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Price and Wealth Dynamics in a Speculative Market with an Arbitrary Number of Generic Technical Traders *

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Abstract

We consider a simple pure exchange economy with two assets, one riskless, yielding a constant return, and one risky, paying a stochastic dividend, and we assume trading to take place in discrete time inside an endogenous price formation setting. Traders demand for the risky asset is expressed as a fraction of their individual wealth and is based on future prices forecast obtained on the basis of past market history.

The general case is studied in which an arbitrary large number of heterogeneous traders operates in the market and any smooth function which maps the infinite information set to the present investment choice is allowed as agent's trading behavior. A complete characterization of equilibria is given and their stability conditions are derived. We find that this economy can only possess isolated generic equilibria where a single agent dominates the market and continuous manifolds of non-generic equilibria where many agents hold finite wealth shares. We show that irrespectively of agents number and of their behavior, all possible equilibria belong to a one dimensional "Equilibria Market Line". Finally we discuss the relative performances of different strategies and the selection principle governing market dynamics.

JEL codes: G12, D83.

Keywords: Asset Pricing Model, CRRA Framework, Equilibria Market Line, Market Selection Principle

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1 Introduction

This work analyzes a simple asset pricing model where an arbitrary number of heterogeneous traders participate in a speculative activity. We consider a simple, pure exchange, two-asset economy. The first asset is a riskless security, yielding a constant return on investment. This security is chosen as the numéraire of the economy. The second asset is a risky equity, paying a stochastic dividend. Trading takes place in discrete time and in each trading period the price of the risky asset is fixed by imposing market clearing condition on the aggregate demand function. Agents participation to the market is described in terms of their individual demand for the risky asset.

We impose two restrictions on the way in which the individual demand of traders is formed. First, we assume that the amount of risky security demanded by a trader is proportional to his wealth. This assumption is consistent with, but not limited to, the maximization of an expected utility function with constant relative risk aversion (CRRA). Second, since the present work is mainly concerned with the effect of speculative behaviors and not of asymmetric evaluation and/or knowledge of the underlying fundamental of the economy, we assume that all traders, when making their investment choices, possess the same public information set. This set is naturally defined to contain the past price returns and the complete characterization of the dividend stochastic process.

While the CRRA framework seems to better fit empirical and experimental evidence (see, for instance the discussions in Levy et al. (2000) and Campbell and Viceira (2002)) and has characterized many among the most important contributions to the theory of economic behavior in speculative environment, it has been partly disregarded in the "agent based" literature. In this field the majority of early contributions consider models where agents investment choices are independent from their wealth (for a review see LeBaron (2000) and Hommes (2005)). In terms of expected utility theory, this amounts to consider constant absolute risk averse (CARA) traders. In fact, the assumption of a CRRA-type behavior does introduce noticeable practical difficulties: if prices are endogenously determined by the investment decisions of a population of CRRA traders, the analysis of the resulting market dynamics requires to account, along the evolution of the economy, for the present wealth of each individual portfolio. Hence, when many different traders operate in the market, this results in a system of substantially higher dimension and leads to a seeming increase in complexity.

Notwithstanding these difficulties, a series of recent papers (Chiarella and He, 2001, 2002; Hens and Schenk-Hoppé, 2005; Amir et al., 2005) started to analytically explore the asymptotic dynamics of CRRA agent based models while another group of contributions (Levy et al., 1994, 2000; Zschischang and Lux, 2001) performed a numerical investigation of the emerging properties. In these models the individual demand of traders is usually obtained through the maximization of a CRRA expected utility[1] based on different estimators in order to reflect different stylized speculative behaviors, like "fundamental", "trend chaser" or "contrarian" attitude. However, the requirement of keeping the dimension of the resulting dynamical system low forces the authors to limit both the set of allowed forecasting functions and the number of strategies present in the market at the same time.

In the present paper, we extend these early investigations in two directions. First, we analyze the aggregate dynamics and asymptotic behavior of the market when *an arbitrary large number* of different traders, each with his own investment behavior, participate to the

[1] They frequently use logarithmic utility function, but the generalization of their analyses and conclusions to generic risk averse power utility is, often, trivial.

trading activity. Second, we do not restrict in any way the procedure used by agents in order to build their forecast about future prices, nor the way in which agents can use this forecast to obtain their present asset demand. In other terms, any smooth function which maps the agent information set to the present investment choice is allowed as agent's trading behavior.

Even if considering an arbitrary number of different agents behaviors leads us to study dynamical systems of an arbitrary large dimension, we are able to provide a complete characterization of market equilibria and a description of their stability conditions in terms of few parameters characterizing the traders investment strategies. In particular, we find that, irrespectively of the number of agents operating in the market and of the structure of their demand functions, only two types of equilibria are possible: generic equilibria, associated with isolated fixed points, where a single agent asymptotically possesses the entire wealth of the economy and non generic equilibria, associated with continuous manifolds of fixed points, where many agents possess a finite shares of the total wealth. We also show, in total generality, that a simple function, the "Equilibria Market Line", can be used to obtain a geometric characterization of both the location of all possible equilibria and the conditions of their stability.

Our general results provide, we believe, a simple and clear description of the principles governing the asymptotic market dynamics resulting from the competition of different trading strategies. As a direct consequence we are able to discuss the validity and limits of the "quasi-optimal selection principle", originally formulated in Chiarella and He (2001) for linear demand functions, when more general traders behaviors are taken into consideration. At the same time, we show how the possible existence of multiple, isolated, locally stable equilibria and the ensuing local nature of traders relative performances can be interpreted as an "impossibility theorem" for the construction of a dominance order relation inside the space of trading strategies.

The present paper is organized as follows. In Section 2 we describe our simple pure-exchange economy, presenting our assumptions and briefly discussing them. We explicitly write the traders inter-temporal budget constraints and lay down the equations governing the dynamics of the market. In Section 3 we present the simple case in which a single trader operates in the market. The Equilibria Market Line is derived, and its use is shortly discussed. The general case in which an arbitrarily large number of traders participates the trading activity is analyzed in Section 4. Our conclusions, and the directions our work will plausibly take in the future, are briefly mentioned in Section 5.

2 Definition of the Model

Consider a simple pure exchange economy, populated by a fixed number N of traders, where trading activities take place in discrete time. The economy is composed by a risk-less asset (bond) giving in each period a constant interest rate $r_f > 0$ and a risky asset (equity) paying a random dividend D_t at the end of each period t. Let the risk-less asset be the numéraire of the economy, so that its price is fixed to 1. The price P_t of the risky asset is determined at each period, on the basis of the aggregate demand, through market-clearing condition. The resulting intertemporal budget constraint is derived below and the main hypotheses, on the nature of the investment choices and of the fundamental process, are discussed. These hypotheses will allow us to derive the explicit dynamical system governing the evolution of the economy.

2.1 Intertemporal Budget Constraint

Let $W_{t,n}$ be the wealth of trader n at time t and let $x_{t,n}$ stands for the fraction of this wealth invested into the risky asset. After the trading session at time $t-1$, agent n possesses $x_{t-1,n}\, W_{t-1,n}/P_{t-1}$ shares of risky asset and $(1-x_{t-1,n})\, W_{t-1,n}$ shares of risk-less security. At this moment he receives the payment of risk-less interest r_f on the wealth invested in the latter and dividends payment D_{t-1} per each risky asset. Therefore, at time t the wealth of agent n, for any notional price P, reads

$$W_{t,n}(P) = (1-x_{t-1,n})\, W_{t-1,n}\,(1+r_f) + \frac{x_{t-1,n}\, W_{t-1,n}}{P_{t-1}}\,(P+D_{t-1}) \tag{2.1}$$

and his individual demand for the risky asset becomes $x_{t,n}\, W_{t,n}(P)/P$. The actual price of the risky asset at time t is fixed so that aggregate demand equals aggregate supply. Assuming a constant supply of risky asset, whose quantity can then be normalized to 1, the price P_t is defined as the solution of the equation

$$\sum_{n=1}^{N} x_{t,n}\, W_{t,n}(P_t) = P_t\ . \tag{2.2}$$

Once the price is fixed, the new portfolios and wealths are determined and, at the end of period t, the dividend D_t and the risk-free interest r_f are paid. At this point the trading session at time $t+1$ can start.

The dynamics defined by (2.1) and (2.2) describes an exogenously growing economy due to the continuous injections of new riskless assets, whose price remains, under the assumption of totally elastic supply, unchanged. It is convenient to remove this exogenous economic expansion from the dynamics of the model. To this purpose we introduce rescaled variables

$$w_{t,n} = W_{t,n}/(1+r_f)^t\ , \quad p_t = P_t/(1+r_f)^t\ , \quad e_t = D_t/(P_t\,(1+r_f))\ . \tag{2.3}$$

Rewriting (2.2) and (2.1) using these variables one obtains

$$\begin{cases} p_t = \displaystyle\sum_{n=1}^{N} x_{t,n}\, w_{t,n} \\ w_{t,n} = w_{t-1,n} + w_{t-1,n}\, x_{t-1,n}\left(\dfrac{p_t}{p_{t-1}} - 1 + e_{t-1}\right) \quad \forall n \in \{1,\ldots,N\} \end{cases}\ . \tag{2.4}$$

These equations give the evolution of state variables $w_{t,n}$ and p_t over time, provided that the stochastic process $\{e_t\}$ is given and the set of investment shares $\{x_{t,n}\}$ is specified. Such dynamics implies a simultaneous determination of the equilibrium price p_t and of the agents' wealths $w_{t,n}$. Due to this simultaneity, the $N+1$ equations in (2.4) define the state of the system at time t only implicitly. Indeed, the N variables $w_{t,n}$, defined in the second equation, appear on the right-hand side of the first, and, at the same time, the variable p_t, defined in the first equation, appears in the right-hand side of the second. For analytical purposes, one has to derive the explicit equations that govern the system dynamics.

2.2 Dynamical System for Wealth and Return

The transformation of the implicit dynamics of (2.4) into an explicit one is not possible in general. Indeed the simultaneous determination of price and wealth in (2.4) entails some restriction on the possible market positions available to agents[2]. We derive this restriction below, but before let us introduce a notation that will prove useful to present the dynamics in a more compact form.

Let a_n be an agent specific variable, dependent or independent from time t. We denote with $\langle a \rangle_t$ the *wealth weighted average* of this variable at time t on the population of agents, i.e.

$$\langle a \rangle_t = \sum_{n=1}^{N} a_n \, \varphi_{t,n} \ , \quad \text{where} \quad \varphi_{t,n} = \frac{w_{t,n}}{w_t} \quad \text{and} \quad w_t = \sum_{n=1}^{N} w_{t,n}. \tag{2.5}$$

The next result gives the condition for which the dynamical system implicitly defined in (2.4) can be made explicit without violating the requirement of positiveness of prices

Proposition 2.1. *From equations* (2.4) *it is possible to derive a map* $\mathbb{R}^{+N} \to \mathbb{R}^{+N}$ *that describes the evolution of traders wealth* $w_{t,n}$ $\forall n \in \{1, \ldots, N\}$ *so that prices* $p_t \in \mathbb{R}^+$ $\forall t$ *remain positive provided that*

$$\Big(\langle x_t \rangle_t - \langle x_t \, x_{t+1} \rangle_t\Big) \Big(\langle x_{t+1} \rangle_t - (1 - e_t)\langle x_t \, x_{t+1} \rangle_t\Big) > 0 \qquad \forall t \quad . \tag{2.6}$$

If previous condition is met, the price growth rate $r_{t+1} = p_{t+1}/p_t - 1$ *reads*

$$r_{t+1} = \frac{\langle x_{t+1} - x_t \rangle_t + e_t \, \langle x_t \, x_{t+1} \rangle_t}{\langle x_t \, (1 - x_{t+1}) \rangle_t} \tag{2.7}$$

and the evolution of wealth, described by the wealth growth rates $\rho_{t+1,n} = w_{t+1,n}/w_{t,n} - 1$, *is given by*

$$\rho_{t+1,n} = x_{t,n} \left(r_{t+1} + e_t\right) = x_{t,n} \, \frac{\langle x_{t+1} - x_t \rangle_t + e_t \, \langle x_t \rangle_t}{\langle x_t \, (1 - x_{t+1}) \rangle_t} \qquad \forall n \in 1, \ldots, N \quad . \tag{2.8}$$

Proof. See appendix A. □

The explicit price dynamics can be obtained from (2.7) in a trivial way but price will be positive only if condition (2.6) is satisfied[3].

