Endogenous Dynamics of Firms and Labor with Large Numbers of Simple Agents

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ENDOGENOUS DYNAMICS OF FIRMS AND LABOR WITH LARGE NUMBERS OF SIMPLE AGENTS

ABSTRACT

A model is described in which simple autonomous agents organize into teams that empirically resemble U.S. firms. The agents work in team production environments, regularly adjust their work effort, and periodically seek better jobs or start new teams when it is in their self-interest. Nash equilibria of the team formation game exist but are unstable. Dynamics are studied using agent computing at full-scale with the U.S. private sector (120 million agents). Stationary distributions arise at the aggregate level despite perpetual adaptation by individuals at the micro level. Such agent adjustments occur for microeconomic reasons without resort to external shocks.

Keywords: firm formation, team production, bounded rationality, unstable Nash equilibria, job tenure distribution, firm size distributions, agent-based computational model, non-equilibrium microeconomics, path-dependence, economic complexity

JEL classification codes: C63, C73, D23, L11, L22

Human beings, viewed as behaving systems, are quite simple. The apparent complexity of our behavior over time is largely a reflection of the complexity of the environment in which we find ourselves. Herbert Simon (1996)

1 Introduction

Over the last decade the annual size of the U.S. private sector work force has fluctuated between 108-116 million employees according to the Bureau of Labor Statistics, 110-120 million from the Census Department. Of this number, 3-4 million change employers each month (Fallick and Fleischman 2001). Over this same period of time there have existed, in any specific year, from 5.7-6.0 million firms with employees. Of this number, roughly 100 thousand go out of business monthly, on average, while a comparable number start up (Fairlie 2012). The turnover in the population of firms indicates significant economic dynamism, associated as it is with entrepreneurship and innovation. Such high levels of monthly turnover in the U.S. economy—1 in 30 to 1 in 40 workers changing employers, 1 in 60 firms terminating its operations—clearly demonstrate a kind of *perpetual flux* at the local economic level, despite what was relatively steady growth at the aggregate level during this period. These high levels of change at the microeconomic level are conventionally interpreted to represent the reallocation of resources to more productive uses, increasing the efficiency of the economy. So how are we best to understand such persistent adjustments and reorganizations? Do they result from technological and related productivity shocks, exogenous to the economy, or are they the direct result of economic decision-making and agent interactions? If we assume that the microeconomy is in general equilibrium then there is no way to get micro-dynamics except by the imposition of external shocks. Alternatively, a microeconomic model that is sufficiently rich that it never settles down to equilibrium at the agent level could, perhaps, produce the kind of micro-dynamics observed empirically. The quest for such a microeconomic specification is the main goal of this paper.

The primary goal of this paper is to build a microeconomic model capable of *endogenously* producing firm and labor dynamics of the size and type experienced by the U.S. economy. In addition to the nearly 3 million people who change jobs in the U.S. each month, nearly 2 million people separate from their employers each month without new jobs, becoming unemployed, while a comparable number move off unemployment into new jobs; another million people either leave the workforce for a spell or else begin a job after being out of the workforce. Overall, this is nearly 8 million labor market events per month at steady-state (Fallick and Fleischman 2001) and these are just the inter-firm ones. These highly variable flows are, of course, affected by the business cycle. Many of the vacancies created by such job-to-job flows are filled by intra-firm job changes, about which there is precious little data. Conservatively, perhaps 10 million distinct separation and hiring events occur each month, involving as much as 8% of the 120 million people in the private sector workforce. Clearly, over the course of a year there is enormous turnover in the matching of people to jobs in the U.S. This paper provides a microeconomic explanation for many of these large flows.

Over the past decade, driven largely by advances in IT, there have appeared increasing amounts of micro-data on U.S. businesses. The model described below reproduces many important features of the empirical data: firm size, age and growth rate distributions, including joint and conditional distributions involving these variables, distributions of job tenure and wages across agents, certain network properties, as well as other quantities. For most of these data the best explanations today are largely phenomenological in nature, with little economic content. Concerning firm sizes, for example, from the early work of Gibrat (1931) and continuing in the efforts of Simon and co-workers (Simon 1955, Simon and Bonini 1958), stochastic growth models have been shown to yield skew firm sizes, following lognormal, Pareto, Yule or similar 'thick-tailed' distributions (Stanley and al. 1995, Kwasnicki 1998, Hashemi 2000, Cabral and Mata 2003, Gabaix and Ioannides 2004, de Wit 2005, Saichev*, et al.* 2010).[1,2] An

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[1] A generation ago Simon caustically critiqued the inability of the neoclassical theory of the firm to plausibly explain the empirical size distribution (Ijiri and Simon 1977: 7-11, 138-140, Simon 1997). Transaction cost (e.g., Williamson 1985) and game theoretic explanations of the firm (e.g., Hart 1995, Zame 2007) are also ambiguous empirically, placing few restrictions on firm sizes and growth rates, for example.

early attempt to add some microeconomics to these stochastic process stories is due to Lucas (1978), who derived Pareto-distributed firm sizes from a Pareto distribution of managerial talent. More recently, Luttmer obtains Zipf-distributed firm sizes in a variety of general equilibrium settings, driven by skewed productivity distributions (Luttmer 2007), or by innovation (Luttmer 2010), or by replication of organizational capital (Luttmer 2011), always mediated in some subtle way by firm entry, and always driven by exogenous shocks. He has attempted to explain, with less empirical success, firm ages (Luttmer 2007) and growth rate variability (Luttmer 2011). Overall, today there do not exist models with microeconomic foundations that can explain the bulk of the emerging microdata on firms. Here I develop just such a model.

The model draws together threads from various theoretical literatures. The notion of a production function is preserved, albeit in a modified form. The model is written at the level of individual agents and incentive problems of the type studied in the principal-agent literature manifest themselves. The agents work in perpetually novel environments, so contracts are incomplete and transaction costs are implicit. Each firm is a coalition of agents, so the general equilibrium approach is relevant. Finally, the ways in which agents make decisions, and firms grow and decline, is in the spirit of evolutionary economics.

Specifically, my model consists of a heterogeneous population of agents with preferences for income and leisure. All production takes place under increasing returns to scale, so agents who work together can produce more output per unit effort than by working alone. However, agents act non-cooperatively:[3] they select efforts that improve their own welfare, and may migrate between firms or start-up new firms when it is advantageous to do so. Analytically, Nash equilibria can be unstable in this environment. Large firms are ultimately unstable because each agent's compensation is imperfectly related to its effort level, making free-riding possible. Highly productive agents eventually leave large firms and such firms

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[2] Sutton's (1998) game theoretic models of bound the extent of intra-industry concentration, constraining the shape of size distributions. He has also studied how growth rate variance depends on size (Sutton 2002).

[3] For a cooperative game theoretic view of firms see Ichiishi (1993).

eventually decline. All firm have finite lives. I study the non-equilibrium dynamics of firms perpetually forming, growing and perishing. It will be shown that *the non-equilibrium regime provides greater welfare than equilibrium.*

Although the model is situated conceptually within existing theories of the firm, the results are developed using agent computing (Holland and Miller 1991, Vriend 1995, Axtell 2000, Tesfatsion 2002). Agents are software objects representing individuals, having behavioral rules governing their interactions. Such models are 'spun' forward in time and regularities emerge from the interactions (e.g., Grimm*, et al.* 2005). The shorthand for this is that macro-structure "grows" from the bottom-up. No equations governing the macro level are specified. Nor do agents have either complete information or correct models for how the economy will unfold. Instead, they glean data inductively from the environment and their social networks—i.e., through direct social interactions—and make imperfect forecasts of economic opportunities. (Arthur 1994). This methodology facilitates modeling agent heterogeneity (Kirman 1992), non-equilibrium dynamics, local interactions (Follmer 1974, Kirman 1997), and bounded rationality (Arthur 1991, Kirman 1993). As we shall see, aggregate stationarity is attained in the model despite perpetual behavioral adjustments and changing employment at the agent level. Thus, microeconomic equilibria are *not* the focal point of the analysis.

2 Team Production and Team Formation

Holmström (1982) formally characterized the equilibria that obtain in team production. These results have been extended in various ways (e.g., Watts 2002). I model a group of agents engaged in team production, each agent contributing a variable amount of effort, leading to variable team output.[4]

Consider a finite set of agents, A, $|A| = n$, each of whom works with an effort level $e_{i\in A} \in [0, \omega_i]$. The total effort of the group is then $E \equiv \sum_{i\in A} e_i$. The group produces output, O, as a function of E, according to $O(E) = aE + bE^{\beta}$, $\beta > 1$. This

[4] The model derives from Canning (1995), Huberman and Glance (1998) and Glance *et al.* (1997).

represents the group's production function.[5] For $b > 0$ there are increasing returns to effort.[6] Increasing returns in production means that agents working together can produce more than they can as individuals.[7] To see this, consider two agents having effort levels e_1 and e_2, with $\beta = 2$. As individuals they produce total output $O_1 + O_2 = a(e_1 + e_2) + b(e_1^2 + e_2^2)$, while working together they make $a(e_1 + e_2) + b(e_1 + e_2)^2$. Clearly this latter quantity is at least as large as the former since $(e_1 + e_2)^2 \geq e_1^2 + e_2^2$. As a compensation rule let us first consider agents sharing total output equally: at the end of each period all output is sold for unit price and each agent receives an O/N share of the total output.[8] Agents have Cobb-Douglas preferences for income and leisure.[9] All time not spent working is spent in leisure, so agent i's utility can be written as a function of its effort, e_i, and the effort of other agents, $E_{\sim i} \equiv E - e_i$ as

$$U_i\left(e_i;\theta_i,\omega_i,E_{\sim i},n\right) = \left(\frac{O\left(e_i;E_{\sim i}\right)}{n}\right)^{\theta_i}\left(\omega_i - e_i\right)^{1-\theta_i}. \quad (1)$$

2.1 Equilibrium of the Team Formation Game

Consider the individual efforts of agents to be unobservable. From team output, O, each agent i determines E and, from its contribution to production, e_i, can figure out $E_{\sim i}$. Agent i then selects effort $e_i^*\left(\theta_i,\omega_i,E_{\sim i},n\right) = \arg\max_{e_i} U_i\left(e_i\right)$. For $\beta = 2$, in symbols, $e_i^*(\theta_i, \omega_i, E_{\sim i}) =$

$$\max\left[0, \frac{-a - 2b\left(E_{\sim i} - \theta_i\omega_i\right) + \sqrt{a^2 + 4b\theta_i^2\left(\omega_i + E_{\sim i}\right)\left[a + b\left(\omega_i + E_{\sim i}\right)\right]}}{2b\left(1+\theta_i\right)}\right]. \quad (2)$$

Note that e^* does not depend on n but does depend on $E_{\sim i}$—the effort put in by the other agents. To develop intuition for the general dependence of e_i^* on its parameters, we plot it for $a = b = \omega_i = 1$ in figure 1, as functions of $E_{\sim i}$ and θ_i.

