f_3\}, \{f_2\}, \{f_1\}$ it is easy to check that the same initial condition 110 will lead to the global optimum 011.

All in all, both multiplicity of social outcomes and agenda-dependence appear to be linked to the specific set of modules which voting is based upon.

Another reason why voting processes can be path-dependent and sub-optimal is because of the well-known voting paradox (de Caritat Marquis de Condorcet (1785), Arrow (1951))[6]: even in the presence of transitive individual preferences, social preferences expressed through some voting rule can be cyclical. In our model this is reflected by the fact that the order through which choice configurations are presented matters in determining the social outcome. However the manipulation of modules may avoid cycling and path dependency even in the case of intransitive social preferences. This result is stated in the next proposition:

Proposition 2: Suppose that for a module C_i social preferences cycle and the outcome of voting processes is path-dependent. A redefinition of module C_i which splits its composing features and suitably aggregates them to other modules can make path dependence disappear.

Discussion: we explain this proposition by providing an example which is a translation in our formalism of the standard textbook case. Consider the case of three agents and three objects with individual preferences expresses by the following table:

Order	**Agent 1**	**Agent 2**	**Agent 3**
1st	x	y	z
2nd	y	z	x
3rd	z	x	y

Cycles in social preferences

It is easy to verify that with these individual preferences, social preferences expressed through majority rule are intransitive and cycle among the three objects: $x \succ^{\Re} y$ and $y \succ^{\Re} z$, but $z \succ^{\Re} x$.

Suppose now that *x,y,z* are three-features objects which we encode according to the following mapping:

$$x \mapsto 000, y \mapsto 100, z \mapsto 010$$

[6] It is worth pointing out that there is no voting paradox in the previously presented example, where path-dependence is only the outcome of finer than optimal modules.

All other combinations of features are dominated by x,y and z for all agents and we suppose, for simplicity, that preferences over them are identical across agents. All in all, individual preferences are given by the following table:

Order	**Agent 1**	**Agent 2**	**Agent 3**
1st	000	100	010
2nd	100	010	000
3th	010	000	100
4th	110	110	110
5th	001	001	001
6th	101	101	101
7th	011	011	011
8th	111	111	111

Modules and intransitivity: 1

It is easy to verify that if voting is based upon the unique module $C = \{\{f_1, f_2, f_3\}\}$ the voting process always ends up in the limit cycle among x,y and z. The same happens is each feature is a separate module: $C = \{\{f_1\}, \{f_2\}, \{f_3\}\}$.

However, if the modules schemes

$$C = \{\{f_1\}, \{f_2, f_3\}\} \text{ or } C = \{\{f_1, f_3\}, \{f_2\}\}$$

are employed, voting always produces the unique global social optimum 010 in both cases. The latter outcome is the most preferred one by agent 3, who can therefore try to have one of these frames adopted. All other modules schemes always determine cycles: the social outcomes 000 and 100 which are the one most preferred by, respectively, agents 1 and 2 cannot be obtained as social optima by any re-framing. They could however with a different encoding.

Consider for instance the following encoding for x, y, z:

$$x \mapsto 100, y \mapsto 010, z \mapsto 001$$

and the following table of individual preferences:

Order	Agent 1	Agent 2	Agent 3
1st	100	010	001
2nd	010	001	100
3th	001	100	010
4th	000	000	000
5th	110	110	110
6th	101	101	101
7th	011	011	011
8th	111	111	111

Modules and intransitivity: 2

Once again we obtain cycles when voting is based upon the unique module $C = \{\{f_1, f_2, f_3\}\}$, if instead each feature is voted as a separate module: $C = \{\{f_1\}, \{f_2\}, \{f_3\}\}$ we have three local optima: $100, 010, 001$ whose basins of attraction depend, both in size and location, upon the agenda. With the modules scheme $C = \{\{f_1\}, \{f_2, f_3\}\}$ we have only the two local optima 100 and 010, while $C = \{\{f_1, f_3\}, \{f_2\}\}$ produces the two local optima 010 and 001 and $C = \{\{f_1, f_3\}, \{f_2\}\}$ produces the two local optima 100 and 001.