Having obtained the explicit dynamics for the evolution of price and wealth one is interested in the asymptotic behavior of the system. In turns out that the dynamics defined by (2.7) and (2.8) does not possess any interesting fixed point. Indeed, if the price and the wealth are constant, one would have $r_{t+1} = \rho_{t+1,n} = 0$ for any t and n. This would imply, in periods when a positive dividend e_t is paid, that $x_{t,n} = 0$ for any n. That is, the only possible fixed point, in terms of price and wealth levels, is the one in which there is no demand for the risky asset. The cause is that the variables rescaling introduced in (2.3) removes the exogenous expansion due to the risk-free interest rate, but not the expansion due to the dividend payment. The

[2]The simplest way to understand this is to consider the case of a single agent. With a little bit of algebra it is easy to show that (2.1) and (2.2) imply that at time t the price should satisfy the equation $P_t = x_t \, (P_t + B_t)$, where B_t denotes the amount of the numéraire available to the agent before trading. Therefore, a positive price requires $x_t < 1$. In the dynamical setting with many agents the matter is more complicated, since both current and previous investment choices of all agents are involved in the determination of price.

[3]In general, it may be quite difficult to check the validity of this condition at each time step. A sufficient condition is provided in Anufriev et al. (2004) where it is shown that if investment choices $x_{t,n}$ are bounded inside the interval $(0, 1)$, *uniformly* with respect to t and n, condition (2.6) is always satisfied.

presence of this expansion suggests to look for possible asymptotic states of *steady growth.* Notice, however, that if the dividend yield e_t depends on price, it is impossible to rewrite the dynamics in Proposition 2.1 in terms of the sole price and wealth returns. This issue is solved as soon as one makes the following

Assumption 1. The dividend yields e_t are i.i.d. random variables obtained from a common distribution with positive support.

This assumption is common to several works in the literature, for instance Chiarella and He (2001), so that a further reason to have this assumption introduced is to maintain comparability with previous investigations. Under Assumption 1, direct dependence on price disappears from (2.7) and (2.8). Consequently, the dynamics of the economy is fully specified in terms of r_t and $\rho_{t,n}$ for $n \in \{1, \ldots, N\}$.

2.3 Agents Investment Functions

In the present work we are mainly concerned with the effect of speculative behaviors on the market aggregate performance. In order to eliminate effects due to asymmetric evaluation and/or knowledge of the underlying fundamental process we model the agents' investment choice as depending on the sole realized price returns, assuming, for any agent, a perfect knowledge of the dividend process. Consequently, the information set $\mathbb{I}$ commonly available to traders at round t reduces to the sequence of past realized returns $\mathbb{I}_{t-1} = \{r_{t-1}, r_{t-2}, \ldots\}$ and we make the following

Assumption 2. For each agent n there exists a differentiable *investment function* f_n which maps the present information set into his investment share:

$$x_{t,n} = f_n(\mathbb{I}_{t-1}) \quad . \tag{2.9}$$

The function f_n in the right-hand side of (2.9) gives a complete description of the investment decision of the n-th agent. The knowledge about the fundamental process, being complete and time invariant, is not explicitly inserted in the information set, rather is considered embedded in the functional form of f_n. Past realizations of the fundamental process do not affect agents' decisions, which, rather, tends to adapt to observed price fluctuations. One can refer to this investment behavior, common in agent based literature (e.g. Brock and Hommes (1998)), as *"technical trading"*, stressing the similarity with trading practices observed in real markets. At the same time, Assumption 2 rules out other possible dependences in the investment function f_n, like an explicit relation of the present investment choice with past investment choices or with investment choices of other traders.

In the majority of models discussed in the literature the investment choice described by (2.9) is obtained as the result of two distinct steps. In the first step agent n, using a set of estimators $\{g_{n,1}, g_{n,2}, \ldots\}$, forms his expectation at time t about the behavior of future prices, $\theta_{n,j}[r_{t+1}] = g_{n,j}(\mathbb{I}_{t-1})$ where $\theta_{\cdot,j}$ stands for some statistics of the returns distribution at time $t+1$, for instance the average return, the variance or the probability that a given return threshold be crossed. With these expectations, using a choice function h_n, possibly derived from some optimization procedure[4], he computes the fraction of the wealth invested in the

[4]The assumption that the demand function of any agent at time t can be written as $x_{t,n}\, W_{t,n}/P_t$, with $x_{t,n}$ independent from his present wealth $W_{t,n}$ and price level P_t, is consistent with a framework in which agents investment decisions are obtained via expected utility maximization with a constant relative risk aversion (CRRA) utility function. Assumption 2 is clearly violated, however, if agents are assumed constant absolute risk aversion (CARA) expected utility maximizers.

risky asset $x_{t+1,n} = h_n(\theta_1, \theta_2, \dots)$. The investment function f_n defined in Assumption 2 is the result of the composition of estimators $\{g_{n,\cdot}\}$ and choice function h_n. This interpretation is both intuitive and common in the economic literature, but, even if perfectly compatible with (2.9), it is not required by our framework. In our model agents are not forced to use some specific predictors, rather they are allowed to map the past return history into the future investment choice, with whatever smooth function they like. Moreover, function f_n, as it is written in (2.9), can be infinite dimensional. In this case, a complete dynamical system for the return dynamics should be, generally speaking, infinite-dimensional as well. This difficulty can be overcome in different ways. For instance, in Anufriev et al. (2004), while leaving the choice function h generic, we confined agents to use, as forecasts functions $\{g_{n,1}, g_{n,2}, \dots\}$, exclusively the expected return $E_n[r_{t+1}]$ and variance $V_n[r_{t+1}]$ obtained from exponentially weighted moving average (EWMA) estimates based on past realized returns. In that case the dynamics can be described with the use of a low dimensional system, even if the information set remains infinite. In fact, this dimensional reduction is always possible provided that the agents forecasting procedure admits a recursive definition.

In what follows, however, we want to consider a more generic situation. We assume that each agent n has his own, so to speak, "memory time span" L_n, so that at each time step his new investment choice is determined as a function of the last L_n return realizations. Apart the requirement that it possesses first order derivatives with respect to the past price returns, we do not restrict this function in any way. Moreover, for the following discussion, L_n must be finite, but can be arbitrary large.

3 Single Agent Case

We start with the analysis of the very special situation in which a single agent operates on the market. The main reason to perform this analysis rests in its relevance for the multi-agent case. Indeed, in the setting with N heterogeneous traders each generic multi-agent equilibrium requires, as necessary condition for stability, the stability of a suitably defined single agent equilibrium[5].

This Section starts laying down the dynamics of the single agent economy as a multidimensional dynamical system of difference equations of the first order. All possible equilibria of the system are identified and the associated characteristic polynomial, which can be used to analyze their stability, derived.

3.1 Dynamical System

In the case of one single agent the dynamical system describing the market evolution can be considerably simplified since the explicit evolution of wealth shares in (2.8) is not needed. As a consequence, the whole system can be described with only $L + 1$ variables: one variable represents the current investment choice x_t and the other variables the L past returns.

The current return can be defined by means of the function in the right hand-side of (2.7):

$$R(x', x, e) = \frac{x' - x + e\, x'\, x}{(1 - x')\, x} \quad , \tag{3.1}$$

[5] Another possible application of the single agent analysis is to provide a succinct description of the aggregate properties of a system with many relatively homogeneous agents. See Anufriev et al. (2004) for a discussion.

where the first variable x' denotes the current (contemporaneous with return) investment choice, and x and e stands for the *previous period* investment choice and dividend yield, respectively.

With such definitions the dynamical system governing the evolution of the economy with a single agent reads

$$\begin{cases} x_{t+1} &= f\big(r_{t,0}, r_{t,1}, \dots, r_{t,L-1}\big) \\ r_{t+1,0} &= R\Big(f\big(r_{t,0}, r_{t,1}, \dots, r_{t,L-1}\big),\ x_t,\ e_t\Big) \\ r_{t+1,1} &= r_{t,0} \\ \quad\vdots & \\ r_{t+1,L-1} &= r_{t,L-2} \end{cases} , \tag{3.2}$$

where $r_{t,l}$ stands for the price return at time $t - l$.

In the rest of this Section we are interested in analyzing the so-called *deterministic skeleton* of this $L + 1$-dimensional system. That is, we substitute the yield by its mean value $\bar{e}$ in order to obtain the deterministic dynamical system which gives, in a sense, the "average" representation of the stochastic dynamics.

3.2 Determination of Equilibria

In the following analysis, in order to give a simple geometrical characterization of equilibria of system (3.2) we will use the following

Definition 3.1. The *Equilibria Market Line* (EML) is the function $l(r)$ defined according to

$$l(r) = \frac{r}{\bar{e} + r} \qquad r \in [-1, \infty)\,. \tag{3.3}$$

Let x^* denotes the agent's wealth share invested in the risky asset at equilibrium and let r^* be the the equilibrium return. In any fixed point the realized returns are constant, so that $r_0 = r_1 = \cdots = r_{L-1} = r^*$. One has the following

Proposition 3.1. *Let $\boldsymbol{x^*} = (x^*; r^*, \dots, r^*)$ be a fixed point of system* (3.2)*.*

(i) The point $\boldsymbol{x^*}$ is a feasible equilibrium, i.e. the equilibrium prices are positive, if either $x^* < 1$ or $x^* \geq 1/(1 - \bar{e})$.

(ii) Equilibrium return r^* satisfies

$$l(r^*) = \frac{r^*}{\bar{e} + r^*} = x^* \tag{3.4}$$

and the equilibrium investment share x^* is defined accordingly to

$$x^* = f(r^*, \dots, r^*) \quad . \tag{3.5}$$

(iii) The equilibrium growth rate of the agent's wealth is given by

$$\rho^* = x^*\,(r^* + \bar{e}) = r^* \tag{3.6}$$

and is equal to the equilibrium price growth rate r^*.

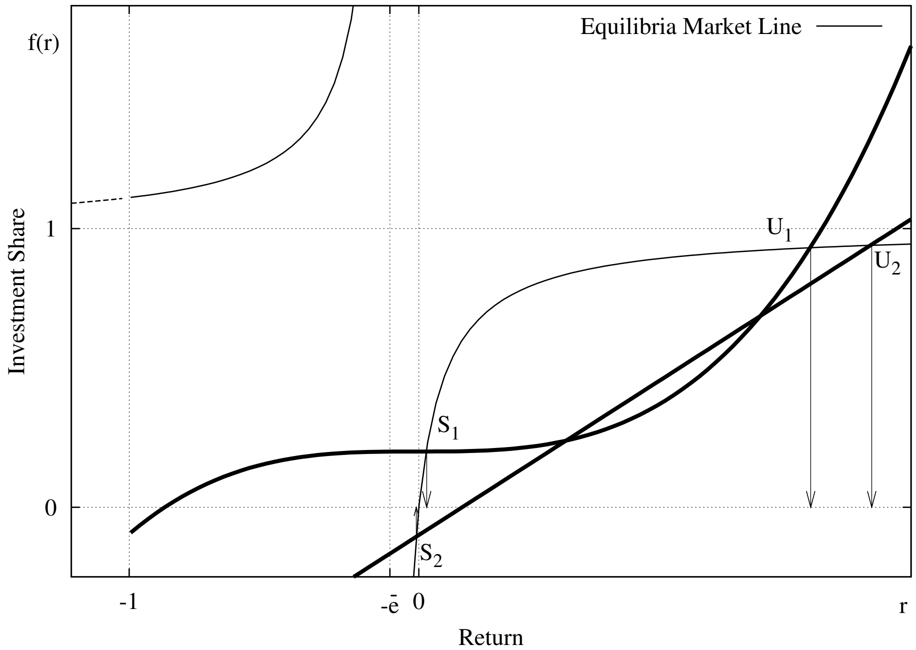

Figure 1: Investment functions (thick lines) based on the last realized return. The equilibria are found as intersections with the EML (thin line). Both functions have two equilibria: S_1 and U_1 the non-linear, S_2 and U_2 the linear.