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5 While $O(E)$ relates inputs to outputs, like a standard production function, the inputs are not explicit choices of a decision-maker, since E results from autonomous agent actions. Thus, $O(E)$ cannot be made the subject of a math program, as in conventional production theory, although, it does describe production possibilities.

6 Increasing returns at the firm level goes back at least to Marshall (1920) and was the basis of theoretical controversies in the 1920s (Sraffa 1926, Young 1928). Recent work on increasing returns is reprinted in Arthur (1994) and Buchanan and Yoon (1994). Colander and Landreth (1999) give a history of the idea.

7 There are many ways to motivate increasing returns, including 'four hands problems': two people working together are able to perform a task that neither could do alone, like carrying a piano up a flight of stairs.

8 The model yields roughly constant total output, so in a competitive market the price of output would be nearly constant. Since there are no fixed costs, agent shares sum to total cost, which equals total revenue. The shares can be thought of as either uniform wages in pure competition or profit shares in a partnership.

9 The online appendix gives a more general model of preferences, yielding qualitatively identical results.

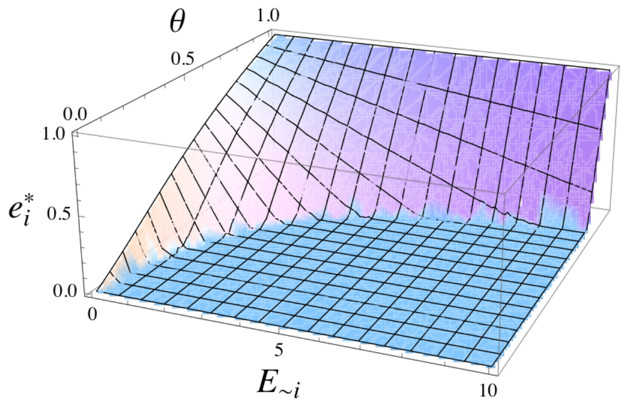

Figure 1: Dependence of e_i^* on $E_{\sim i}$ and θ for $a = b = \omega_i = 1$

The optimal effort level decreases monotonically as 'other agent effort,' $E_{\sim i}$, increases. For each θ_i there exists some $E_{\sim i}$ beyond which it is rational for agent i to put in no effort. For constant returns, e_i^* decreases linearly with $\theta_i = -1$.

Equilibrium in a group corresponds to each agent working with effort e_i^* from equation 2, using $E_{\sim i}^*$ in place of $E_{\sim i}$ such that $E_{\sim i}^* = \sum_{i \neq i} e_j^*$. This leads to:

***Proposition 1*:** Nash equilibria exist in any group.

Proposition 2: There exists a set of agent efforts that Pareto dominate Nash equilibrium, a subset of which are Pareto optimal. These efforts (a) need larger effort levels than the Nash equilibrium, and (b) are not individually rational.

(Proofs of propositions 1 and 2 are in the appendix.) This effort region that Pareto-dominates Nash equilibrium is where firms live.

Singleton Firms

The $E_{\sim i} = 0$ solution of (2) corresponds to agents working alone in single agent firms. For this case the expression for the optimal effort level is

$$e^*(\theta,\omega) = \frac{-a + 2b\theta\omega + \sqrt{a^2 + 4b\theta^2\omega(a + b\omega)}}{2b(1+\theta)}. \qquad (3)$$

For $\theta = 0$, $e^* = 0$ while for $\theta = 1$, $e^* = \omega$. For $\theta \in (0, 1)$ it is easily shown that the optimal effort is greater than for constant returns.

Homogeneous Groups

Consider a group composed of agents of the same type (identical θ and ω). In a homogeneous group each agent works with the same effort in equilibrium, determined from (2) by substituting $(n\text{-}1)\ e_i^*$ for $E_{\sim i}$, and solving for e^*, yielding:

$$e^*(\theta,\omega,n)=\frac{2b\theta\omega n-a\big(\theta+n(1-\theta)\big)+\sqrt{4b\theta^2\omega n(a+b\omega n)+a^2\big(\theta+n(1-\theta)\big)^2}}{2bn\big(2\theta+n(1-\theta)\big)}. \quad (4)$$

These efforts are shown in figure 3a as a function of θ, with $a = b = \omega = 1$ and various n. Figure 3b plots the utilities for $\theta \in \{0.5, 0.6, 0.7, 0.8, 0.9\}$ versus n.

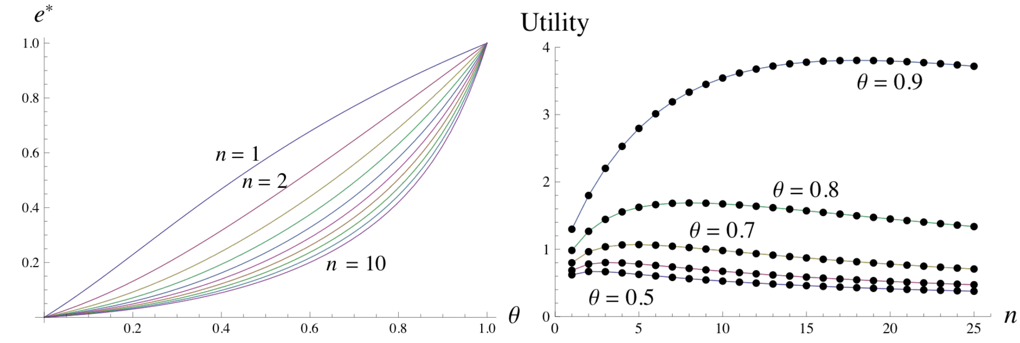

Figure 3: Optimal effort (a) and utility (b) in homogeneous groups vs. θ and n, with $a = b = \omega = 1$

Note that each curve in figure 3b is single-peaked, so there is an optimal group size for every θ. This size is shown in figure 4a for two values of ω.

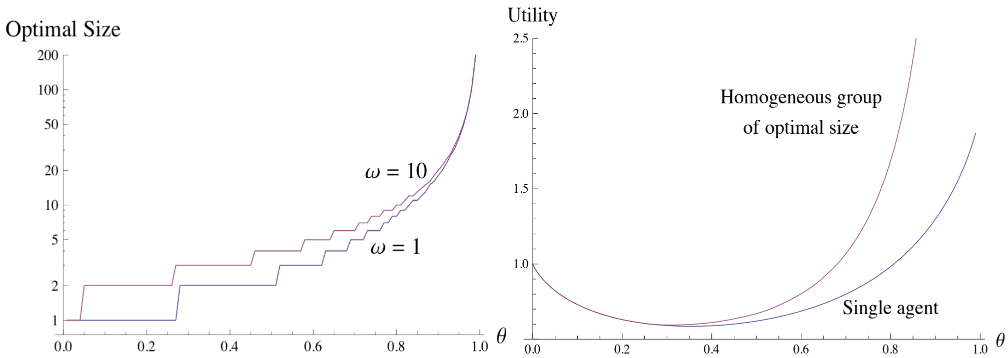

Figure 4: Optimal size (a) and utility (b) in homogeneous groups as functions of θ; $a = b = 1$; $\omega = 1, 10$

Optimal group sizes rise quickly with θ (note log scale). Utilities in groups are shown in figure 4b. Gains from being in a team are greater for high θ agents.[10]

2.2 Stability of Nash Equilibrium, Dependence on Team Size

A unique Nash equilibrium always exists but for sufficiently large group size it is unstable. To see this, consider a team out of equilibrium, each agent adjusting its effort. As long as the adjustment functions are decreasing in $E_{\sim i}$ then one expects the Nash levels to obtain. Because aggregate effort is a linear

[10] For analytical characterization of an equal share (partnership) model with perfect exclusionary power see Farrell and Scotchmer (1988); an extension to heterogeneous skills is given by Sherstyuk (1998).

combination of individual efforts, the adjustment dynamics can be conceived of in aggregate terms. In particular, the total effort level at time $t + 1$, $E(t+1)$, is a decreasing function of $E(t)$, as depicted notionally in figure 5 for a five agent firm, with the dependence of $E(t+1)$ on $E(t)$ shown as piecewise linear.

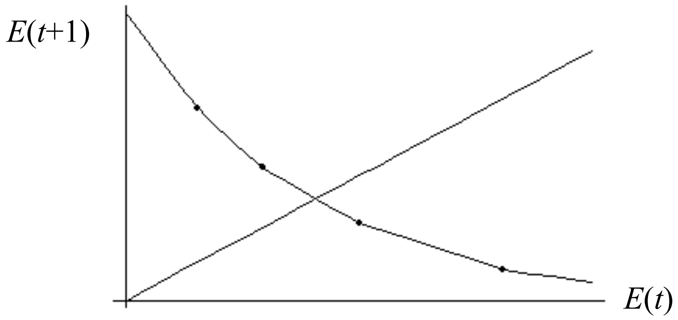

Figure 5: Phase space of effort level adjustment, $n = 5$

The intersection of this function with the 45° line is the equilibrium total effort. However, if the slope at the intersection is less than –1, the equilibrium will be unstable. Every team has a maximum stable size, dependent on agent θs.

Consider the n agent group in some state other than equilibrium at time t, described by the vector of effort levels, $e(t) = (e_1(t), e_2(t), ..., e_n(t))$. Now suppose that at $t + 1$ each agent adjusts its effort level using (2), a 'best reply' to the previous period's value of $E_{\sim i}$,[11]

$$e_i(t+1) = \max\left[0, \frac{-a - 2b\left(E_{\sim i}(t) - \theta_i\omega_i\right) + \sqrt{a^2 + 4b\theta_i^2\left(\omega_i + E_{\sim i}(t)\right)\left[a + b\left(\omega_i + E_{\sim i}(t)\right)\right]}}{2b\left(1+\theta_i\right)}\right].$$

Each agent adjusts its effort, resulting an n-dimensional dynamical system, and:

Proposition 3: All teams are unstable for sufficiently large group size.