Another interesting issue that we can analyze concerns the time required to reach the social outcome. When the space of alternatives is large, an exhaustive voting process which compares every alternative against all the others can be time-consuming, costly or even unfeasible in any reasonable time scale. In such cases the voting process must necessarily consider only a subset of the conceivable alternatives and there is no certainty that such a subset contains the most desirable alternatives. A trade-off may arise between the time required to make the choice and the social quality of the choice itself. Decision modules define a balance in this trade-off: the smaller the modules the faster a the social outcome is reached. In fact the number of alternatives examined is exponential in the size of the largest module in the modules scheme. Moreover, as we will show in the next section, extensive voting based upon large modules has a high probability of generating intransitive cycles, whereas this probability decreases with smaller decision modules. Thus smaller decision modules make decisions possible (as they tend to reduce the chances of cycles) and faster (as they reduce the number of comparisons between pairs of alternatives) but they do so at the cost of creating a power of determination of the social outcome.

It is easy to show that only under very restrictive assumptions of separability of preferences for all agents does this power vanishes and social outcomes are independent of the choice of the decision modules.

5 Majority voting with random preferences

In the previous section we have shown that by manipulation of decision modules we can deform the social selection surface in such a way as to modify the number and location of social optima and as to make some "perverse" phenomena which have been widely discussed in the literature appear or disappear. Among such phenomena we have discussed intransitivities and cycles in social preferences. In other words we have shown the possibility and dependence upon categorization and framing of non ergodicity, intransitivity, social preference reversal.

An interesting and related question however is to try and measure how likely or plausible such phenomena are, that is to ask questions like, e.g.: a) how many local optima are we likely to encounter? b) how different and/or distant from each other are such local optima? c) how does the number and location of local optima change with a modification of categories? d) how likely are cycles?

Such questions could be addressed either empirically by means for instance of laboratory experiments or theoretically. In this paper we limit ourselves to a preliminary investigation of the latter by means of computer simulations. We simulate in fact the above described voting model for populations of randomly generated agents, i.e. agents whose order relation over the elements of the set X is totally random ("...de gustibus non est disputandum"). In this section we present a preliminary set of simulations.

In the first benchmark simulation we consider a set of 8 binary features and therefore a space 256 configurations, on which a population of 99 random agents vote following the majority rule. All the results we present here and below – unless otherwise specified – are averages over 1,000 repetitions of a simulation all with the same parameters but a different randomly generated population.

We have tested the following module decompositions:

- $C_1 = \{\{1, 2, 3, 4, 5, 6, 7, 8\}\}$
- $C_2 = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}$
- $C_4 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}$

- $C_8 = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\}\}$

The following table presents a summary of results:

Modules	**N. of cases with optima**	**Average n. of social optima**	**N. of cases with cycles**	**Average cycle length**
$\mathbf{C_1}$	47	1 (0)	953	39.61 (13.88)
$\mathbf{C_2}$	940	3.93 (1.45)	1000*	4.67 (1.38)
$\mathbf{C_4}$	1000	9.19 (2.33)	1000**	4.03 (1.09)
$\mathbf{C_8}$	1000	15.66 (3.05)	318**	3.11 (0.48)

Modules, local optima and cycles
(N=8, N. agents=99, 1000 repetitions)

(* indicates that *some* cases present cycles for some initial conditions and local optima for others; ** indicates that *all* cases present cycles for some initial conditions and local optima for others)

The table shows that for the modules scheme C_1, that is a single decision module containing all the features, we have almost always intransitive cycles and that these cycles are rather long (almost 40 different choice configuration on average). Only in about 5% of the randomly generated populations do we obtain a social decision problem which does present cycles but a single social optimum, which is obviously always achieved by voting based on C_1. All in all, intransitive social cycles are the rule in all but a small number of cases.

If we instead take the other extreme, i.e. the set of finest modules C_8, in 682 out of 1000 populations we do not observe cycles, but voting ends in a local optimum. On average there are 15.66 local optima[7] (with standard deviation 3.05). In the remaining 318 cases we observe that voting can end up either on a local optimum or in a cycle depending upon the initial condition. In particular, in these cases in which we observe cycles, the latter are the outcome in – on average – 42.83 (standard deviation 32.58) out of the 256

[7] We have also carried out some simulations with 10 and 12 features, where the number of local optima for the finest modules is around 40 and around 150 respectively. The number of local rapidly increases with the number of features.

possible starting conditions. When they appear, cycles are short, consisting on average in about 3 configurations. All in all, cycles are not very frequent, but on the other hand we have a considerable number of local optima, whose selection depends upon the agenda.