Proof. Item *(i)* follows directly from condition (2.6) written at equilibrium. In order to get (3.5) one has to plug the equilibrium values of the variables in the first equation in (3.2). From the second equation of (3.2) one has

$$r^* = R(x^*, x^*, \bar{e}) = \bar{e}\,\frac{x^*}{1 - x^*} \quad .$$

Inverting this relation to obtain x^* as a function of r^* and using (3.5) one gets (3.4). Finally, from (2.8) using the previous relations one has $\rho^* = x^*(r^* + \bar{e}) = l(r^*)(r^* + \bar{e}) = r^*$. □

The first item states that economically meaningful equilibria are characterized by values of the investment share inside the intervals $(-\infty, 1)$ or $[1/(1 - \bar{e}), +\infty)$. This is equivalent to the restriction $r^* \geq -1$. The second item justifies the introduction of the Equilibria Market Line in Definition 3.1: all equilibria of system (3.2) can be found as the intersections of the EML with the full symmetrization of function f, i.e. with the restriction of this function to the one dimensional subspace defined by the $L - 1$ equations $r_0 = r_1 = \cdots = r_L$. Notice that for $r^* = -\bar{e}$ no bounded equilibria exist. This is a general property of the system with N agents defined in (2.7) and (2.8). If at equilibrium the positive dividend yield were offset by a negative price return, the wealth of each agent would be constant over time and the investment share would increase with a rate $1 - \bar{e}$. Finally, the third item states that the growth rate of the agent's wealth coincides with the price growth rate. It is interesting that the interrelation between the total return $r^* + \bar{e}$ and the investment in equilibrium x^* is such that the total wealth grows with a rate which does not directly dependent on the dividend yield.

As a first example of the application of Proposition 3.1 consider investment functions which are one-dimensional functions of the sole last return (i.e. $L = 1$). In Fig. 1 two functions of this type are shown (thick lines) together with the hyperbolic curve representing the EML defined in (3.3) (thin line). The intersections of the investment function with the EML are the possible equilibria of the system. The ordinate of the intersection gives the value of equilibrium investment share x^*, while the abscissa gives the equilibrium return r^*. One can distinguish between three qualitatively different scenarios. In equilibria with $r^* \in [-1, -\bar{e})$ the investment in the risky asset is characterized by negative return $r^* + \bar{e} < 0$. In these equilibria the agent maintains a long position in the risky asset ($x^* > 1$) so that his wealth return ρ^* is negative. If $r^* \in (-\bar{e}, 0)$ the capital gain on the risky asset is negative, nevertheless the gross return $r^* + \bar{e}$ is positive due to the dividend yield. In these equilibria the agent maintains a short position in the risky asset ($x^* < 0$) and therefore it is again $\rho^* < 0$. Equilibrium S_2 for the linear investment function in Fig. 1 is of such kind. Finally, if $r^* \in (0, +\infty)$ the price return is positive, the agent position is characterized by a fixed fraction of wealth invested in the risky asset $x^* \in (0, 1)$ and his wealth return is positive. This is the case of equilibrium U_2 of the linear investment function and equilibria S_1 and U_1 of the nonlinear function[6].

A second "geometrical" example is presented in Fig. 2. The two-dimensional surface represents the investment function $f(r_0, r_1) = |r_0|\,(r_0 + 0.4\,(r_0 - r_1))$ which depends on the two last realized returns (i.e. $L = 2$). Here r_0 stands for the last period return while r_1 is the return of the period before the last. The thick line on the function surface is the intersection of the investment function with the "symmetric" plane defined by the condition $r_0 = r_1$. On the same plane the curve relative to EML $l(r)$ is also drawn. The intersections of these two curves define all possible equilibria. In Fig. 2 there is one trivial equilibrium with zero return and a second equilibrium with positive price return r^* and equilibrium investment share $x^* = f(r^*) = |r^*|\,r^*$.

The same analysis can be applied, unmodified, to the investment functions with higher values of L. The bottom line of these examples and of the previous discussion is that the agent's memory span L is irrelevant for the question of the existence and location of equilibria: only the restriction of the investment function f on the "symmetric" plane is relevant and, in all cases, the equilibria are located on the one dimensional EML and can be presented in a diagram analogous to Fig. 1.

3.3 Stability Conditions of Equilibria

As the next natural step we move to discuss the stability conditions of the equilibria that has been identified in the previous Section. We derive the stability conditions from the analysis of the roots of the characteristic polynomial associated with the Jacobian of system (3.2) computed at equilibrium. The characteristic polynomial does, in general, depend on the behavior of the individual investment function f in an infinitesimal neighborhood of the equilibrium $\boldsymbol{x^*}$. This dependence can be summarized with the help of the following

Definition 3.2. The *stability polynomial* $P(\mu)$ of the investment function f in $\boldsymbol{x^*}$ is

$$P_f(\mu) = \frac{\partial f}{\partial r_0}\,\mu^{L-1} + \frac{\partial f}{\partial r_1}\,\mu^{L-2} + \cdots + \frac{\partial f}{\partial r_{L-2}}\,\mu + \frac{\partial f}{\partial r_{L-1}}\,, \tag{3.7}$$

where the derivatives of f are computed in point $(r^*, \ldots, r^*)$.

Using the previous definition the equilibrium stability conditions can be formulated in terms of the equilibrium return r^*, and of the slope of the Equilibria Market Line in equilibrium

$$l'(r^*) = \frac{\bar{e}}{(\bar{e} + r^*)^2}\,. \tag{3.8}$$

[6]Remember that the analysis is performed with respect to rescaled variables as defined in (2.3)

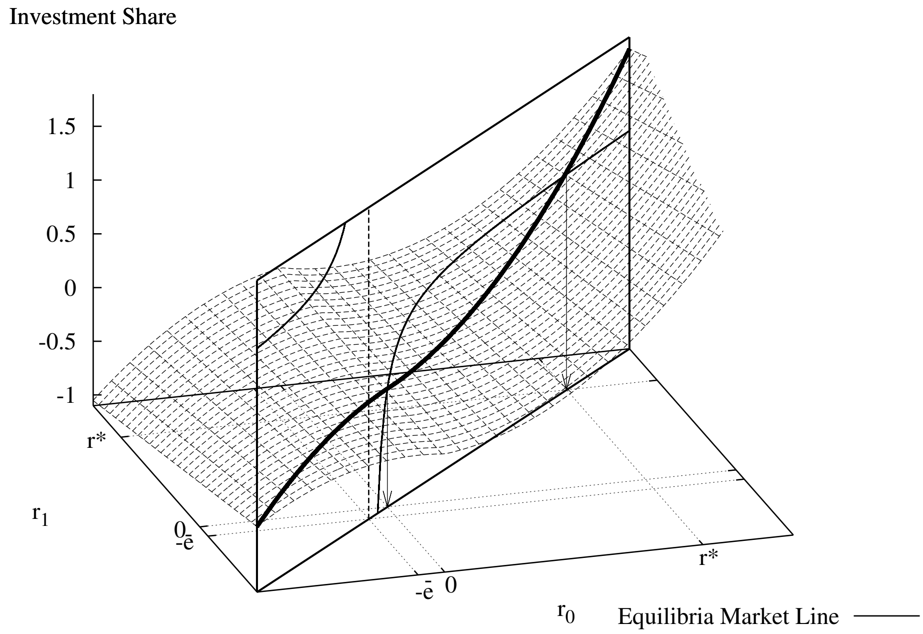

Figure 2: Investment function based on the last two realized returns $f(r_{t-1}, r_{t-2})$ and its intersection (thick line) with the plane $r_{t-1} = r_{t-2}$. Equilibria are found on this plane as intersections with the EML (thin line).

The following applies

Proposition 3.2. *The fixed point $\boldsymbol{x}^* = (x^*; r^*, \ldots, r^*)$ of system* (3.2) *is (locally) asymptotically stable if all the roots of the polynomial*

$$Q(\mu) = \mu^{L+1} - \frac{P_f(\mu)}{r^*\, l'(r^*)} \Big(\left(1 + r^*\right) \mu - 1 \Big) \quad , \tag{3.9}$$

are inside the unit circle.

The equilibrium $\boldsymbol{x}^*$ is unstable if at least one of the roots of $Q(\mu)$ lies outside the unit circle.

Proof. The condition above is a direct consequence of the characteristic polynomial of the Jacobian matrix at equilibrium. See appendix B for a derivation. □

Once investment function f is known, polynomial $P(\mu)$ and, in turn, polynomial $Q(\mu)$ can be explicitly derived. The analysis of the roots of $Q(\mu)$ can be used to reveal the role of the different parameters in stabilizing or destabilizing a given equilibrium. For illustrative purposes we present here the explicit analysis in the simplest case[7] in which $L = 1$.

3.3.1 Example: Naïve forecast.

As an example consider the agent with a memory time span of a single lag, so that his investment function reads $x_{t+1} = f(r_t)$. In terms of the two-step interpretation of investment

[7]Results for larger classes of investment functions are presented in Anufriev et al. (2004)

function discussed in Sec. 2.3, the agent in this example can be thought to have *naïve* forecast of the form $E[r_{t+1}] = r_t$.

When $L = 1$, expression in (3.9) reduces to a second degree polynomial. The multipliers are the roots of

$$\mu^2 - \frac{f'(r^*)}{l'(r^*)\, r^*} \Big(\left(1 + r^*\right) \mu - 1 \Big) = 0 \; .$$

The following can be derived from well known stability conditions for a two dimensional system fixed point

Proposition 3.3. *The fixed point* $\boldsymbol{x^*} = (x^*; r^*)$ *of system* (3.2) *with* $L = 1$ *is (locally) asymptotically stable if*

$$\frac{f'(r^*)}{l'(r^*)} \frac{1}{r^*} < 1 \, , \qquad \frac{f'(r^*)}{l'(r^*)} < 1 \qquad \textit{and} \qquad \frac{f'(r^*)}{l'(r^*)} \frac{2 + r^*}{r^*} > -1 \quad . \tag{3.10}$$

where $f' = df(r^*)/dr$. *This fixed point undertakes* Neimark-Sacker, fold *or* flip *bifurcation if the first, the second or the third inequality in* (3.10) *turns to equality, respectively.*

The stability region defined by the inequalities in (3.10) is shown in Fig. 3 in coordinates r^* and $f'(r^*)/l'(r^*)$. The second coordinate is the relative slope of the investment function at equilibrium with respect to the slope of the Equilibrium Market Line. Notice that if the slope of f at the equilibrium increases, the system tends to lose its stability. In particular, the second inequality in (3.10) requires the slope of investment function to be smaller than the slope of function $l(r)$.

As an example let us look at the equilibria in Fig. 1. Concerning the nonlinear function, one can immediately see that equilibrium U_1 is unstable since the slope of the function is higher than the slope of the EML. On the contrary, the slope of the investment function in S_1 is very small, so that, presumably, this equilibrium is stable. For the linear function, equilibrium U_2 is clearly unstable while S_2 can result stable. As can be seen from Fig. 3, if the slope of the first function in S_1 increased, this equilibrium would lose stability through a Neimark-Sacker bifurcation. The increase of the slope of the second function in S_2 would instead lead to a flip bifurcation.

4 Economy with Many Agents

This Section extends the previous results to the case of a finite, but arbitrarily large, number of heterogenous agents. Each agent n possesses his own investment function f_n based on a finite number L_n of past market realizations. Without loss of generality, however, we can assume that the memory spans of the different function f_n are all the same and equal to the largest span $L = \max\{L_1, \dots L_N\}$, so that each investment function can be thought as having exactly L arguments

$$x_{t+1,n} = f_n(r_t, r_{t-1}, \dots, r_{t-L+1}) \; . \tag{4.1}$$

This section is organized as the previous one. It starts with the derivation of the $2N + L - 1$ dimensional stochastic dynamical system which describes the evolution of the economy and continues with the identification of all possible equilibria of the associated deterministic skeleton and the analysis of their stability.