Proof: Stability is assessed from the eigenvalues of the Jacobian matrix:[12]

$$J_{ij} \equiv \frac{\partial e_i}{\partial e_j} = \frac{1}{1+\theta_i}\left\{-1 + \theta_i^2 \frac{a + 2b\left(\omega_i + E_{\sim i}^*\right)}{\sqrt{a^2 + 4b\theta_i^2\left(\omega_i + E_{\sim i}^*\right)\left[a + b\left(\omega_i + E_{\sim i}^*\right)\right]}}\right\}, \quad (5)$$

[11] Effort adjustment functions that are decreasing in $E_{\sim i}$ and increasing in θ_i yield qualitatively similar results; see appendix A. While this is a dynamic strategic environment, agents make no attempt to deduce optimal multi-period strategies. Rather, at each period they myopically 'best respond'. This simple behavior is sufficient to produce very complex dynamics, suggesting sub-game perfection is implausible.

[12] Technically, agents who put in no effort do not contribute to the dynamics, so the effective dimension of the system will be strictly less than n when such agents are present.

while J_{ii} = 0. Since each $\theta_i \in [0, 1]$ it can be shown that $J_{ij} \in [-1,0]$, and J_{ij} is monotone increasing with θ_i,. The *RHS* of (5) is independent of *j*, so each row of the Jacobian has the same value off the diagonal, i.e., $J_{ij} \equiv k_i$ for all $j \neq i$. Overall,

$$J = \begin{bmatrix} 0 & k_1 & \cdots & k_1 \\ k_2 & 0 & \cdots & k_2 \\ \vdots & & \ddots & \vdots \\ k_n & \cdots & k_n & 0 \end{bmatrix},$$

with each of the $k_i \leq 0$. Stability of equilibrium requires that this matrix's dominant eigenvalue, λ_0, have modulus strictly inside the unit circle. It will now be shown that this condition holds only for sufficiently small group sizes. Call ρ_i the row sum of the i^{th} row of *J*. It is well-known (Luenberger 1979: 194-195) that $\min_i \rho_i \leq \lambda_0 \leq \max_i \rho_i$. Since the rows of *J* are comprised of identical entries

$$(n-1)\min_i k_i \leq \lambda_0 \leq (n-1)\max_i k_i. \tag{6}$$

Consider the upper bound: when the largest $k_i < 0$ there is some value of *n* beyond which $\lambda_0 < -1$ and the solution is unstable. Furthermore, since large k_i corresponds to agents with high θ_i, it is these agents who determine group stability. From (6), compute the maximum stable group size, $N^{\max}$, by setting $\lambda_0 =$ -1 and rearranging:

$$n^{\max} \leq \left\lfloor \frac{\max_i k_i - 1}{\max_i k_i} \right\rfloor, \tag{7}$$

where $\lfloor z \rfloor$ refers to the largest integer less than or equal to *z*. Groups larger than $n^{\max}$ will never be stable, that is, (7) is an upper bound on group size. □

For either *b* or $E_{\sim i} \gg a$, such as when $a \sim 0$, $k_i \approx (\theta_i - 1)/(\theta_i + 1)$. Using this together with (7) we obtain an expression for $n^{\max}$ in terms of preferences

$$n^{\max} \leq \left\lfloor \frac{2}{1-\max_i \theta_i} \right\rfloor. \tag{8}$$

The agent with *highest* income preference thus determines the maximum stable group size. Other bounds on λ_0 can be obtained via column sums of *J*. Noting the i^{th} column sum by γ_i, we have $\min_i \gamma_i \leq \lambda_0 \leq \max_i \gamma_i$, which means that

$$\sum_{i=1}^{n} k_i - \min_i k_i \le \lambda_0 \le \sum_{i=1}^{n} k_i - \max_i k_i . \qquad (9)$$

These bounds on λ_0 can be written in terms of the group size by substituting $n\,\bar{k}$ for the sums. Then an expression for n^{max} can be obtained by substituting $\lambda_0 = 1$ in the upper bound of (9) and solving for the maximum group size, yielding

$$n^{max} \le \left\lfloor \frac{\max_i k_i - 1}{\bar{k}} \right\rfloor . \qquad (10)$$

The bounds given by (7) and (10) are the same (tight) for homogeneous groups.

These calculations are performed for all θ in figure 7. The maximum stable size is green, with the smallest size at which instability occurs (red). The lower (magenta) line is the optimal firm size (figure 4a), which is clearly very near the stability boundary, meaning optimally-sized firms are unstable to the addition of even a single agent.

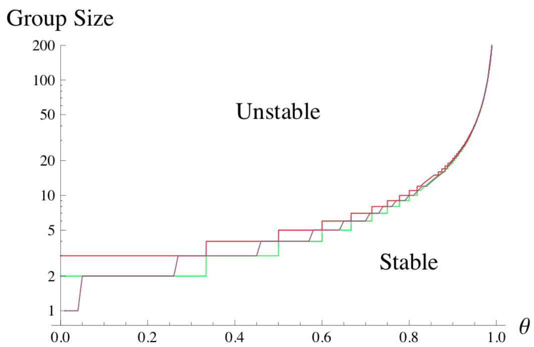

Figure 6: Unstable Nash equilibria in homogeneous groups having income preference θ

This is reminiscent of the 'edge of chaos' literature, for systems poised at the boundary between order and disorder (Levitan*, et al.* 2002).

Unstable Equilibria and Pattern Formation Far From Agent Level Equilibriua

Unstable equilibria may be viewed as problematical if one assumes agent level equilibria are *necessary* for social regularity. But games in which optimal strategies are cycles have long been known (e.g., Shapley 1964, Shubik 1997). Solution concepts can be defined to include such possibilities (Gilboa and Matsui 1991). Agent level equilibria are *sufficient* for macro-regularity, but not *necessary*. When agents are learning or in combinatorially rich environments, as here, fixed points seem unlikely. Non-equilibrium models in economics include

Papageorgiou and Smith (1983) and Krugman (1996).[13]

Firms are inherently dynamic. They age, internal dynamics shift, some agents leave, new ones arrive, everyone adjusts.[14] Indeed, there is vast turnover: of the largest 5000 U.S. firms in 1982, in excess of 65% of them no longer existed as independent entities by 1996 (Blair, *et al.* 2000)! 'Turbulent' is apropos for such volatility (Beesley and Hamilton 1984, Ericson and Pakes 1995).

3 Computational Implementation with Simple Agents

The motivation for a computational model is simple. Since equilibria of the team formation game are unstable, what are its non-equilibrium dynamics? Do the dynamics contain firm formation patterns that are recognizable vis-a-vis actual firms? Such patterns can be difficult to discern analytically, leaving computational models as a practical way of studying them. In what follows we find that such patterns *do* exist and are closely related to data.

3.1 Set-Up of the Computational Model

In the analytical model above the focus is a single group. In the computational model many groups will form within the agent population. The set-up for the computational model is just like the analytical model. Total output of a firm consists of both constant and increasing returns. Preferences and endowments, θ and ω respectively, are heterogeneous across agents. When agent i acts it searches over $[0, \omega_i]$ for the effort maximizing its next period utility. Because many firms will arise in the computational model, it is necessary to specify how agents move between firms. Each agent has an exogenous social network, an Erdös-Renyi graph, consisting of ν_i other agents. It evalustes (a) staying in its current firm, (b) joining ν_i other firms, in essence an on-the-job search over its social network (Granovetter 1973, Montgomery 1991), and (c) starting up a new firm, choosing the option that yields greatest utility. Since agents evaluate only a small number of firms their information is limited. We use

[13] Non-equilibrium models are better known and well-established in other sciences, e.g., in mathematical biology the instabilities of certain PDE systems are the basis for pattern formation (Murray 1993).

[14] Arguments against firm equilibrium include Kaldor (1972, 1985), Moss (1981) and Lazonick (1991).

120 million agents, roughly the size of the U.S. private sector. In each period about 5 million agents are activated, corresponding to one calendar month, calibrated by job search frequency (Fallick and Fleischman 2001). The 'base case' parameterization was developed by applying a heuristic optimization method to seek good fits to the many empirical data described in the next three subsections. This technique generated numerical values that were then rounded to simple rational numbers, summarized in table 2.[15] The fit of the model to the data produced by these parameters is not noticeably different from the parameter values produced by the optimization code.

Model Attribute	Value
number of agents	120,000,000
constant returns coefficient, a	*uniform* on [0, 1/2]
increasing returns coefficient, b	*uniform* on [3/4, 5/4]
increasing returns exponent, β	*uniform* on [3/2, 2]
distribution of preferences, θ	*uniform* on (0, 1)
endowments, ω	1
compensation rule	equal shares
number of neighbors, ν	*uniform* on [2,6]
agent activations per period	4,800,000 or 4% of total agents
time calibration: one model period	one month of calendar time
initial condition	all agents in singleton firms

Table 2: 'Base case' configuration of the computational model

The model's execution can now be summarized in pseudo-code:

```
•  INSTANTIATE and INITIALIZE time, agents, firms, and data objects;
•  WHILE time < final time DO:
      o  FOR each agent, activate it with 4% probability:
            ▪  Compute e* and U(e*) in current firm;
            ▪  Compute e* and U(e*) for starting up a new firm;
            ▪  FOR each firm in the agent's social network:
                  •  Compute e* and U(e*);
            ▪  IF current firm not best choice, leave current firm:
                  •  IF start-up firm is best: form start-up;
                  •  IF another firm is best: join other firm;
      o  FOR each firm:
            ▪  Sum agent inputs and then do production;
            ▪  Distribute output;
      o  IF in stationary state COLLECT monthly statistics;
      o  INCREMENT time and reset periodic statistics;
•  COLLECT final statistics.
```

The essential feature of this model is that it is specified at the level of individuals, thus it is 'agent-based'. It is important to emphasize that it is *not* a numerical

15 For model attributes with random values, each agent or firm is given a realization when it is initialized.

model: there are no (explicit) equations governing the aggregate level; the only equations present are for agent decisions. “Solving” an agent model means iterating it to see what patterns emerge (cf. Axtell 2000).