With modules scheme C_4 we always (all 1000 repetitions) observe the coexistence of cycles and local optima in the same social decision problem, depending upon the initial condition. On average, out of the 256 initial conditions, 128.85 (st. dev. 28.26) lead to a cycle and the remaining to a local optimum. In the latter event, the average number of local optima is 9.19.

Finally, with modules scheme C_2 we observe 60 repetitions in which we observe only cycles for all 256 initial conditions, whereas in the 940 remaining case cycles appear on average for 206.53 (dev. st. 28.61) initial conditions. The other initial conditions lead to one out of about 4 local optima. Also in this case cycles tend to be short, as they are made on average of 4.67 configurations.

To summarize, we observe a very clear trade-off between the presence of cycles and the number of local optima. When large modules are employed, cycles are very likely to occur. The likelihood rapidly drops when finer and finer modules are employed, but in parallel the number of local optima increases. This implies that a social outcomes becomes well defined (and as already mentioned can be reached in a shorter time) but which social outcome strongly depends upon the specific modules employed and the sequence in which they are examined.

We also have checked whether local optima tend to concentrate in particular parts of the space, that is if, for a single repetition of the simulation, local optima are somehow similar, in the sense that they display at least for some features the same value. All tests reject this hypothesis: the distribution of local optima in the space of configuration appears as indistinguishable form a randomly generated one.

If we decrease the number of agents we do not observe any difference for the one module C_1 case, while for finer modules we observe a slow increase in the number of local optima and a decrease in the frequency of cycles. For instance with 9 agents and the eight finest modules (C_8) the number of local optima increases on average to 16.89 and cycles appear in 284 repetitions, and in those cases on average only 34 initial conditions lead to a cycle. With only three agents the average number of local optima is 20.01 (st. dev. 3.15) and cycles appear in 176 out of 1000 repetitions, and in the latter only for 30.52 out of 256 initial conditions.

Less heterogeneity among agents seems therefore to reduce the likelihood of cycles [8]

Finally we can test what happens if we decrease the number of features, i.e. the "complexity" of the problem. The following table presents the results of analogous simulations with 99 agents on a "simpler" decision problem with only four features and the three modules schemes:

- $C_1 = \{\{1,2,3,4\}\}$
- $C_2 = \{\{1,2\},\{5,6\}\}$
- $C_4 = \{\{1\},\{2\},\{3\},\{4\}\}$

Modules	N. of cases with optima	Average n. of social optima	N. of cases with cycles	Average cycle length
$\mathbf{C_1}$	369	1 (0)	631	5.02 (1.78)
$\mathbf{C_2}$	932	1.64 (0.69)	702*	3.87 (1.41)
$\mathbf{C_4}$	988	9.19 (2.33)	75*	3.23 (0.79)

Modules, local optima and cycles

(N=4, N. agents=99, 1000 repetitions)

(* indicates that *some* cases present cycles for some initial conditions and local optima for others;)

Results are in line with those of the previous table. Of course we observe a considerable decrease in the number of local optima and length of cycles due to the vast decrease of the size of the combinatorial search space. We also observe an overall decrease in the occurrence of cycles for all categorizations.

[8] In order to be more precise on the role of heterogeneity we plan to define a measure of inter-agent heterogeneity and run simulations with more or less heterogenous agents, but keeping equal their number.

6 Conclusions

Trying to sum up the results thus obtained, we submit the following tentative and preliminary conclusions together with some hypotheses for future research.

1. we believe that our work casts some light and provides some precise tools to investigate the relation between the possibility of aggregating individual preferences, their structure and the existence of some centralized form of power. With respect to this point see the results in section 5, showing that the possibility of constructing aggregate states is to some extent founded upon a fairly strong categorization performed by underlying pre-choice institution. Institutionless choice tends to produce the impossibility of aggregation. To sum up with a *mot d'esprit*: you will not get any society out of a primordial broth of individuals.

2. as a matter of fact, any act of choice takes place within an institutionally framed scenario which, at a minimum, constructs a set of alternatives.

3. the very construction process is far from being neutral neither with respect to individual choice nor to the selection of social outcomes.

4. it thus follows that every social actor fulfilling the social function of framing decision problems, in the aforementioned sense, enjoys a fundamental power of influencing the selection of social outcomes.

We would like to think about our "modest proposal" as a a first step towards a serious consideration of power as an economic and not merely political issue.

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