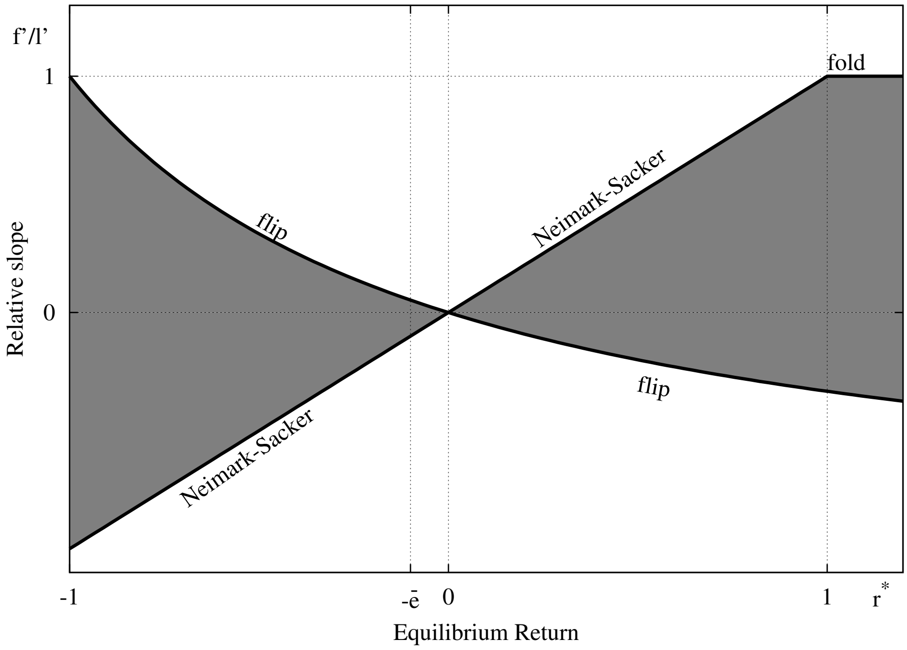

Figure 3: Stability region (gray) and fixed point bifurcations for system (3.2) with $L = 1$. The parameter space has coordinates r^* and $f'(r^*)/l'(r^*)$.

4.1 Dynamical System

If there is more than one agent on the market, the evolution of agents wealth is not decoupled from the system and, consequently, all N equations in (2.8) are relevant for the dynamics. In this case it is convenient to rewrite the system using the individual share of the total wealth $\varphi_{t,n}$ defined in (2.5). The dynamics of price in terms of these variables is provided by the following

Lemma 4.1. *Under the conditions of Proposition 2.1, the price growth rate* (2.7) *reads:*

$$r_{t+1} = \frac{\left\langle x_{t+1} - x_t + e_t\, x_t\, x_{t+1} \right\rangle_t}{\left\langle x_t\,(1 - x_{t+1}) \right\rangle_t} \,, \tag{4.2}$$

while the agents' wealth shares evolve accordingly to

$$\varphi_{t+1,n} = \varphi_{t,n}\, \frac{1 + (r_{t+1} + e_t)\, x_{t,n}}{1 + (r_{t+1} + e_t) \left\langle x_t \right\rangle_t} \qquad \forall n \in \{1, \ldots, N\} \quad . \tag{4.3}$$

Proof. See appendix C. □

The first-order dynamical system associated with (4.2) and (4.3) with investment functions $x_{t+1,n} = f_n(r_t, r_{t-1}, \ldots, r_{t-L})$ as in (4.1) is defined in terms of the following $2N + L - 1$ independent variables

$$x_{t,n} \quad \forall n \in \{1, \ldots, N\}\,; \qquad \varphi_{t,n} \quad \forall n \in \{1, \ldots, N-1\}\,; \qquad r_{t,l} \quad \forall l \in \{0, \ldots, L-1\}\,, \tag{4.4}$$

where $r_{t,l}$ denotes the price return at time $t - l$. Notice that only $N - 1$ wealth shares are needed. Indeed, at any time step t, it is $\sum_{n=1}^{N} \varphi_{t,n} = 1$ so that $\varphi_{t,N} = 1 - \sum_{n=1}^{N-1} \varphi_{t,n}$. The dynamics of the system is provided by the following

Lemma 4.2. *The* $2N+L-1$ *dynamical system defined by* (4.2) *and* (4.3) *in terms of the variables in* (4.4) *reads*

$$
\begin{array}{rl}
\mathcal{X}: & \left[\begin{array}{lcl}
x_{t+1,1} & = & f_1\big(r_{t,0},\ldots,r_{t,L-1}\big)\\
\vdots & \vdots & \vdots\\
x_{t+1,N} & = & f_N\big(r_{t,0},\ldots,r_{t,L-1}\big)
\end{array}\right.\\
\mathcal{W}: & \left[\begin{array}{lcl}
\varphi_{t+1,1} & = & \Phi_1\Big(x_{t,1},\ldots,x_{t,N};\varphi_{t,1},\ldots,\varphi_{t,N-1};e_t;\\
& & \qquad R\big(f_1(r_{t,0},\ldots,r_{t,L-1}),\ldots,f_N(r_{t,0},\ldots,r_{t,L-1});\\
& & \qquad\qquad x_{t,1},\ldots,x_{t,1};\varphi_{t,1},\ldots,\varphi_{t,N-1};e_t\big)\Big)\\
\vdots & \vdots & \vdots\\
\varphi_{t+1,N-1} & = & \Phi_{t,N-1}\Big(x_{t,1},\ldots,x_{t,N};\varphi_{t,1},\ldots,\varphi_{t,N-1};e_t;\\
& & \qquad R\big(f_1(r_{t,0},\ldots,r_{t,L-1}),\ldots,f_N(r_{t,0},\ldots,r_{t,L-1});\\
& & \qquad x_{t,1},\ldots,x_{t,N};\varphi_{t,1},\ldots,\varphi_{t,N-1};e_t\big)\Big)
\end{array}\right.\\
\mathcal{R}: & \left[\begin{array}{lcl}
r_{t+1,0} & = & R\Big(f_1(r_{t,0},\ldots,r_{t,L-1}),\ldots,f_N(r_{t,0},\ldots,r_{t,L-1});\\
& & \qquad x_{t,1},\ldots,x_{t,N};\varphi_{t,1},\ldots,\varphi_{t,N-1};e_t\Big)\\
r_{t+1,1} & = & r_{t,0}\\
\vdots & \vdots & \vdots\\
r_{t+1,L-1} & = & r_{t,L-2}
\end{array}\right.
\end{array}, \tag{4.5}
$$

where

$$
\begin{aligned}
&R\Big(y_1,y_2,\ldots,y_N;x_1,x_2,\ldots,x_N;\varphi_1,\varphi_2,\ldots,\varphi_{N-1};e\Big)=\\
&=\frac{\sum_{n=1}^{N-1}\varphi_n\big(y_n\,(1+e\,x_n)-x_n\big)+\Big(1-\sum_{n=1}^{N-1}\varphi_n\Big)\,\big(y_N\,(1+e\,x_N)-x_N\big)}{\sum_{n=1}^{N-1}\varphi_n\,x_n\,(1-y_n)+(1-\sum_{n=1}^{N-1}\varphi_n)\,x_N\,(1-y_N)}
\end{aligned} \tag{4.6}
$$

and

$$
\begin{aligned}
&\Phi_n\big(x_1,x_2,\ldots,x_N;\varphi_1,\varphi_2,\ldots,\varphi_{N-1};e;R\big)=\\
&\quad=\varphi_n\,\frac{1+x_n\,(R+e)}{1+(R+e)\,\Big(\sum_{m=1}^{N-1}\varphi_m\,x_m+\big(1-\sum_{m=1}^{N-1}\varphi_m\big)\,x_N\Big)}\qquad\forall n\in\{1,\ldots,N-1\}\,.
\end{aligned} \tag{4.7}
$$

Proof. We ordered the equations to obtain three separated blocks: $\mathcal{X}$, $\mathcal{W}$ and $\mathcal{R}$. In block $\mathcal{X}$ there are N equations defining the investment choices of agents. Block $\mathcal{W}$ contains $N-1$ equations describing the evolution of the wealth shares. Finally, block $\mathcal{R}$ is composed by L equations which describe the evolution of the return. In the last block equations are in ascending order with respect to the time lag.

The set $\mathcal{X}$ is immediately obtained from the definition of the investment functions (4.1). The first equation of block $\mathcal{R}$ is (4.2) rewritten in terms of variables (4.4) using (4.6) and (2.5), while the remaining equations are just the result of a "lag" operation. Notice that (4.6) reduces to (3.1) in the case of a single agent. Finally, the evolution of wealth shares described in block $\mathcal{W}$ is obtained from (4.3) once the notation introduced in (2.5) is explicitly expanded. Notice that, due to the presence of function R in the last expression, all functions Φ_n depend on the same set of variables as R. □

The rest of this Section is devoted to the analysis of the *deterministic skeleton* of (4.5): we replace the yield realizations $\{e_t\}$ by their mean value $\bar{e}$ and analyze the equilibria of the resulting deterministic system.

4.2 Determination of Equilibria

The characterization of fixed points of system (4.5) is in many respect similar to the single agent case discussed above. Let $\boldsymbol{x^*} = (x_1^*, \ldots, x_N^*; \varphi_1^*, \ldots, \varphi_{N-1}^*; r^*, \ldots, r^*)$ denotes a fixed point where r^* is the equilibrium return[8] and x_n^* and φ_n^* stay for the equilibrium value of the investment function and the equilibrium wealth share of agent n, respectively. Let us introduce the following

Definition 4.1. Agent n is said to *"survive"* in $\boldsymbol{x^*}$ if his equilibrium wealth share is strictly positive, $\varphi_n^* > 0$. Agent n is said to *"dominate"* agent n' in $\boldsymbol{x^*}$ if $\varphi_{n'}^*/\varphi_n^* = 0$. An agent n who dominates, at equilibrium, any other agent $n' \neq n$ is said to *"dominate"* the economy.

One can recognize the parallel between our definition above and the framework developed in DeLong et al. (1991). Indeed, we adopt here the deterministic version of the concepts of survivance and dominance used in that paper. The following statement characterizes all possible equilibria of system (4.5).

Proposition 4.1. *Let $\boldsymbol{x^*}$ be a fixed point of the deterministic skeleton of system* (4.5). *Two mutually exclusive cases are possible:*

(i) ***Single agent survival.*** *In $\boldsymbol{x^*}$ only one agent survives and, therefore, dominates the economy. Without loss of generality we can assume this agent to be agent* 1 *so that for the equilibrium wealth shares one has*

$$\varphi_n^* = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases} \quad . \tag{4.8}$$

Equilibrium return r^* is determined as the solution of

$$l(r^*) = f_1(r^*, \ldots, r^*) \quad , \tag{4.9}$$

while the equilibrium investment shares are defined according to

$$x_n^* = f_n(r^*, \ldots, r^*) \qquad \forall n \in \{1, \ldots, N\}. \tag{4.10}$$

The wealth growth rate of the survivor at equilibrium is given by

$$\rho_1^* = x_1^* \left(r^* + \bar{e}\right) = r^* \tag{4.11}$$

end is equal to the equilibrium price return.

(ii) ***Many agents survival.*** *In $\boldsymbol{x^*}$ more than one agent survives. Without loss of generality one can assume that the agents with non-zero wealth shares are the first k agents (with $k > 1$) so that the equilibrium wealth shares satisfy*

$$\varphi_n^* = \begin{cases} 1 & \text{if } n \leq k \\ 0 & \text{if } n > k \end{cases} \quad , \qquad \sum_{n=1}^{k} \varphi_n^* = 1 \quad . \tag{4.12}$$

[8]Remember that return in equilibrium r^* cannot be equal to $-\bar{e}$ as mentioned in Section 3.2.

The first k agents possess the same investment share $x^*_{1\diamond k}$ at equilibrium

$$x^*_1 = x^*_2 = \cdots = x^*_k = x_{1\diamond k} \quad . \tag{4.13}$$

and equilibrium return r^* must simultaneously satisfy the following set of k equations

$$l(r^*) = f_n(r^*, \ldots, r^*) = x^*_{1\diamond k} \quad \forall n \in \{1, \ldots, k\} \quad . \tag{4.14}$$

The equilibrium investment shares of the last $N-k$ agents are defined according to

$$x^*_n = f_n(r^*, \ldots, r^*) \quad \forall n \in \{k+1, \ldots, N\} \quad . \tag{4.15}$$

The wealth growth rate of the survivors at equilibrium is given by

$$\rho^*_n = x^*_{1\diamond k}\,(r^* + \bar{e}) = r^*\;, \quad \forall n \in \{1, \ldots, k\} \tag{4.16}$$

end is equal to the equilibrium price return.

Proof. From block $\mathcal{X}$ one immediately has (4.10) and (4.15). From block $\mathcal{W}$ using (4.7) and the condition $r^* + \bar{e} \neq 0$ one obtains

$$\varphi^*_n = 0 \quad \text{or} \quad \sum\nolimits_{m=1}^{N-1} \varphi^*_m\, x^*_m + \left(1 - \sum\nolimits_{m=1}^{N-1} \varphi^*_m\right) x^*_N = x^*_n \quad \forall n \in \{1, \ldots, N-1\} \quad . \tag{4.17}$$

Finally, from the first row of block $\mathcal{R}$ it is

$$r^* = \bar{e}\,\frac{\sum_{n=1}^{N-1} \varphi^*_n {x^*_n}^2 + \left(1 - \sum_{n=1}^{N-1} \varphi^*_n\right) {x^*_N}^2}{\sum_{n=1}^{N-1} \varphi^*_n\, x^*_n\,(1 - x^*_n) + \left(1 - \sum_{n=1}^{N-1} \varphi^*_n\right) x^*_N\,(1 - x^*_N)}\;. \tag{4.18}$$

The previous set of equations admits two types of solutions, depending on how many equilibrium wealth shares are different from zero: if one or many.