3.2 Aggregate Dynamics

Initially, agents work alone. As each is activated it discovers it can do better working with another agent to jointly produce output. Over time some groups expand as agents find it welfare-improving to join them, while others contract as their agents discover better opportunities elsewhere. New firms are born as discontented agents form start-ups. Overall, once an initial transient passes, an approximately stationary macrostate emerges.[16] In this macro steady-state agents continue to adjust their efforts and change jobs, causing firms to evolve.

Number of Firms and Average Firm Size

The number of firms varies over time, due both to entry—agents leaving extant firms for start-ups—and the demise of failing firms; figure 7 is a typical time series from the model’s steady-state. About 6 million firms in the U.S. have employees and about this number are shown in figure 7 (blue). There are ~100K startups with employees in the U.S. monthly (Fairlie 2012), like the green line in figure 7; exits are in red. Note the higher variability in firm exit than entry.

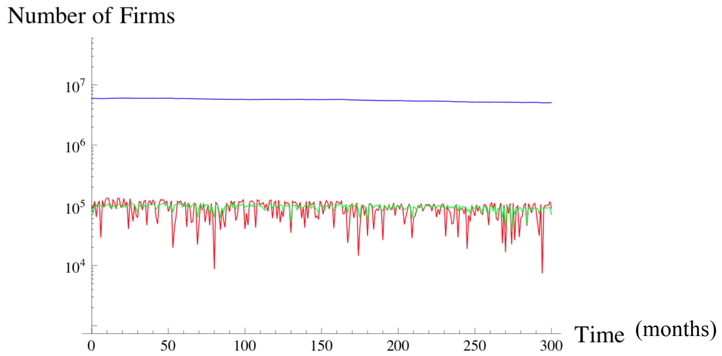

Figure 7: Typical time series for the total number of firms (blue), new firms (green), and exiting firms (red) over 25 years (300 months); note higher volatility in exits.

Since the number of agents is fixed and the number of firms is almost constant, average firm size, the blue line in figure 8, is roughly constant at 20 agents/firm.

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[16] Movies are available at http://www.css.gmu.edu/groups/firms/wiki/5e8cb/Movies.html.

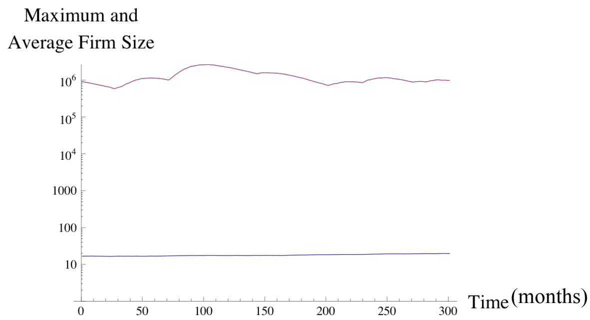

Figure 8: Typical time series for average firm size (blue) and maximum firm size (magenta)

The 120 million workers in the U.S. private sector, in 6 million firms, implies 20 workers/firm. Also shown in figure 8 is the largest firm (red), which fluctuates.

Typical Effort, Income and Utility Levels

Agents who work together improve upon their singleton utility levels with reduced effort. This is the essence of firms, as shown in figure 9.

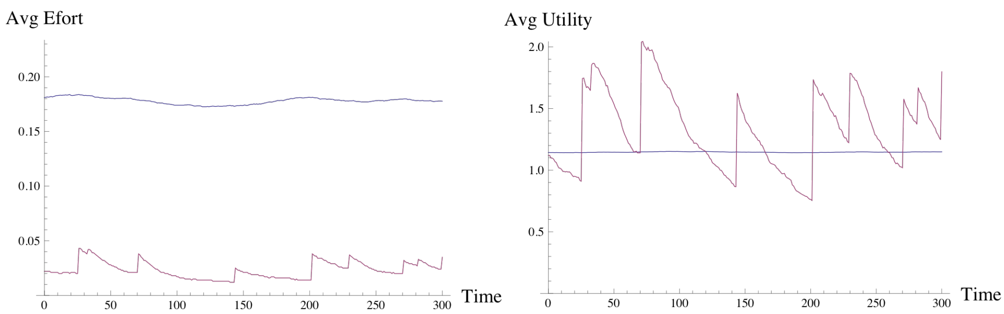

Figure 9: Typical time series for (a) average effort level in the population (blue) and in the largest firm (magenta), (b) average utility (blue) and in the largest firm (magenta)

While efforts in large firms fluctuate, average effort is quite stable (figure 9a). Much of the dynamism in the 'large firm' time series is due to the identity of the largest firm changing. Figure 9b shows that the average utility in the population (blue) is usually exceeded by that in the largest firm (red). Occasionally utility in large firms falls below average, signaling that the large firm is in trouble.

Labor Flows

In real economies people change jobs with, what is to some, "astonishingly high" frequency (Hall 1999: 1151). Job-to-job switching (aka employer-to-employer flow), represents 30-40% of labor turnover, substantially higher than unemployment flows (Davis*, et al.* 1996, Fallick and Fleischman 2001, Davis*, et al.* 2006, Faberman and Nagypál 2008, Nagypál 2008, Davis*, et al.* 2012).

Moving between jobs is basic to the model. In figure 10 the level of monthly job changing in the run of the model described in figures 7-9 is shown (blue)—steady at just over 3 million/month—along with measures of jobs created (red) and jobs destroyed (green). Job creation occurs in firms with net monthly hiring, while job destruction takes place when firms lose workers (net). Note the high volatility in job destruction, about 4x that of job creation, comparable to U.S. data.

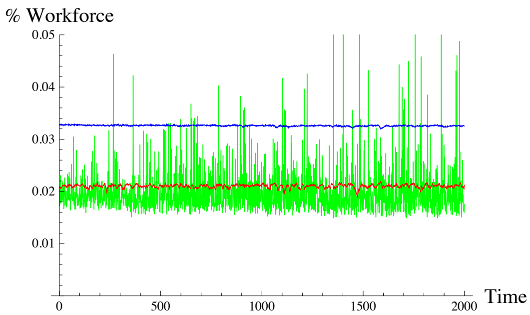

Figure 10: Typical monthly job-to-job changes (blue), job creation (red) and destruction (green)

Overall, figures 7-10 develop intuition about typical dynamics of firm formation, growth and dissolution. They are a 'longitudinal' picture of typical micro-dynamics of agents and firms. We now turn to cross-sectional properties.

3.3 Firms in Cross-Section: Sizes, Ages and Growth Rates

Watching individual firms form, grow, and die in the model movies (see footnote 16), one readily sees up the ‘lumpiness’ of the output, with a few big firms, more medium-sized ones, and lots of small ones.

Firm Sizes (by Employees and Output)

At any instant there exists a distribution of firm sizes in the model. At steady-state, firm sizes reach a skew configuration, with a few large firms and larger numbers of progressively smaller ones. Typical output from the model is shown in figure 11 for firm size measured two ways. The modal firm size is 1 employee, the median is between 3 and 4, and the mean is 20. Empirical data on U.S. firms have comparable statistics. Specifically, for firm size S, the complementary cumulative Pareto distribution function, $F_S^{\,C}(s)$ is

$$\Pr\left(S \geq s_i\right) \equiv F_S^C\left(s;\alpha,s_0\right) = \left(\frac{s_0}{s}\right)^{\alpha}, s \geq s_0, \alpha > 0 . \qquad (11)$$

where s_0 is the minimum size, unity for size measured by employees. The U.S.

data are well fit by $\alpha \approx$ -1.06 (Axtell 2001), the line in figure 11a. The Pareto is a power law, and for $\alpha = 1$ is known as Zipf's law.

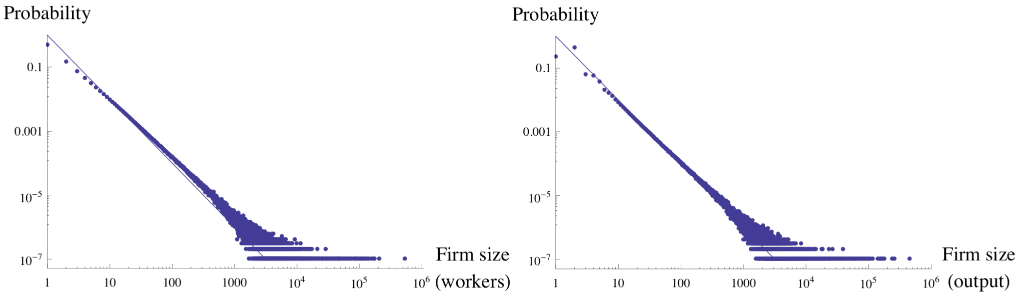

Figure 11: Stationary firm size distributions (pmfs) by (a) employees and (b) output

A variety of explanations for power laws have been put forward.[17] Common to these is the idea that such systems are far from (static) equilibrium at the microscopic level. Our model is non-equilibrium at the agent level with agents regularly changing jobs. Note that power laws fit the *entire distribution* of firm sizes. Simon (1977) argued that such highly skew distributions are so odd as to constitute *extreme* hypotheses. That this model reproduces this peculiar distribution is strong evidence it captures some essence of firm dynamics.[18]

Labor Productivity

Firm output per employee is productivity. Figure 12 is a plot of average firm output as a function of firm size. Fitting a line by several distinct methods indicates that output scales linearly with size, implying constant returns to scale.

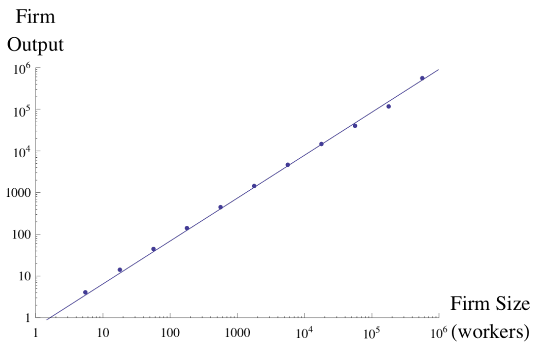

Figure 12: Constant returns at the aggregate level despite increasing returns at the micro-level

Approximately constant returns is also a feature of the U.S. output data; see Basu and Fernald (1997). That *constant returns* occur at the *aggregate* level despite

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[17] Bak (1996: 62-64), Marsili and Zhang (1998), Gabaix, (1999), Reed (2001), and Saichev *et al*. (2010).

[18] At least it is preferable to models of identical (e.g., Robin 2011) or unit size firms (e.g., Shimer 2005).

increasing returns at the *micro*-level suggests the difficulties of making inferences across levels. An explanation of why this occurs is apparent. As the increasing returns-induced advantages that accrue to a firm with size are consumed by free riding behavior, agents migrate to more productive firms. Each agent who changes jobs 'arbitrages' the returns across firms. Since output per worker represents wages in our simple model there is little wage-size effect (Brown and Medoff 1989, Even and Macpherson 2012).

While average labor productivity is constant across firms, there is substantial variation in productivity, as given by the distribution in figure 13.

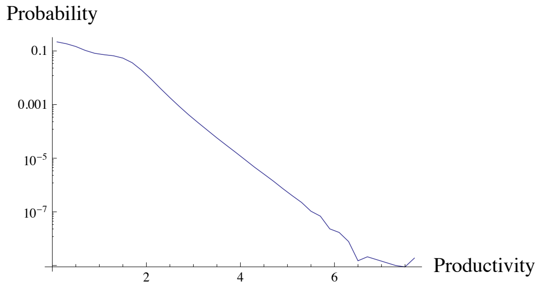

Figure 13: Labor productivity distribution

Average productivity is about 0.7 with a standard deviation of 0.6, and in the semilog coordinates of figure 13 labor productivities are approximately exponentially-distributed, at least the larger ones, not Pareto-distributed as has become a fashionable specification among theorists (Helpman 2006). Interestingly, small and large firms have about the same productivity distribution.

Firm Ages

Using data from the BLS Business Employment Dynamics program, figure 14 gives the age distribution (*pmf*) of U.S. firms, in semi-log coordinates, with each colored line representing the distribution in a recent year.

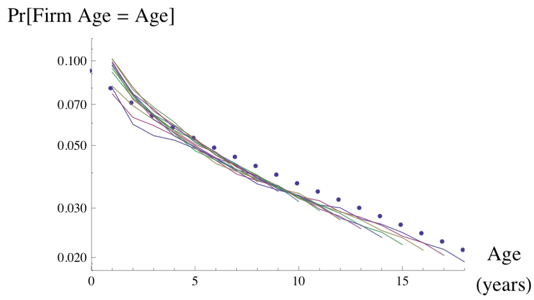

Figure 14: Firm age distributions (*pmf*s), U.S. data 2000-2011 (lines) and model output (points); source: BLS (www.bls.gov/bdm/us_age_naics_00_table5.txt) and author calculations

Model output is overlaid on the raw data as points and agrees reasonably well. While the exponential distribution (Coad 2010) is a rough approximation, the curvature is important, indicating that failure probability depends on age. Average firm lifetime and standard deviation are about 17 years in these figures.

Joint Distribution of Firms by Size and Age

With unconditional size and age distributions now analyzed, their joint distribution is shown in figure 15, a normalized histogram in *log* probabilities.

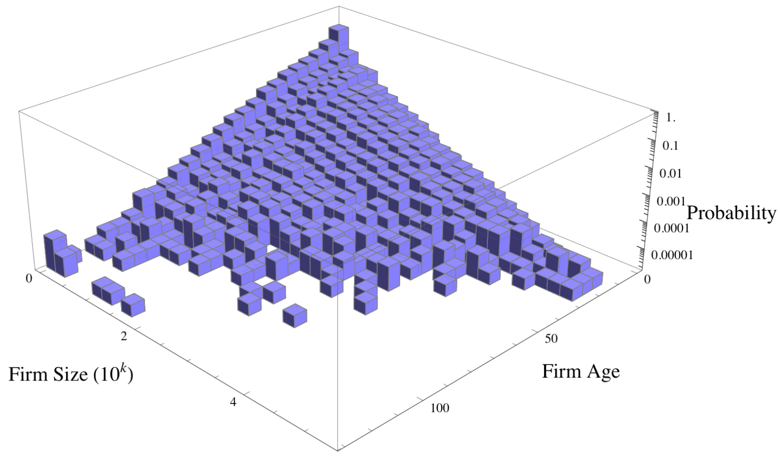

Figure 15: Histogram of the steady-state distribution of firms by *log*(size) and age

Note that log probabilities decline approximately linearly as a function of age and *log* firm size. From the BLS data one can determine average firm size conditional on firm age. In figure 16a these data are plotted for five recent years, starting with 2005, each year its own line. To first order there is a linear relation between firm size and age: firms that are 10 years old have slightly more than 10 employees on average, firms 20 years old have 20 employees, 30 year old firms have roughly 30 employees, and so on. Model output are the dots.

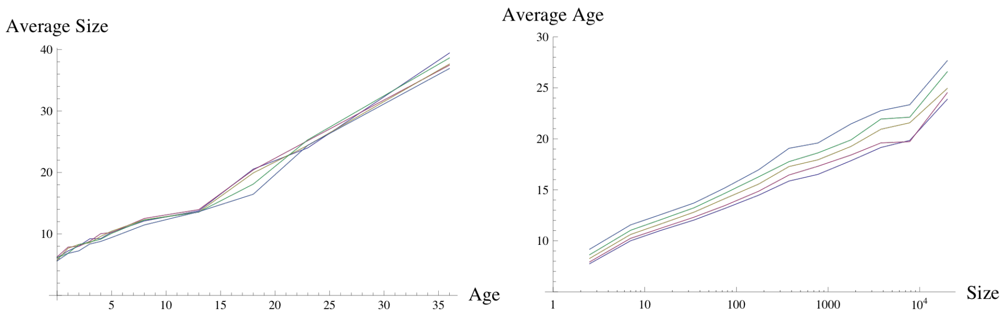

Figure 16: (a) Average firm size by age bins in the U.S. for 2005-2009 and the model;
(b) average firm age by size bins in the U.S. and the model; source: BLS and author calculations

The conditional in the other direction—the dependence of average age on firm size—is shown in figure 16b in semilog coordinates. To first order, average age increases linearly with log size: firms with 10 employees are on average 10 years old, firms with 100 employees average nearly 15 years of age, and firms with 1000 employees are roughly 20 years old, on average. The model (dots) yields a similar result: linearly increasing age with *log* size.

Firm Survival Rates by Age and Size

If firm ages were exactly exponentially distributed then the survival probability would be constant, independent of age (Barlow and Proschan 1965). Curvature in figure 14 indicates that survival probability does depend on age. Empirically it is well-known that survival probability *increases* with age (Evans 1987, Hall 1987). In figure 17 firm survival probabilities over recent years are shown for U.S. companies (lines) with points being model output. Firm survival rates also rise with firm size in both the U.S. data and the model.

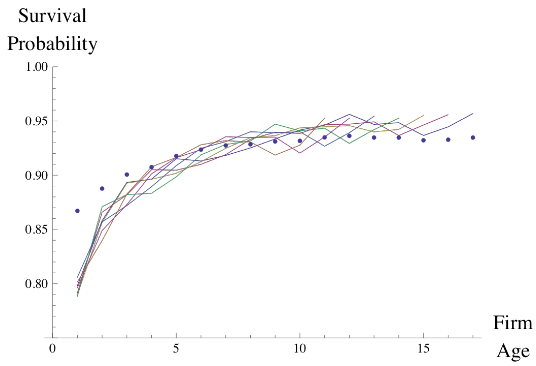

Figure 17: Firm survival probability increases with firm age, U.S. data 1994-2000 (lines) and model (points), and firm size; source: BLS and author calculations

Firm Growth Rates

Calling a firm's size at time t, S_t, a common specification of firm growth rate is $G_{t+1} \equiv S_{t+1}/S_t$. This raw growth rate has support on $\boldsymbol{R}_+$ and is right skew, since there is no upper limit to how much a firm can grow yet it cannot shrink by more than its current size. The quantity $g_{t+1} \equiv ln(G_{t+1})$ has support on $\boldsymbol{R}$ and is nearly symmetric. Gibrat's (1931) proportional growth model—all firms have the same growth rate distribution—implies that G_t is lognormally distributed (e.g., Sutton 1997), meaning g_t is Gaussian. In the basic growth model these distributions are not stationary as their variance grows with time. Adding firm birth and death processes can lead to stationary firm size distributions (see de Wit (2005)).

Gaussian specifications for g were common in IO for many years (e.g., Hart and Prais 1956, Hymer and Pashigian 1962), based on samples of firms. Stanley *et al*. (1996) reported that data on g for all publicly-traded U.S. manufacturing firms (Compustat) were well-fit by the Laplace distribution, which is heavier-tailed than the Gaussian.[19] Subsequently, growth rates for European pharmaceuticals (Bottazzi*, et al.* 2001), Italian and French manufacturers (Bottazzi*, et al.* 2007, Bottazzi*, et al.* 2011), and all U.S. establishments (Teitelbaum and Axtell 2005) were shown to be Laplacian; Schwarzkopf (2011) argues that g is stable.

Representations of Laplace and stably-distributed firm growth rates depart from the usual central limit theorem (Bottazzi and Secchi 2006): when the number of summands is geometrically distributed the Laplace distribution results (Kotz*, et al.* 2001) while heavier-tails yield stable laws (Schwarzkopf 2010).

Empirically, the Subbotin or exponential power distribution is useful as it embeds both the Laplace and Gaussian distributions. Its *pdf* has the form

$$\frac{\eta}{2\sigma_g\Gamma(1/\eta)}\exp\left[-\left(\frac{g-\bar{g}}{\sigma_g}\right)^{\eta}\right], \tag{12}$$

where $\bar{g}$ is the average log growth rate, σ_g is proportional to the standard deviation, and η is a parameter; η = 2 is the normal distribution, η = 1 the Laplace. Semilog plots of (12) vs g yield distinctive 'tent-shaped' figures for $\eta \approx 1$, parabolas for $\eta = 2$. Empirical estimates often yield $\eta \leq 1$ (Perline*, et al.* 2006, Bottazzi*, et al.* 2011).[20] Overall, g has several empirical characteristics:

1. Typically, there is more variance for negative g, i.e., firm decline, corresponding to more variability in job destruction than job creation (Davis*, et al.* 1996), requiring an asymmetric Subbotin (Perline*, et al.* 2006).
2. While Mansfield (1962), Birch (1981), Evans (1987) and Hall (1987) all show that average growth declines with firm size, or at least is positive for

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[19] For g Laplace-distributed, G follows the log-Laplace distribution, a kind of double-sided Pareto distribution (Reed 2001), a combination of the power function distribution on (0, 1) and the Pareto on (1, ∞).