To derive the first type of solutions assume (4.8). In this case (4.17) is satisfied for all agents. From (4.18) one has $x^*_1 = r^*/(\bar{e} + r^*)$ which together with (4.10) leads to (4.9).

To derive the second type of solutions assume (4.12). In this case, the second equality of (4.17) must be satisfied for any $n \leq k$. Since its left-hand side does not depend on n, a $x^*_{1\diamond k}$ must exist such that $x^*_1 = \cdots = x^*_k = x^*_{1\diamond k}$. Substituting $x^*_n = 0$ for $n > k$ and $x^*_n = x^*_{1\diamond k}$ for $n \leq k$ in (4.18) one gets $x^*_{1\diamond k} = r^*/(\bar{e} + r^*)$. The equilibrium return r^* is implicitly defined combining this last relation with (4.15) for $n \leq k$.

The equilibrium wealth growth rate of the survivors is immediately obtained from (2.8) and from (4.10) or (4.15) for the single survivor and the many survivors case, respectively.

□

Strictly speaking, item *(i)* of the previous Proposition can be seen as a particular case of item *(ii)*. Nevertheless, the nature of the two situations is deeply different. In the first case, when a single agent survives, Proposition 4.1 defines a precise value for each component (x^*, φ^* and r^*) of the equilibrium $\boldsymbol{x^*}$, so that a single point is uniquely determined. In the second case, on the contrary, there is a residual degree of freedom in the definition of the equilibrium: while r^* and investment shares x^*'s are uniquely defined, the only requirement on the equilibrium wealth shares of the surviving agents is the fulfillment of the second equality in (4.12). Consequently, item *(ii)* does not define a single equilibrium point but an equilibria hyperplane in the parameter space. The particular fixed point eventually chosen by the system will depend on the initial conditions. In the next Section we will see that the partially indeterminate nature of the equilibria in the case of many survivors will have a major effect also on their stability.

The differences among the two cases of Proposition 4.1 does not only regard the geometrical nature of the *locus* of equilibria. Indeed, while in the first case no requirements are imposed on the behavior of the investment function of the different agents, in the second type of solutions all the investment shares $x_1^*, \ldots, x_k^*$ must *at the same time* be equal to a single value $x_{1\diamond k}^*$. The equilibrium with $k > 1$ survivors exists only in the particular case in which the k investment functions $f_1, \ldots, f_k$ satisfy this restriction. Consequently, an economy composed by N agents having generic, so to speak "randomly defined", investment functions, has probability zero of displaying any equilibrium with multiple survivors. In other terms, the many survivors equilibria are non-generic.

Both types of multi-agent equilibria derived in Proposition 4.1 are strictly related to "special" single-agent equilibria. As in the single agent case, the growth rate of the total wealth is equal to the equilibrium price return and is determined by the growth rate of those agents who survive in the equilibrium. Moreover, the determination of the equilibrium return level r^* for the multi-agent case in (4.9) or (4.14) is identical to the case where the agent, or one of the agents, who would survive in the multi-agent equilibrium, is present alone in the market. An useful consequence of this fact is that the geometrical interpretation of market equilibria presented in Section 3.2 can be extended to illustrate how equilibria with many agents are determined. As an example consider Fig. 1 and suppose that the two investment functions shown there belong to two agents who are simultaneously operating on the market. According to Proposition 4.1 all possible equilibria can be found as intersections of one of the functions with the Equilibria Market Line (c.f. (4.9) and (4.14)). In this example there are four possible equilibria. In two of them (S_1 and U_1) the first agent, with non-linear investment function, survives such that $\varphi_1^* = 1$ (and obviously $\varphi_2^* = 0$). In the other two equilibria (S_2 and U_2) is the second agent, with linear investment function, who survives so that, in these points, $\varphi_1^* = 0$. In each equilibrium, the intersection of the investment function of the surviving agent with the Equilibria Market Line gives both equilibrium return r^* and the equilibrium investment share of the survivor. The equilibrium investment share of the other agent can be found, accordingly to (4.15), as the intersection of his own investment function with the vertical line passing through the equilibrium return. Since two investment functions shown in Fig. 1 do not possess common intersections with the EML, the equilibria with more than one survivors are impossible. Two examples of investment functions which allow for multiple survivors equilibria are reported in Fig. 4. The common intersection of different investment functions with the equilibria market line define the multiple survivors equilibria.

4.3 Stability Conditions of Equilibria

This Section presents two propositions relevant for the stability analysis of the equilibria defined in Proposition 4.1. The first Proposition provides the stability region in the parameters space for the generic case of one single survivor. The non-generic case of many survivors is addressed in the second Proposition, where the destabilizing effect of the existence of an entire hyperplane of equilibria is revealed. Since the proofs of these Propositions require quite cumbersome algebraic manipulations, we provide below only their statements and refer the reader to Appendix D for the intermediate Lemmas and the final proofs. The discussion concerning economic interpretation of these Propositions and analysis of their consequences for the aggregate behavior of the system are postponed to the next Section.

For the generic case of a single survivor equilibrium we have the following

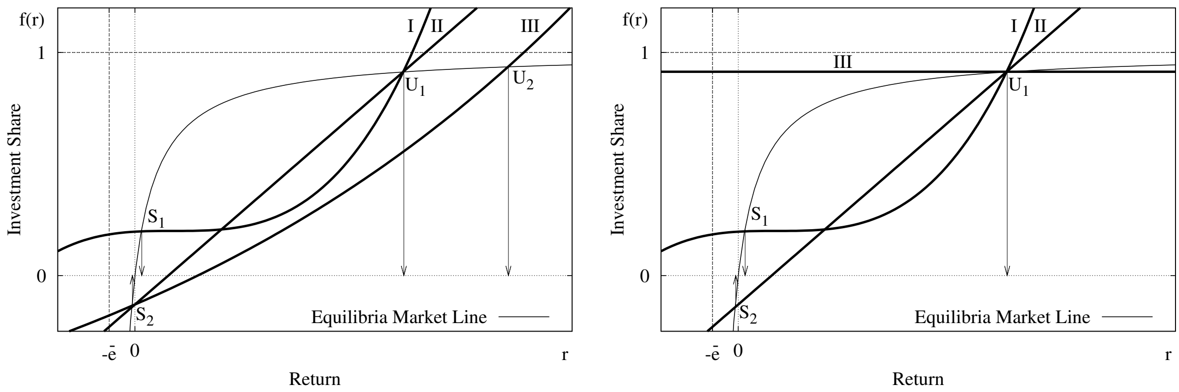

Figure 4: Non-generic situations with 3 agents operating on the market. **Left panel**: In equilibria S_2 and U_1 two agents survive. **Right panel**: In equilibrium U_1 all three agents survive.

Proposition 4.2. *Let $\boldsymbol{x}^*$ be a fixed point of* (4.5) *associated with a single survivor equilibrium. Without loss of generality we can assume that the survivor is the first agent, so that*

$$\varphi_n^* = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases} \quad .$$

Denote with $P_{f_1}(\mu)$ the $(L-1)$-dimensional stability polynomial associated with the investment function of the first agent f_1. With the above hypothesis, point $\boldsymbol{x}^*$ is (locally) asymptotically stable if the two following conditions are met:

1) *all the roots of polynomial*

$$Q_1(\mu) = \mu^{L+1} - \frac{(1+r^*)\,\mu - 1}{r^*\, l'(r^*)}\, P_{f_1}(\mu) \quad , \tag{4.19}$$

are inside the unit circle.

2) *the equilibrium investment shares of the non-surviving agents satisfy the following relations*

$$-2 - r^* < x_n^* \left(r^* + \bar{e}\right) < r^* \,, \quad 1 < n \leq N \,. \tag{4.20}$$

The equilibrium $\boldsymbol{x}^*$ is unstable if at least one of the roots of polynomial in (4.19) *is outside the unit circle or if at least one of the inequalities in* (4.20) *holds with the opposite (strict) sign.*

Thus, the stability condition for a generic fixed point in the multi-agent economies is twofold. First, comparison between $Q_1(\mu)$ and polynomial $Q(\mu)$ from (3.9) implies that equilibrium should be "self-consistent", i.e. remain stable even if any non-surviving agent would be removed from the economy. This is however not enough. A further requirement comes from the two inequalities in (4.20). In particular, according to the second inequality, the wealth growth rate of those agents who do not survive in the stable equilibrium should be strictly less than the wealth growth rate of the survivors r^*. In those equilibria where $r^* > -\bar{e}$ the surviving agent must be the most aggressive and invest a higher wealth share in the risky asset. On the other hand, in those equilibria where $r^* < -\bar{e}$ the survivor has to be the least aggressive.

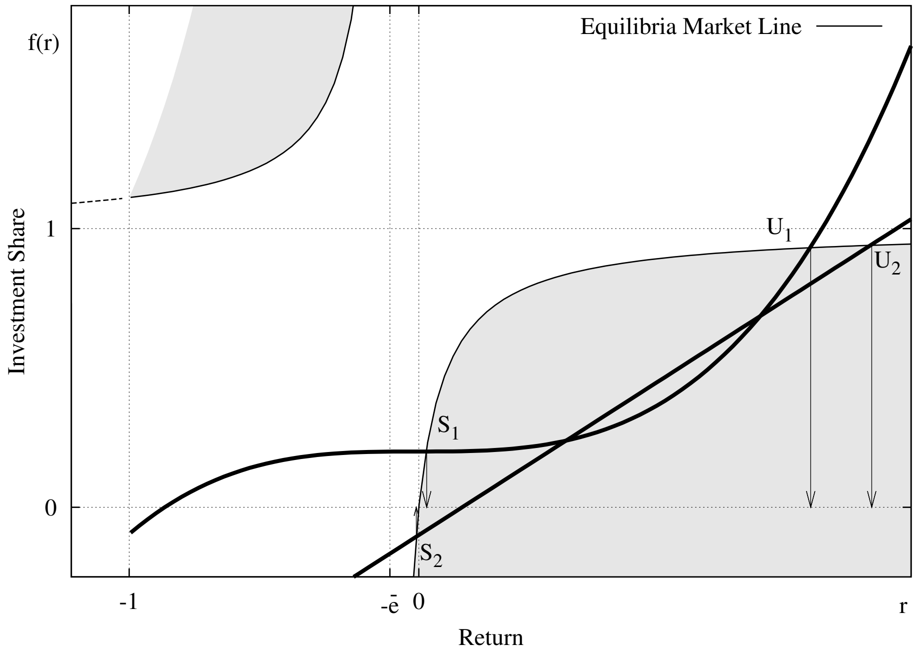

Figure 5: Four equilibria in the market with two agents. The region where condition (4.20) is satisfied is shown gray.

The EML "plot" can be used to obtain a geometrical illustration of the previous Proposition. In Fig. 5 we draw again the two investment functions discussed in Section 3. Let us now suppose that they are both present on the market at the same time. The region where the additional condition (4.20) is satisfied is reported in gray. In Section 4.2 we found four possible equilibria: S_1, S_2, U_1 and U_2. First notice that the dynamics cannot be attracted by U_1 or U_2. Since these equilibria were unstable in the respective single-agent cases, they cannot be stable when both agents are present in the market. Assume that S_1 and S_2 are stable equilibria when the first and second function, respectively, are present alone in the market. Then, from Proposition 4.2, it follows that S_1 is the only stable equilibrium of the system with two agents. Notice, indeed, that in the abscissa of S_1, i.e. for the equilibrium return, the linear investment function of the non-surviving agent passes below the investment function of the surviving agent and belongs to the gray area. On the contrary, in the abscissa of S_2, the investment function of the non-surviving agent has greater value and does not belong to the gray area. Consequently, this equilibrium is unstable.