[20] An alternative definition of G is $2(S_{t+1} - S_t)/(S_t + S_{t+1})$, making $G \in [-2, 2]$ (Davis*, et al.* 1996). Although advantageous because it keeps exiting and entering firms in datasets for one additional period, it is objectionable because it makes distinguishing Laplace from normally-distributed growth rates difficult.

small firms and negative for large firms, there is evidence this an artifact of the specification of g (Haltiwanger*, et al.* 2011, Dixon and Rollin 2012).

3. Mansfield (1962), Evans (1987), Hall (1987) and Stanley *et al*. (1996) all show that growth rate variance declines with firm size, on average in the first three cases, for the full distribution in the latter. This is significant insofar as it vitiates Gibrat's simple growth rate specification: all firms are *not* subject to the same growth rate distribution—large firms face less variable growth.
4. Average growth falls with age (Haltiwanger*, et al.* 2008, 2011).
5. Over longer time periods g tends to become more normal (Perline*, et al.* 2006), i.e., η increases with the duration over which growth is measured.

With this as background, figure 18 shows distributions of g emanating from the model for seven classes of firm sizes, from small (blue) to large (purple) ones.

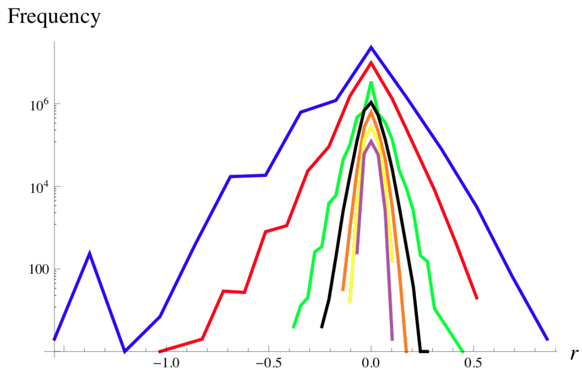

Figure 18: Distributions of annual g, depending on size, from the model; sizes 8-15 (blue), 16-31 (red), 32-63 (green), 64-127 (black), 128-255 (orange), 256-511 (yellow), and 512-1023 (purple)

Overall, $\bar{g}$ is very close to 0.0 (no growth) and figure 19a shows its dependence on size (blue). The red line is an alternative definition of G (see footnote 30).

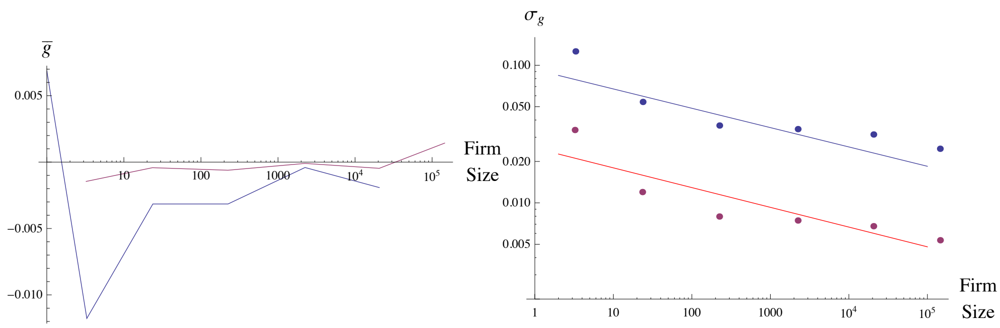

Figure 19: Dependence of the (a) mean and (b) standard deviation of g on firm size, in agreement with Dixon and Rollin [2012] for (a) and Stanley *et al.* [1996] for (b)

The variability of g declines with firm size in figure 18, and figure 19b shows how. Stanley *et al*. (1996) find that the std. dev. in g decreases with size like $s^{-\tau}$, and estimate τ = 0.16 ± 0.03 for size based on employees (data from Compustat manufacturing firms). We get τ = 0.14 ± 0.02 (blue and red lines). For τ = 0.5 the central limit theorem would apply; τ = 0 means that variance is not a function of size. Several explanations for this dependence have been proposed (Buldyrev*, et al.* 1997, Amaral*, et al.* 1998, Sutton 2002, Wyart and Bouchaud 2002, Klette and Kortum 2004, Fu*, et al.* 2005, Luttmer 2007, Riccaboni*, et al.* 2008).

Firm growth rates decline with age, as mentioned above. Figure 20 is a smoothed histogram of output, the insets depict $\bar{g}$ and the s.d. of g vs. age.

At any instant of time, some firms are growing and others are declining. However, growing firms shed workers and declining firms do some hiring. In figure 21 the left panel represents empirical data on the U.S. economy (Davis*, et al.* 2006), and shows that growing firms hire in excess of the separations they suffer, while declining firms keep hiring even when separations are the norm.

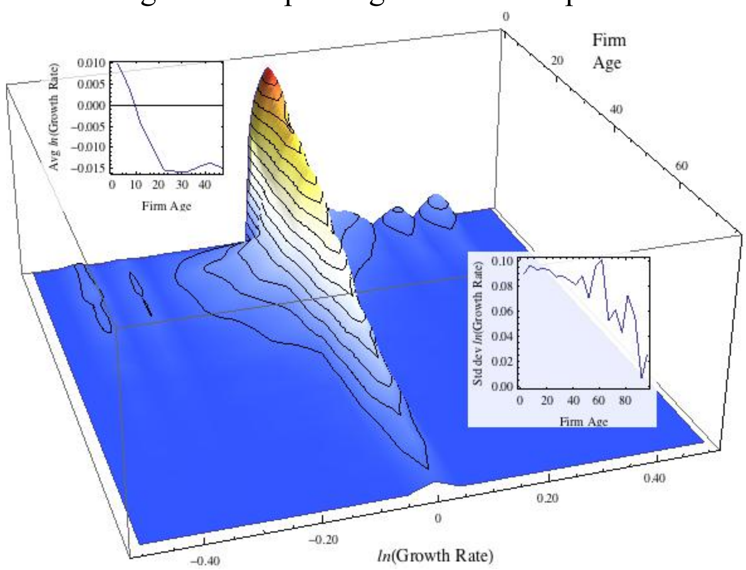

Figure 20: Smoothed histogram of firm growth rates as a function of firm age; the dependence of the mean and standard deviation of g on firm age are shown in the two insets

In the right panel are data from my model, and clearly firms can both gain and lose workers. Note that the ‘hiring’ line in the two figures looks comparable, but the ‘separations’ line is different, with too few separations in the model.

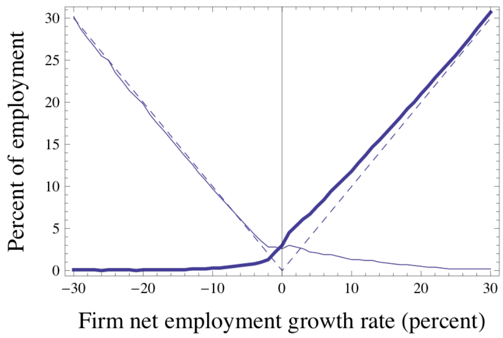

Figure 21: Labor transitions as a function of firm growth rate

Having explored firms cross-sectionally, we next turn to agents.

3.4 Agents in Cross-Section: Income, Job Tenure, Employment

In this section we quantify worker behavior in the aggregate steady-state. Obviously, each worker's circumstances changes periodically, but at the population level there emerge robust statistical features.

Income Distribution

While income and wealth are famously heavy-tailed (Pareto 1971 [1927], Wolff 1994), *wages* are less so. A recent empirical examination of U.S. adjusted gross incomes—primarily salaries, wages and tips—argues that an exponential distribution fits the data below \$125K, while a power law better fits the upper tail (Yakovenko and Rosser 2009). Figure 22 gives the model income distribution.

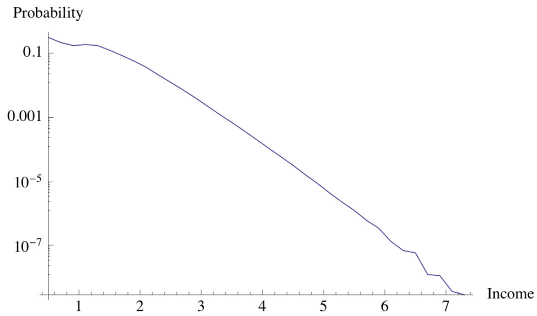

Figure 22: Income distribution (arbitrary units)

Since incomes are nearly linear in semi-log coordinates, they are approximately exponentially-distributed. Although there is not room to analyze these data further, it is the case that incomes increase with preferences for income, θ, and are independent of firm size and age.

Job Tenure Distribution

Job tenure in the U.S. has a median of just over 4 years and a mean of about

8.5 years (BLS Job Tenure 2010). The counter-cumulative distribution for 2010 is figure 23a (points) with the straight line being the estimated exponential distribution. The model job tenure counter-cumulative distribution is figure 23b.

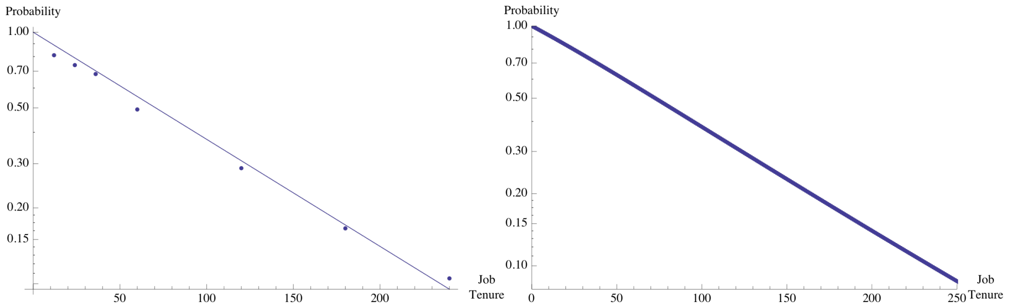

Figure 23: Job tenure (months) is exponentially-distributed (a) in the U.S. and (b) in the model; source: BLS and author calculations

The base case of the model is calibrated to make these distributions coincide. That is, the number of agent activations per period is specified in order to bring these two figures into agreement, thus defining the meaning of one unit of time in the model, here a month. The many other dimensions of the model having to do with time—e.g., firm growth rates, ages—derive from this basic calibration.