Let us move now to consider the non-generic case, when k different agents survive in the equilibrium. The following applies

Proposition 4.3. *A fixed point* $\boldsymbol{x}^*$ *of* (4.5) *belonging to a* $k-1$*-dimensional manifold of* k*-survivors equilibria defined by* (4.12),(4.14) *and* (4.15) *is never hyperbolic.*

The non-hyperbolic submanifold is the $k-1$*-dimensional hyperplane generated by the following eigenvectors*

$$\boldsymbol{u}_n = \Big(\underbrace{0,\ldots,0}_{N};\underbrace{0,\ldots,0,1,0,\ldots,0}_{k-1},-1,\underbrace{0,\ldots,0}_{N-k};\underbrace{0,\ldots,0}_{L}\Big), \quad n \in \{N+1,\ldots,N+k-1\}$$

with 1 *in the* n*'th place,* -1 *in the* $N+k$*-th place and* 0 *elsewhere. These vectors correspond to a change in the relative wealths of the survivors.*

Let $P_{f_n}(\mu)$ be the stability polynomial of investment function f_n. The equilibrium $\boldsymbol{x}^*$ is stable with respect to perturbation orthogonal to the non-hyperbolic manifold if the two following conditions are met:

1) *all the roots of polynomial*

$$Q_{1\diamond k}(\mu) = \mu^{L+1} - \frac{(1+r^*)\,\mu - 1}{r^*\, l'(r^*)} \sum_{n=1}^{k} \varphi_n^*\, P_{f_n}(\mu) \quad , \tag{4.21}$$

are inside the unit circle.

2) *the equilibrium investment shares of the non-surviving agents satisfy to the following relations*

$$-2 - r^* < x_n^*\,(r^* + \bar{e}) < r^* \ , \quad k < n \leq N \ . \tag{4.22}$$

The orthogonal perturbations are unstable if at least one of the roots of polynomial in (4.21) *is outside the unit circle or if at least one of the inequalities in* (4.22) *holds with the opposite (strict) sign.*

The non-hyperbolic nature of the equilibria with many survivors turns out to be a direct consequence of their non-unique specifications. The motion of the system along the $k-1$ dimensional subspace consisting of the continuum of equilibria leaves the aggregate properties of the system invariant so that all these equilibria can be considered equivalent. Proposition 4.3 also provides the stability conditions for perturbations in the hyperplane orthogonal to the non-hyperbolic manifold formed by equivalent equilibria. The polynomial $Q_{1\diamond k}(\mu)$ is quite similar to the corresponding polynomial in Proposition 4.2, except that one has to weight the stability polynomial of the different investment functions $P_{f_k}(\mu)$ with the weights corresponding to the relative wealth of survivors in the equilibrium. At the same time, the constraint on the investment shares in (4.22) is identical to the one obtained in (4.20). In particular, similar to the case with one survivor, in those equilibria where $r^* > -\bar{e}$ all surviving agents must be more aggressive than those who do not survive. In those equilibria where $r^* < -\bar{e}$ the investment behaviors of survivors have to be less aggressive than the behavior of any non-surviving trader.

4.4 Market Selection and Asymptotic Dominance

In this Section, using the geometric interpretation based on EML "plot", we try to understand some relevant implications of Proposition 4.2 about the asymptotic behavior of the model and its global properties. The following discussion is confined to the generic case of equilibria with a single surviving trader.

The first implication concerns the aggregate dynamics of the economy. Let us consider a stable many-agent equilibrium $\boldsymbol{x}^*$. Let us suppose that r^* is the equilibrium return in $\boldsymbol{x}^*$ and that the first agent is dominating. Then his wealth return is equal to $\rho_1^* = r^*$ and this is also the asymptotic growth rate of the *total* wealth. Then, we can interpret the second requirement of Proposition 4.2 as saying that, in the dynamical competition, those agent survives who allows the economy to grow with the highest possible rate. Indeed, if any other agent $n \neq 1$ survived, the economy would have grown with a rate $x_n^*\,(r^* + \bar{e})$, which, since $x_n^* < x_1^*$, is lower than ρ_1^*. To put the same statement in negative terms, the economy will never end up in equilibria where its growth rate is lower than what it would be if the survivor were substituted by some other agent. One could call this result an *optimal selection principle* since it clearly states the market endogenous selection towards the best aggregate outcome.

Notice, however, that this selection does not apply to the whole set of equilibria, but only to the subset formed by equilibria associated with stable fixed points in the single agent case (c.f. (4.21)). For instance, with the investment functions shown in Fig. 5, the dynamics will never end up in U_2, even if this is the equilibrium with the highest possible return. But at least inside this restricted subset of equilibria that would be stable in the single agent case, is the selection of market optimal? Does the market choose the equilibrium with the highest growth rate, among all *stable single-agent* equilibria? If one looks at Fig. 5, the answer seems affirmative. In that case the market prefers the stable equilibrium S_2 to the unstable S_1. However, even if this "quasi-optimal" selection may be realized for some particular set of investment functions (like the ones in Fig. 5 and those considered in Chiarella and He (2001)), it does not apply in general. A simple counter example is provided by a single investment function possessing *multiple stable equilibria*. Consider for instance the nonlinear function in the left panel of Fig. 6. For this investment function both S_L and S_H are stable. Now suppose that an agent possessing this function competes on the market with other agents which are more risk averse than him and always invest smaller shares of wealth in the risky asset. An example of more risk averse behavior is provided by the linear investment function in the same plot. In this situation, the two equilibria of the nonlinear function remain stable and the riskier agent will ultimately dominate the market. But which equilibria will the market select? It only depends on the initial condition. There are no guarantee that the market will and up in S_H, the highest return equilibrium. Then, the quasi-optimal selection principle in the sense of Chiarella and He (2001) is violated.

The existence of multiple equilibria also leads to a second interesting implication of Proposition 4.2, the fact that the dominance of one investment behavior on another is a *local* property and, consequently, depends on the initial conditions. Consider again the investment function in the left panel of Fig. 6 and add a second agent with constant investment function, to obtain the situation shown in the right panel of Fig. 6. The entry of the new agent changes the possible equilibria, which become the points S and S_H. Notice, however, that in these two equilibria different agents dominate the market. If the market before the entry of the new agent was in S_H, the first agent still remains the more aggressive, and the entry of the new agent does not affect his dominant position. On the other hand, if the equilibrium before the entry was S_L, this equilibrium becomes unstable and the system will tend to move away from it. The ensuing dynamic could ultimately choose the investment function of the new entrant as the dominant one. This simple example suggests that, at least inside our framework, the definition of a dominance order relation on the space of trading strategies is impossible.

5 Conclusion

This paper extends the analysis presented in Chiarella and He (2001) and introduces novel results concerning the characterization and stability of equilibria in speculative pure exchange economies with heterogeneous traders.

Let us shortly review the assumptions we made and our achievements in order to sketch the possible future lines of research. We considered a simple analytical framework using a minimal number of assumptions (2 assets and Walrasian price formation). We modeled agents as speculative traders and we imposed the constraint that their participation to the trading activity is described by an individual demand function proportional to their wealth. Moreover, we assumed that agents form their individual demand decisions on the predictions about future price returns obtained from the publicly available past prices history. With

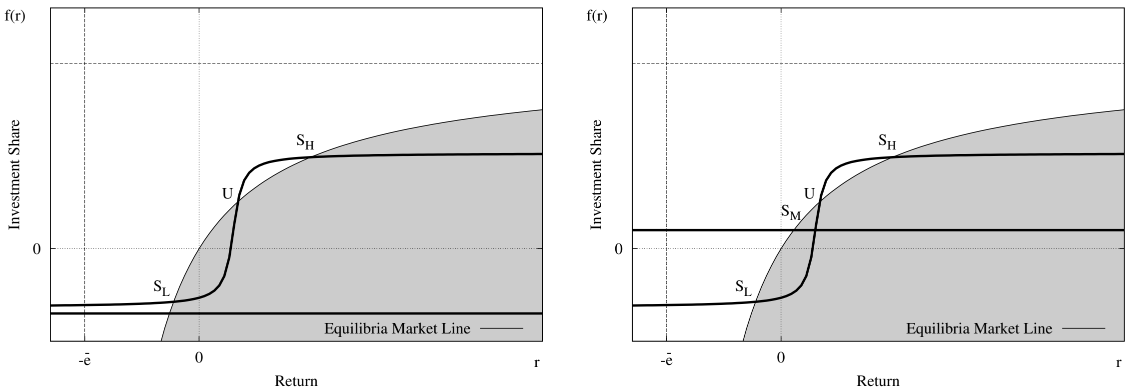

Figure 6: **Left Panel**: Non-linear investment function leading to multiple equilibria. S_H and S_L are stable while U is unstable. A second linear investment function always lays below the first. **Right Panel**: The linear investment function is raised above S_L. As a result, market now possesses a new stable equilibria, S_M, while S_L looses its stability. In the stable equilibria S_H and S_M two different agents survive.

prescribed but arbitrary specification about the agents' behavior, the feasible dynamics of the economy (i.e. the dynamics for which prices stay always positive) can then be described as a multi-dimensional dynamical system.

In such framework we started with a single agent case and presented the fixed point stability analysis of the corresponding system. Then we moved to the general framework with an arbitrary number of agents and showed that the conditions for the existence of fixed points and the conditions for their stability are related to the corresponding conditions in the situation with one single agent. We found that different scenarios are possible: in the generic case, the system possesses isolated equilibria where one single trader dominates the others and ultimately captures the entire market. Alternatively, in the non-generic case in which traders investment functions satisfy a special set of constraints, the system can possesses a continuous manifold of equilibria associated with non-hyperbolic fixed points. In these non-generic equilibria many agents possess a finite amount of the total wealth of the economy.

The present analysis can be extended in many directions. First of all, even if we proved that the existence of multiple equilibria is possible, the dynamics in this case remains to be unveiled. Probably numerical methods can be effectively applied to clarify the role of initial conditions, the determinants of the relative size of the attraction domains for different equilibria, etc. These methods can be also used to study the dynamics in the cases when there are no stable equilibria.

Second, one may ask what are the consequences of the optimal selection principle for a market in which the set of strategies is not "frozen", but instead is evolving in time, plausibly following some adaptive process. For instance, one can assume that agents imitate the behavior of other traders (see e.g. Kirman (1991)) or that they update strategies according to recent relative performances (see e.g. Brock and Hommes (1998)). In such cases, those situations which we referred as "non-generic" above may become, instead, typical. Proposition 4.3 can be considered only a first step in the analysis of such situations.

Third, inside our general framework, numerous different specifications of the traders strategies are possible, in addition to the ones analyzed here. They range from the evaluation of the "fundamental" value of the asset, possibly obtained from a private source of information, to a strategic behavior that try to keep in consideration the reaction of other market participants to the revealed individual choices. The analysis of the consequences of the introduction of such strategies on the optimal selection principle may, ultimately, refute the statement about the impossibility of defining a dominance relation among strategies.

APPENDIX

A Proof of Proposition 2.1

Plugging the expression for $w_{t,n}$ from the second equation in (2.4) into the right hand-side of the first equation, assuming $p_{t-1} > 0$ and, consistently with (2.6), $p_{t-1} \neq \sum x_{t,n}\, x_{t-1,n}\, w_{t-1,n}$, one obtains

$$p_t = \left(1 - \frac{1}{p_{t-1}} \sum x_{t,n}\, x_{t-1,n}\, w_{t-1,n}\right)^{-1} \left(\sum x_{t,n}\, w_{t-1,n} + (e_{t-1} - 1) \sum x_{t,n}\, w_{t-1,n}\, x_{t-1,n}\right) =$$

$$= p_{t-1} \frac{\sum x_{t,n}\, w_{t-1,n} + (e_{t-1} - 1) \sum x_{t,n}\, w_{t-1,n}\, x_{t-1,n}}{\sum x_{t-1,n}\, w_{t-1,n} - \sum x_{t,n}\, x_{t-1,n}\, w_{t-1,n}} =$$

$$= p_{t-1} \frac{\langle x_t \rangle_{t-1} - \langle x_{t-1}\, x_t \rangle_{t-1} + e_{t-1} \langle x_{t-1}\, x_t \rangle_{t-1}}{\langle x_{t-1} \rangle_{t-1} - \langle x_{t-1}\, x_t \rangle_{t-1}} \quad ,$$

where the second equality comes from the first equation in (2.4) rewritten for time $t-1$. Condition (2.6) is immediately obtained imposing $p_t > 0$. Then the price return and wealth return for each agent n at time t can be derived straightforwardly.

B Proof of Proposition 3.2

The $(L+1) \times (L+1)$ Jacobian matrix $\boldsymbol{J}$ of system (3.2) reads

$$\left\| \begin{array}{ccccccc} 0 & \frac{\partial f}{\partial r_0} & \frac{\partial f}{\partial r_1} & \frac{\partial f}{\partial r_2} & \dots & \frac{\partial f}{\partial r_{L-2}} & \frac{\partial f}{\partial r_{L-1}} \\ R^x & R^f \frac{\partial f}{\partial r_0} & R^f \frac{\partial f}{\partial r_1} & R^f \frac{\partial f}{\partial r_2} & \dots & R^f \frac{\partial f}{\partial r_{L-2}} & R^f \frac{\partial f}{\partial r_{L-1}} \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{array} \right\| , \tag{B.1}$$

where

$$R^x = \frac{\partial R(x^*, x^*)}{\partial x} = -\frac{1}{x^*\,(1 - x^*)} , \qquad R^f = \frac{\partial R(x^*, x^*)}{\partial x'} = \frac{1 + r^*}{x^*\,(1 - x^*)} . \tag{B.2}$$

The stability condition of equilibrium are provided by the following

Lemma B.1. *The characteristic polynomial $P_J(\mu)$ of system* (3.2) *in the equilibrium $\boldsymbol{x}^*$ is*

$$P_J(\mu) = (-1)^{L-1}\left(\mu^{L+1} - \frac{(1+r^*)\mu - 1}{x^*(1-x^*)} P_f(\mu)\right) \tag{B.3}$$

where $P_f(\mu)$ denotes the stability polynomial of function f introduced in (3.7)*.*

Proof. Consider (B.1) and introduce $(L+1)\times(L+1)$ identity matrix $\boldsymbol{I}$. Expanding the determinant of $\boldsymbol{J} - \mu\,\boldsymbol{I}$ by the elements of the first column and using Lemma E.1 one has

$$\begin{aligned}
\det\big(\boldsymbol{J} - \mu\,\boldsymbol{I}\big) &= (-\mu)\,(-1)^{L-1}\left(\left(R^f\,\frac{\partial f}{\partial r_0} - \mu\right)\mu^{L-1} + R^f\,\frac{\partial f}{\partial r_1}\,\mu^{L-2} + \cdots + R^f\,\frac{\partial f}{\partial r_{L-1}}\right) - \\
&\quad - R^x\,(-1)^{L-1}\left(\frac{\partial f}{\partial r_0}\,\mu^{L-1} + \frac{\partial f}{\partial r_1}\,\mu^{L-2} + \cdots + \frac{\partial f}{\partial r_{L-2}}\,\mu + \frac{\partial f}{\partial r_{L-1}}\right) = \\
&= (-1)^{L-1}\left(\mu^{L+1} - \left(\mu R^f + R^x\right)\sum_{k=0}^{L-1}\frac{\partial f}{\partial r_k}\,\mu^{L-1-k}\right) \qquad ,
\end{aligned}$$

which, using relations in (B.2) and definition of stability polynomial in (3.7) reduces to (B.3). ☐

Using the relationship $l'(r^*) = x^*(1-x^*)/r^*$ it is immediate to see that, apart from irrelevant sign, (B.3) is identical to (3.9).

C Proof of Lemma 4.1

The expression for r_{t+1} in (4.2) is identical to the one given in (2.7). From (2.8) it is

$$w_{t+1,n} = w_{t,n}\,(1 + x_{t,n}\,(r_{t+1} + e_t))\ ,$$

and dividing both sides by the total wealth at time $t+1$ one gets

$$\begin{aligned}
\varphi_{t+1,n} &= \frac{w_{t,n}}{\sum_m w_{t+1,m}}\,(1 + x_{t,n}\,(r_{t+1} + e_t)) = \\
&= \frac{w_{t,n}}{\sum_m w_{t,m} + (r_{t+1} + e_t)\sum_m x_{t,m} w_{t,m}}\,(1 + x_{t,n}\,(r_{t+1} + e_t)) = \\
&= \frac{\varphi_{t,n}}{1 + (r_{t+1} + e_t)\sum_m x_{t,m}\varphi_{t,m}}\,(1 + x_{t,n}\,(r_{t+1} + e_t))\ .
\end{aligned}$$

D Proofs of Propositions 4.2 and 4.3

Before proving Propositions 4.2 and 4.3 we need some preliminary results. The Jacobian matrix of the deterministic skeleton of system (4.5) is a $\big(2N+L-1\big)\times\big(2N+L-1\big)$ matrix. Using the block structure introduced in Section 4.1 it is separated in nine blocks

$$\boldsymbol{J} = \begin{Vmatrix} \frac{\partial \mathcal{X}}{\partial \mathcal{X}} & \frac{\partial \mathcal{X}}{\partial \mathcal{W}} & \frac{\partial \mathcal{X}}{\partial \mathcal{R}} \\ \frac{\partial \mathcal{W}}{\partial \mathcal{X}} & \frac{\partial \mathcal{W}}{\partial \mathcal{W}} & \frac{\partial \mathcal{W}}{\partial \mathcal{R}} \\ \frac{\partial \mathcal{R}}{\partial \mathcal{X}} & \frac{\partial \mathcal{R}}{\partial \mathcal{W}} & \frac{\partial \mathcal{R}}{\partial \mathcal{R}} \end{Vmatrix}\ , \tag{D.1}$$

The block $\partial\mathcal{X}/\partial\mathcal{X}$ is a $N \times N$ matrix containing the partial derivatives of the agents' present investment choices with respect to the agents' past investment choices. According to (2.9) the investment choice of any agent does not explicitly depend on the investment choices in previous period and it is

$$\left[\frac{\partial \mathcal{X}}{\partial \mathcal{X}}\right]_{n,m} = \frac{\partial f_n}{\partial x_m} = 0 \;, \qquad 1 \leq n, m \leq N$$

and this block is a zero matrix.

The block $\partial\mathcal{X}/\partial\mathcal{W}$ is a $N \times (N-1)$ matrix containing the partial derivatives of the agents' investment choices with respect to the agents' wealth shares. According to (2.9) this is a zero matrix and

$$\left[\frac{\partial \mathcal{X}}{\partial \mathcal{W}}\right]_{n,m} = \frac{\partial f_n}{\partial \varphi_m} = 0 \;, \qquad 1 \leq n \leq N \;, \quad 1 \leq m \leq N-1 \;.$$

The block $\partial\mathcal{X}/\partial\mathcal{R}$ is a $N \times L$ matrix containing the partial derivatives of the agents' investment choices with respect to the past returns

$$\left[\frac{\partial \mathcal{X}}{\partial \mathcal{R}}\right]_{n,l} = \frac{\partial f_n}{\partial r_{l-1}} = f_n^{r_{l-1}} \;, \qquad 1 \leq n \leq N \;, \quad 1 \leq l \leq L$$

The block $\partial\mathcal{W}/\partial\mathcal{X}$ is $(N-1) \times N$ matrix containing the partial derivatives of the agents' wealth shares with respect to the agents' investment choices. It is

$$\left[\frac{\partial \mathcal{W}}{\partial \mathcal{X}}\right]_{n,m} = \frac{\partial \varphi_n}{\partial x_m} = \Phi_n^{x_m} + \Phi_n^{R} \cdot R^{x_m} \;, \qquad 1 \leq n \leq N-1 \;, \quad 1 \leq m \leq N \tag{D.2}$$

where $R^{x_m} = \partial R/\partial x_m$, $\Phi_n^{x_m} = \partial \Phi_n/\partial x_m$ and $\Phi_n^R = \partial \Phi_n/\partial R$.

The block $\partial\mathcal{W}/\partial\mathcal{W}$ is a $(N-1) \times (N-1)$ matrix containing the partial derivatives of the agents' wealth shares with respect to the agents' wealth shares. It is

$$\left[\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\right]_{n,m} = \frac{\partial \varphi_n}{\partial \varphi_m} = \Phi_n^{\varphi_m} + \Phi_n^{R} \cdot R^{\varphi_m} \;, \qquad 1 \leq n, m \leq N-1 \;, \tag{D.3}$$

where $\Phi_n^{\varphi_m} = \partial \Phi_n/\partial \varphi_m$, $R^{\varphi_m} = \partial R/\partial \varphi_m$.

The block $\partial\mathcal{W}/\partial\mathcal{R}$ is a $(N-1) \times L$ matrix containing the partial derivatives of the agents' wealth share with respect to lagged returns. It is

$$\left[\frac{\partial \mathcal{W}}{\partial \mathcal{R}}\right]_{n,l} = \frac{\partial \varphi_n}{\partial r_{l-1}} = \Phi_n^{R} \cdot \sum_{m=1}^{N} R^{f_m} f_m^{r_{l-1}} \;, \quad 1 \leq n \leq N-1 \;, \quad 1 \leq l \leq L \;, \tag{D.4}$$

where $R^{f_n} = \partial R/\partial y_n$.

The block $\partial\mathcal{R}/\partial\mathcal{X}$ is the $L \times N$ matrix containing the partial derivatives of the lagged returns with respect to the agents' investment choices. Its structure is simple and reads

$$\left[\frac{\partial \mathcal{R}}{\partial \mathcal{X}}\right] = \begin{Vmatrix} R^{x_1} & R^{x_2} & \dots & R^{x_N} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{Vmatrix} \;.$$

The block $\partial\mathcal{R}/\partial\mathcal{W}$ is the $L \times (N-1)$ matrix containing the partial derivatives of the lagged returns with respect to the agents' wealth shares and reads

$$\left[\frac{\partial \mathcal{R}}{\partial \mathcal{W}}\right] = \begin{Vmatrix} R^{\varphi_1} & R^{\varphi_2} & \dots & R^{\varphi_{N-1}} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{Vmatrix} \;.$$

The block $\partial\mathcal{R}/\partial\mathcal{R}$ is the $L \times L$ matrix containing the partial derivatives of the lagged returns with respect to themselves

$$\left[\frac{\partial\mathcal{R}}{\partial\mathcal{R}}\right] = \left\| \begin{array}{cccc} R^{r_0} & R^{r_1} & \dots & R^{r_{L-1}} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{array} \right\| \quad ,$$

where

$$R^{r_l} = \sum_{m=1}^{N} R^{f_m} f_m^{r_l} \quad . \tag{D.5}$$

Lemma D.1. *Let $\boldsymbol{x}^*$ be an equilibrium of system* (4.5). *The Jacobian matrix computed in this point $\boldsymbol{J}(\boldsymbol{x}^*)$ has the following structure*

$$\left\| \begin{array}{cccccc|cccccc|cccc}
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & f_1^{r_0} & \dots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & f_N^{r_0} & \dots & f_N^{r_{L-2}} & f_N^{r_{L-1}} \\
\hline
\Phi_1^{x_1} & \dots & \Phi_1^{x_k} & 0 & \dots & 0 & \Phi_1^{\varphi_1} & \dots & \Phi_1^{\varphi_k} & \Phi_1^{\varphi_{k+1}} & \dots & \Phi_1^{\varphi_{N-1}} & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\Phi_k^{x_1} & \dots & \Phi_k^{x_k} & 0 & \dots & 0 & \Phi_k^{\varphi_1} & \dots & \Phi_k^{\varphi_k} & \Phi_k^{\varphi_{k+1}} & \dots & \Phi_k^{\varphi_{N-1}} & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \Phi_{k+1}^{\varphi_{k+1}} & \dots & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \Phi_{N-1}^{\varphi_{N-1}} & 0 & \dots & 0 & 0 \\
\hline
R^{x_1} & \dots & R^{x_k} & 0 & \dots & 0 & 0 & \dots & 0 & R^{\varphi_{k+1}} & \dots & R^{\varphi_{N-1}} & R^{r_0} & \dots & R^{r_{L-2}} & R^{r_{L-1}} \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0
\end{array} \right\|$$

Proof. At equilibrium it is

$$\begin{aligned}
\Phi_n^R &= \varphi_n^* \, \frac{x_n^* - x_{1\diamond k}^*}{1 + r^*} \quad , \\
\Phi_n^{x_m} &= \varphi_n^* \, (\delta_{n,m} - \varphi_m^*) \, \frac{\bar{e} + r^*}{1 + r^*} \quad , \\
\Phi_n^{\varphi_l} &= \frac{1}{1 + r^*} \Big(\delta_{n,l} \big(1 + x_n^*(r^* + \bar{e})\big) - \varphi_n^*(r^* + \bar{e})(x_l^* - x_N^*) \Big) \, ,
\end{aligned} \tag{D.6}$$

where $n, l \in \{1, \dots, N-1\}$, $m \in \{1, \dots, N\}$ and $\varphi_N^* = 1 - \sum_{j=1}^{N-1} \varphi_j^*$. Then from (4.12) and (4.13) it follows that $\Phi_n^R = 0$ for any agent n and