Employment as a Function of Firm Size and Age

Because the model's firm size distribution by employees is approximately right (figure 11a), it is also the case that employment as a function of firm size also comes out about right. But the dependence of employment on firm age is not obvious. In figure 24 we count the number of employees in firms as a function of age. About half of American private sector workers are in firms younger than 28 years of age. The first panel are the U.S. data, available online via BLS BDM, shown as a counter-cumulative distribution of employment by firm age, while the second is the same plot using output from the model.

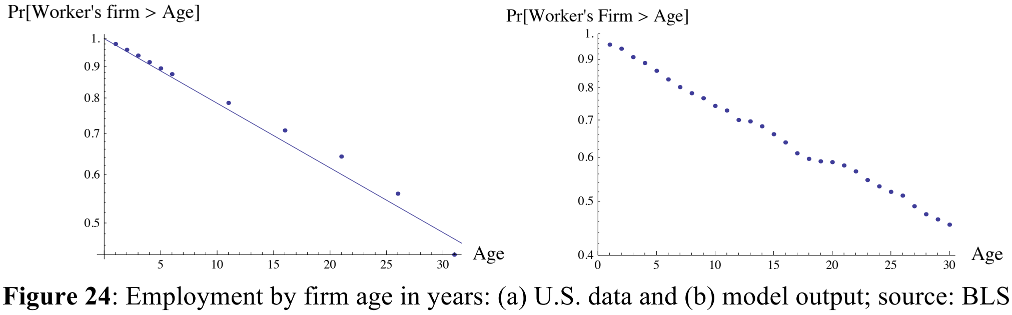

Figure 24: Employment by firm age in years: (a) U.S. data and (b) model output; source: BLS

These two panels show broad agreement between the model and the data.

3.5 Inter-firm Worker Movement: The Labor Flow Network

In the model, as in the real world, workers regularly migrate from one job to another. Here we ask whether there is any *persistent structure* to such migrations. To answer this question we use a graph theoretic representation of inter-firm labor flows. Let each firm be a node (vertex) in such a graph, and an edge exists between two firms if a worker has migrated between the firms. Elsewhere we have called this the *labor flow network* (Guerrero and Axtell 2013). In figure 25 we show the properties of this network for the base case parameterization of the model. The upper left panel gives the degree distribution, while the upper right is the distribution of edge weights. The lower left panel the clustering coefficient as a function of degree, while the lower right panel is the assortativity as a function of degree.

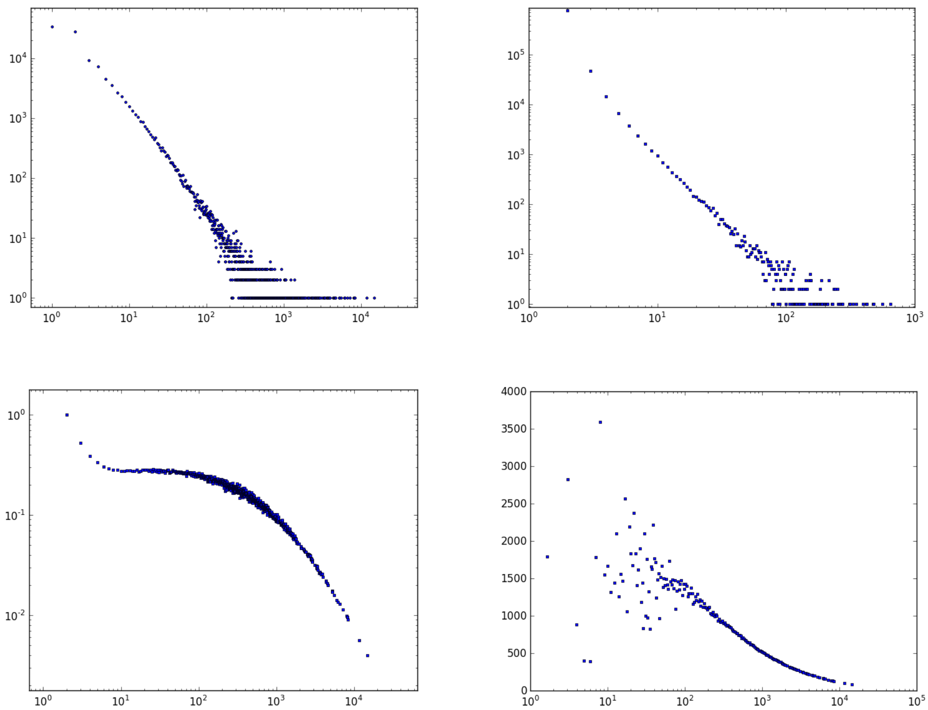

Fig 25: Properties of the labor flow network: (a) degree distribution, (b) edge weight distribution, (c) clustering coefficient, and (d) assortativity; insets: data from Finland (blue) and Mexico (red)

The insets in the upper right corner of each panel are data for Finland. The model closely reproduces these empirical data.

3.6 Agent Welfare in Endogenous Firms

Each time an agent is activated it seeks higher utility, which is bounded from below by the singleton utility. Therefore, it must be the case that all agents prefer the non-equilibrium state to one in which each is working alone—the state of all firms being size one is Pareto-dominated by the dynamical configurations above.

To analyze welfare of agents, consider homogeneous groups of maximum stable size. Associated with such groups are the utility levels shown in figure 4b above. Figure 26 starts out as a recapitulation of figure 4b: a plot of the optimal utility for both singleton firms as well as optimal size homogeneous ones, as a function of θ. Overlaid on these smooth curves is the cross-section of utilities in realized groups. The main result here is that most agents prefer the non-equilibrium world to the equilibrium outcome with homogeneous groups.

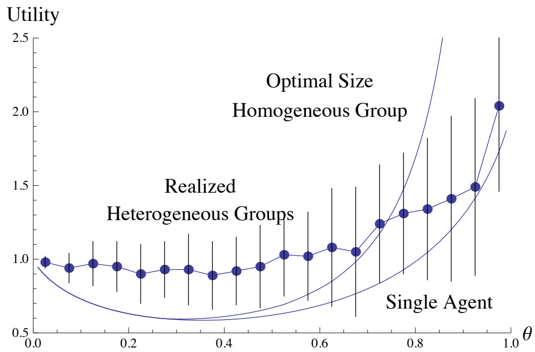

Figure 26: Utility in 1 agent firms, in optimal homogeneous firms, and realized firms, by θ

4 Robustness of the Results

In this section the base model of table 2 is varied and the effects described. One specification found to have no effect on the model in the long run is the initial condition. Starting the agents in groups seems to modify only the duration of the initial transient. The main lesson of this section is that, while certain behavioral and other features can be added to this model and the empirical character of the results preserved, relaxation of any of the basic specifications of the model, individually, is sufficient to break its deep connection to the data.

Against this simple model it is possible to mount the following critique. Since certain stochastic growth processes are known to yield power law distributions, perhaps the model described above is simply a complicated way to generate randomness. That is, although the agents are behaving purposively, this may be just noise at the macro level. If agent behavior were truly random, would this too yield realistic firms? We have investigated this in two ways. First, imagine that agents randomly select whether to stay in their current firm, leave for another firm, or start-up a new firm, while still picking an optimal effort where they end up. It turns out that this specification yields only small firms, under size 10. Second, if agents select the best firm to work in but then choose an effort level at random, again nothing like skew size distributions arise. These results suggest that any systematic departure from (locally) purposive behavior is unrealistic.

Next let us look at how the number of agents maters. While the base case of the model has been realized for 120 million agents, figure 27 gives the dependence of the largest firm realized vs. population size.

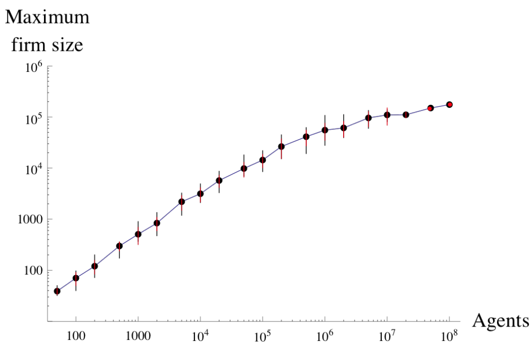

Figure 27: Largest firm size realized as a function of the number of agents

The maximum firm size rises sub-linearly with the size of the population.

Next, consider alternative agent activation schemes. While it is well-known that *synchronous* activation can produce anomalous output (Huberman and Glance 1993), while for the *asynchronous* activation model there can be subtle effects based on whether agents are activated randomly or uniformly (Axtell*, et al.* 1996). The same effect has been found here but it primarily affects firm growth rates (Axtell 2001).

How does the specification of production matter? Of the three parameters that specify the production function, a, b and β, as increasing returns are made stronger, larger firms are realized and average firm size increases. For $\beta > 2$, very large firms arise; these are 'too big' empirically.[21]

Are the results presented above robust to different kinds of agent heterogeneity? Preferences are distributed uniformly on (0,1) in the base case. This yields a certain number of agents having extreme preferences: those with $\theta \approx 0$ are leisure lovers and those with $\theta \approx 1$ love income. Other distributions (e.g., beta, triangular) were investigated and found to change the results very little. Removing agents with extreme preferences from the population can modify the main findings quantitatively. If agent preferences are too homogeneous the model output is qualitatively different from the empirical data. Finally, CES preferences do not alter the general character of the results. Overall, the model is insensitive to preferences as long as they are sufficiently heterogeneous.

Social networks play an important role in the model. In the base case each agent has 2 to 4 friends. This number is a measure of the size of an agent's search or information space, since the agent queries these other agents when active to assess the feasibility of joining their firms. The main qualitative impact of increasing the number of friends is to slow model execution.

However, when agents query *firms* for jobs something different happens. Asking an agent about a job may lead to working at a big firm. But asking a firm at random usually leads to small firms and empirically-irrelevant model output.

How does the equal sharing rule matter for the results above? Here alternative compensation rules are investigated involving pay in proportion to effort:[22]

$$U_i^p\left(e_i;\theta_i,E_{\sim i}\right)=\left(\frac{e_i}{E_{\sim i}+e_i}O\left(e_i,E_{\sim i}\right)\right)^{\theta_i}\left(\omega_i-e_i\right)^{1-\theta_i}$$

Interestingly, this change, when implemented globally across the entire economy, leads to a breakdown in the basic model results, with one giant firm forming. The reason for this is that there are great advantages from the increasing returns to being in a large firm and if everyone is compensated in proportion to their effort

[21] The model can occasionally 'run away' to a single large firm for β in this range.