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{X}}\right]_{n,m} = \begin{cases} \Phi_n^{x_m} & m, n \leq k \\ 0 & \text{otherwise} \end{cases}$$

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{W}}\right]_{n,m} = \begin{cases} 0 & n > k \, , \quad n \neq m \\ \Phi_n^{\varphi_m} & \text{otherwise} \end{cases}$$

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{R}}\right]_{n,m} = 0 \, , \quad \forall n, m \, .$$

At equilibrium we also have for $m \in \{1, \dots, N\}$:

$$R^{x_m} = -\,\varphi^*_m \,\frac{1}{x^*_{1\diamond k}\,(1-x^*_{1\diamond k})}\,, \qquad R^{\varphi_m} = x^*_m\,(r^* + \bar{e})\,\frac{x^*_m - x^*_{1\diamond k}}{x^*_{1\diamond k}\,(1-x^*_{1\diamond k})}\,, \tag{D.7}$$

so that $R^{x_m} = 0$ for $m > k$ and $R^{\varphi_m} = 0$ for $m \le k$. The structure above immediately follows. $\square$

Lemma D.2. *The characteristic polynomial P_J of the matrix $\boldsymbol{J}(\boldsymbol{x}^*)$ can be reduced to the following form*

$$P_J(\mu) = (-1)^{N+L}\,\mu^{N-1}\,(1-\mu)^{k-1}\prod_{j=k+1}^{N}\left(\frac{1+x^*_j\,(r^*+\bar{e})}{1+r^*} - \mu\right) \left(\mu^{L+1} - \frac{(1+r^*)\mu - 1}{x^*_{1\diamond k}(1-x^*_{1\diamond k})}\sum_{j=1}^{k}\varphi^*_j\,P_{f_j}(\mu)\right) \tag{D.8}$$

where P_{f_n} is the stability polynomial associated to the n-th investment function as defined in (3.7).

Proof. The following proof is constructive: we will identify in succession the factors appearing in (D.8). At each step, a set of eigenvalues is found and the problem is reduced to the analysis of the residual matrix obtained removing the rows and columns associated with the relative eigenspace. In this way the dimension of the analyzed matrix is progressively reduced.

Consider the Jacobian matrix in Lemma D.1. The last $N-k$ columns of the left blocks contain only zero entries so that the matrix possesses eigenvalue 0 with (at least) multiplicity $N-k$. Moreover, in each of the last $N-1-k$ rows in the central blocks the only non-zero entries are on the diagonal. Consequently, $\Phi^{\varphi_j}_j$ for $k+1 \le j \le N-1$ are eigenvalues of the matrix, with multiplicity (at least) one. A first contribution to the characteristic polynomial is then determined as

$$(-\mu)^{N-k}\prod_{j=k+1}^{N-1}(\Phi^{\varphi_j}_j - \mu) = (-\mu)^{N-k}\prod_{j=k+1}^{N-1}\left(\frac{1+x^*_j(r^*+\bar{e})}{1+r^*} - \mu\right) \tag{D.9}$$

where we used (D.6) to compute $\Phi^{\varphi_j}_j$ at equilibrium.

In order to find the remaining part of the characteristic polynomial we eliminate the rows and column associated to the previous eigenvalues to obtain

$$\left\|\begin{array}{ccc|ccc|cccc}
0 & \dots & 0 & 0 & \dots & 0 & f^{r_0}_1 & \dots & f^{r_{L-2}}_1 & f^{r_{L-1}}_1 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & f^{r_0}_k & \dots & f^{r_{L-2}}_k & f^{r_{L-1}}_k \\
\hline
\Phi^{x_1}_1 & \dots & \Phi^{x_k}_1 & \Phi^{\varphi_1}_1 & \dots & \Phi^{\varphi_k}_1 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\Phi^{x_1}_k & \dots & \Phi^{x_k}_k & \Phi^{\varphi_1}_k & \dots & \Phi^{\varphi_k}_k & 0 & \dots & 0 & 0 \\
\hline
R^{x_1} & \dots & R^{x_k} & 0 & \dots & 0 & R^{r_0} & \dots & R^{r_{L-2}} & R^{r_{L-1}} \\
0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0
\end{array}\right\| \tag{D.10}$$

Consider the "central" block: since it has only zeros above and below, it is a "diagonal" block and its eigenvalues are at the same time eigenvalues of the whole matrix. To compute these eigenvalues, notice that from (D.6) it is

$$\Phi^{\varphi_m}_n = \begin{cases} 1 + \varphi^*_n\,v & \text{if } n \le k \text{ and } n = m \\ \varphi^*_n\,v & \text{if } n, m \le k \text{ and } n \ne m \end{cases}, \qquad \text{where} \qquad v = -\big(x^*_{1\diamond k} - x^*_N\big)\,\frac{\bar{e}+r^*}{1+r^*}\,.$$

so that the central block can be rewritten as

$$\|\boldsymbol{I_1} + v\boldsymbol{\varphi} \qquad \dots \qquad \boldsymbol{I_k} + v\boldsymbol{\varphi}\|$$

where $\boldsymbol{I_j}$ is the j-th column vector of the $k \times k$ identity matrix, and $\boldsymbol{\varphi} = (\varphi_1^*, \dots, \varphi_k^*)^t$ is the column vector of equilibrium market shares. Using the multilinear property of the determinant one has

$$|(1-\mu)\boldsymbol{I_1} + v\boldsymbol{\varphi} \quad \dots \quad (1-\mu)\boldsymbol{I_k} + v\boldsymbol{\varphi}| =$$
$$(1-\mu)^k \, |\boldsymbol{I}_1 \quad \dots \quad \boldsymbol{I}_k| + (1-\mu)^{k-1} \, v \sum_{j=1}^{k} |\boldsymbol{I}_1 \quad \dots \quad \boldsymbol{I}_{j-1} \quad \boldsymbol{\varphi} \quad \boldsymbol{I}_{j+1} \quad \dots \quad \boldsymbol{I}_k|$$

where zero determinant terms containing more than one $\boldsymbol{\varphi}$ column has been discarded. The first contribution on the right hand side is the identity matrix, while the matrices in the summation are identity matrices with one column replaced by $\boldsymbol{\varphi}$. Then the previous expression reduces to

$$(1-\mu)^k + (1-\mu)^{k-1} v \sum_{j=1}^{k} \varphi_j^* = (1-\mu)^{k-1}(1-\mu+v) = (1-\mu)^{k-1}\left(\frac{1+x_N^*(r^*+\bar{e})}{1+r^*} - \mu\right) . \quad \text{(D.11)}$$

Having identified the eigenstructure of the central block we can eliminate it so that the final factor of the characteristic polynomial can be found from the computation of the determinant of the matrix

$$\boldsymbol{M}(k, L-1) = \left|\begin{array}{ccc|cccc} -\mu & \dots & 0 & f_1^{r_0} & \dots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & -\mu & f_k^{r_0} & \dots & f_k^{r_{L-2}} & f_k^{r_{L-1}} \\ \hline R^{x_1} & \dots & R^{x_k} & R^{r_0} - \mu & \dots & R^{r_{L-2}} & R^{r_{L-1}} \\ 0 & \dots & 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 & -\mu \end{array}\right|$$

We compute this determinant in a recursive way. Consider the expansion by the determinant of the minors of the elements of the first column. The minor of the first element $-\mu$ is a matrix with a structure similar to $\boldsymbol{M}(k, L-1)$. This is the final matrix obtained if the first agents were not among the survivors. Let us denote its determinant with $\boldsymbol{M}(k-1, L-1)$. The minor associated with R^{x_1} has a left upper block with $k-1$ entries equal to $-\mu$ below the main diagonal. This block generates a contribution μ^{k-1} to the determinant and once its columns and rows are eliminated, one remains with a matrix of the type in (E.1). Applying Lemma E.1 one then has

$$\boldsymbol{M}(k, L-1) = (-\mu)\, \boldsymbol{M}(k-1, L-1) + (-1)^k R^{x_1} \mu^{k-1} \, (-1)^{L-1} \, P_{f_1}(\mu) \; ,$$

where P_{f_1} is the stability polynomial associated with the first investment function. Applying recursively the relation above, the dimension of the determinant is progressively reduced. At the end one remains with the lower right block of the original matrix, which is again a matrix similar to (E.1). Applying once more Lemma E.1 one has for $\boldsymbol{M}(k, L-1)$ the following

$$(-1)^{L-1+k} \, \mu^{k-1} \sum_{j=1}^{k} R^{x_j} \, P_{f_j}(\mu) + (-1)^{L-1+k} \, \mu^k \left(\sum_{j=0}^{L-1} R^{r_j} \mu^{L-1-j} - \mu^L\right) .$$

Using (D.7) for R^{x_j} in equilibrium, using (D.5) for R^{r_j} and also computing at equilibrium:

$$R^{f_n} = \varphi_n^* \, \frac{1+r^*}{x_{1\diamond k}^* \, (1 - x_{1\diamond k}^*)} \; ,$$

we can simplify the last expression and get

$$(-1)^{L-1+k}\,\mu^{k-1}\left(\frac{(1+r^*)\mu-1}{x^*_{1\diamond k}(1-x^*_{1\diamond k})}\sum_{j=1}^{k}\varphi^*_j\,P_{f_j}(\mu)-\mu^{L+1}\right)\quad . \tag{D.12}$$

Finally, the product of (D.9), (D.11) and (D.12) gives (D.8). □

Using the characteristic polynomial of the Jacobian matrix it is straightforward to derive the equilibrium stability conditions mentioned in Section 4.3.

Case of one survivor: Proof of Proposition 4.2

If $k=1$ the characteristic polynomial (D.8) reduces to

$$P_J(\mu)=(-1)^{N+L}\,\mu^{N-1}\prod_{j=2}^{N}\left(\frac{1+x^*_j\,(r^*+\bar{e})}{1+r^*}-\mu\right)\left(\mu^{L+1}-\frac{(1+r^*)\mu-1}{x^*_1(1-x^*_1)}\,P_{f_1}(\mu)\right)\ .$$

¿From the expression of the derivative of the EML at equilibrium $l'(r^*)$ one can see that last factor corresponds to the polynomial Q_1 in (4.19). The conditions in (4.20) are derived from the requirement

$$\left|\frac{1+x^*_j\,(r^*+\bar{e})}{1+r^*}\right|<1\qquad j>1\ ,$$

and the Proposition is proved.

Case of many survivors: Proof of Proposition 4.3

In the case of $k>1$ survivors the characteristic polynomial in (D.8) possesses a unit root with multiplicity $k-1$. Consequently, the fixed point is non-hyperbolic. The eigenspace associated to eigenvector 1 is a subspace of the central block in (D.10). One can see by direct computation that the $k-1$ linearly independent vectors $\boldsymbol{u}_n$ defined in the Proposition form a base of this space. Since this space does not depend on system parameters, it is immediate to realize that it does constitute not only the tangent space to the non-hyperbolic manifold, but the manifold itself. The polynomial $Q_{1\diamond k}(\mu)$ in (4.21) is the last factor in (D.8) while conditions (4.22) are obtained by imposing

$$\left|\frac{1+x^*_j\,(r^*+\bar{e})}{1+r^*}\right|<1\qquad j>k+1\ .$$

E Determinant of auxiliary matrix

The following is useful for the stability analysis of the different systems considered in this paper

Lemma E.1.

$$\begin{vmatrix} x_1 & x_2 & x_3 & \dots & x_{n-1} & x_n \\ 1 & -\mu & 0 & \dots & 0 & 0 \\ 0 & 1 & -\mu & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\mu & 0 \\ 0 & 0 & 0 & \dots & 1 & -\mu \end{vmatrix} = (-1)^{n-1}\sum_{k=1}^{n} x_k\,\mu^{n-k}\quad , \tag{E.1}$$

Proof. Consider this determinant as a sum of elements from the first row multiplied on the corresponding minor. The minor of element x_k is a block-diagonal matrix consisting of two blocks. The upper-left block is an upper-diagonal matrix with 1's on the diagonal. The lower-right block is a lower-diagonal matrix with $-\mu$'s on the diagonal. The determinant of this minor is equal to $(-\mu)^{n-1-k}$ and the relation above immediately follows. $\square$

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