[22] Encinosa *et al*. (1997) studied compensation systems empirically for team production environments in medical practices. They find that "group norms" are important in determining pay practices. Garen (1998) empirically links pay systems to monitoring costs. More recent work is Shaw and Lazear (2008).

level no one can do better away from the one large firm. Thus, while there is a certain ‘perfection’ in the microeconomics of this compensation, it completely breaks all connections of the model to empirical data.

Next consider a mixture of compensation schemes, with workers paid partially in proportion to how hard they work and partially based on total output.

$$U_i(e_i) = fU_i^e(e_i) + (1-f)U_i^p(e_i) = \left(\frac{f}{n^{\theta_i}} + \frac{(1-f)e_i^{\theta_i}}{(E_{\sim i} + e_i)^{\theta_i}} \right) \left[O(e_i, E_{\sim i}) \right]^{\theta_i} (\omega_i - e_i)^{1-\theta_i}.$$

Parameter f moves compensation from ‘equal’ or ‘proportional’. This can be solved analytically for $\beta = 2$, but is long and messy. Experiments with $f \in [½, 1]$ reveal that the qualitative character of the model is not sensitive to f.

5 Summary and Conclusions

A model of firm formation has been analyzed mathematically, studied computationally, and tested empirically. Stable equilibrium configurations of firms *do not exist* in this model. Rather, agents constantly adapt to their economic circumstances, changing firms when it is in their self-interest to do so. This model, consisting of simple agents in an environment of increasing returns, is sufficient to generate macro-statistics on firm size, growth rates, ages, job tenure, and so on, that closely resemble U.S. data. Overall, firms are vehicles through which agents realize greater utility than they would achieve by working alone. The general character of these results is robust to variations in model specifications. However, it is possible to sever connections to empirical data with agents who are too homogeneous, too random, or too rational.

5.1 Emergence of Firms, Out of Microeconomic Equilibrium

The main result of this research is to connect an explicit microeconomic model of team formation to emerging micro-data on the population of U.S. business firms. Agent behavior is specified at the micro-level with firms emerging at a meso-level, and the population of firms studied at the aggregate level. This micro-to-macro picture has been created with agent computing, realized at full-scale with the U.S. private sector.[23] However, despite the vast scale of the model, its specification is actually very *minimal*, so spare as to seem quite unrealistic[24]—no product markets are modeled, no prices computed, no consumption represented. How is it that such a stripped-down model could ever resemble empirical data?

This model works because its *dynamics* capture elements of the real world more closely than the *static equilibrium* models conventional in the theory of the firm. This is so despite our agents being myopic and incapable of figuring out anything remotely resembling optimal multi-period strategies. Two defenses of such simple agents are clear. First, the environments in which the agents find themselves are *combinatorially too complex* for even highly capable agents to compute rational behaviors. There are just too many possible coalition structures, so each agent finds itself in perpetually novel circumstances.[25] Second, the strategic environment is *dynamically too complex* for agents to make accurate forecasts, even in the short run:[26] agents are constantly moving between firms, new firms are forming, and although the macro-level is stationary there is constant flux and adaptation locally.

More generally, equating social equilibrium with agent-level equilibrium, common throughout the social sciences, is problematical (Foley 1994, Axtell 2014). While the goal of social science is to explain *aggregate* regularities, agent-level equilibria are commonly treated as *necessary* when in fact they are only *sufficient*—micro- and macro-worlds are commonly viewed as *homogeneous* with respect to equilibrium. But macroscopic regularities that have the character of statistical equilibria—stationary distributions, for instance—may have two conceptually distinct origins. When equilibrium at the agent level is achieved, perhaps as stochastic fluctuations about one or more deterministic equilibria (e.g., Young 1993), then there is a definite sense in which macro-stationarity is a direct

[23] It is folk wisdom that agent models are 'macroscopes,' illuminating macro patterns from the micro rules.
[24] In this it is reminiscent of Gode and Sunder and zero-intelligence traders (Gode and Sunder 1993).
[25] Anderlini and Felli (1994) assert the impossibility of complete contracts due to the complexity of nature.
[26] Anderlini (1998) describes the kinds of forecasting errors that are intrinsic in such environments.

consequence of micro-equilibrium. But when there do not exist stable agent-level equilibria, the assumption of homogeneity across levels is invalid, yet it may nonetheless be the case that regularities and patterns will appear at the macro-level. Furthermore, when stable equilibria exist but require an amount of time to be realized that is long in comparison to the economic process under consideration, one may be better off looking for regularities in the long-lived transients. This is particularly relevant to coalition formation games in large populations, where the number of coalitions is given by the unimaginably vast Bell numbers, making it unlikely that anything like optimal coalitions could ever be realized. Perpetual flux in the composition of groups must result, leading to the conclusion that microeconomic equilibria have little explanatory power.

5.2 Theories of the Firm Versus a Theory of Firms

Extant theories of the firm are steeped in this kind of micro-to-macro homogeneity. They begin innocuously enough, with firms conceived as being composed of a few actors. They then go on to derive firm performance in response to strategic rivals, uncertainty, information processing constraints, and so on. But these derivations interpret the overall performance of many-agent groups and organizations in terms of a few agents in equilibrium,[27] and have little connection to the kinds of empirical regularities documented above.[28]

There are two senses in which our model is a theory of firms. First, from a purely descriptive point of view, the model reproduces many facts. Theories of the firm able to explain more than a few of these facts do not exist.[29] Nor are most theories sufficiently explicit to be operationalized in software—although stated at the microeconomic level, the focus on equilibrium leaves behavior away from equilibrium unspecified.[30] In the language of Simon (1976), these theories are substantively rational, not procedurally so. Or, if micro-mechanisms are

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27 Least guilty of this charge is the evolutionary paradigm.

28 For example, neither Shy (1995) nor Cabral (2000) mention of size and growth rate distributions!

29 A variety of models target one of these desiderata, often the firm size distribution (e.g., Kwasnicki 1998).

30 I began this work with the expectation of drawing heavily on extant theory. While I did not expect to be able to turn Coase's elegant prose into software line-for-line, I did expect to find significant guidance on the micro-mechanisms of firm formation. These hopes were soon dashed.

given, the model is not quantitatively related to data (e.g., Kremer 1993, Rajan and Zingales 2001), or else the model generates the wrong patterns (e.g., Cooley and Quadrini (2001) who get *exponential* firm sizes). The second sense in which my model is a theory of firms is that agent models are *explanations* of the phenomena they reproduce.[31] In the philosophy of science an explanation is defined with respect to a theory.[32] A theory has to be general enough to provide explanations of whole classes of phenomena, while not being so vague that it can rationalize all phenomena. Each parameterization of an agent-based model is an instance of a more general agent 'theory'. Executing an instance yields patterns that can be compared to data, thus making the instance falsifiable.[33]

My 'explanation' for firms is simple: purposive agents in increasing returns environments form transient coalitions; freedom of movement between such coalitions 'arbitrages' away super linear returns and induces firms to compete for talent, Suitably parameterized, empirically-salient firms result. Someday a mathematical derivation from the micro (agent) level to the macro (firm population) level—through the meso (firm) level—may appear, but for now we must content ourselves with the *discovery* that the latter result from the former.

This model is a first step toward a more realistic, dynamical theory of the firm, one with explicit micro-foundations. Clearly this approach yields empirically rich results. These results are produced computationally. Typical uses of computers by economists today are to numerically solve equations (Judd 1998) or mathematical programs, to run regressions (Sala-i-Martin 1997), or to simulate stochastic processes—all complementary to conventional theorizing. The way computer power is being harnessed here is different. Agent computing facilitates heterogeneity, so representative agents are not needed (Kirman 1992). It encourages use of behavioral specifications featuring direct (local) interactions, so networks are natural (Kirman 1997). Agents possess a limited amount of

[31] According to Simon (Ijiri and Simon 1977: 118): "To 'explain' an empirical regularity is to discover a set of simple mechanisms that would produce the former in any system governed by the latter."

[32] This is the so-called deductive-nomological (D-N) view of explanation; see Hempel (1966).

[33] In models that are intrinsically stochastic, multiple realizations must be made to find robust regularities.

information and are of necessity boundedly rational, since full rationality is computationally intractable (Papadimitriou and Yannakakis 1994). Aggregation happens, as in the real world, by summing up numerical quantities, without concern for functional forms (of utility and production functions). Macro-relationships *emerge* and are not limited *a priori* to what the 'armchair economist' (Simon 1986) can first imagine and then solve for analytically. There is no need to postulate the attainment of equilibrium since one merely interrogates a model's output for patterns, which may or may not include stable equilibria. Indeed, agent computing is a natural technique for studying economic processes that are far from (agent-level) equilibrium. This paper has merely scratched the surface of the seemingly rich vein at the intersection of large-scale agent computing and economics.

5.3 Economics of Computation

We have entered the age of *computational synthesis*. Across the sciences researchers have begun to create fundamental structures and phenomena in their fields using large-scale computation. In chemistry, complex molecules are synthesized digitally in order to study their structure and properties computationally (Lewars 2011), before they are manufactured in the lab. In biology, whole cell simulation, a grand challenge problem involving thousands of genes and millions of molecules, has recently been shown to be possible (Karr*, et al.* 2012). In physics and engineering, fluid mechanical turbulence has resisted mathematical analysis despite the governing equations being well-known, yet is increasingly well understood through computational methods that explicitly represent the dynamic structures that arise spontaneously in such flows (Hoffman and Johnson 2007). In climate science whole Earth models couple atmospheric and ocean circulation dynamics to study global warming at ever-higher spatio-temporal resolution (Lau and Polshay 2013). In neuroscience accurate simulation of millions of neurons is now possible, leading to the drive for whole brain models (Markram 2006, 2012).

Economists and other social scientists are now utilizing 'big data' in a variety of ways (Lazer*, et al.* 2009) but have yet to systematically embark on the computationally synthetic research program of building social structures and institutions at full scale with real economies. For instance, in macroeconomics, DSGE models still employ a representative consumer and a representative firm.

More than a generation ago an empirically-rich computational model of a specific firm was created and described in *A Behavioral Theory of the Firm* (Cyert and March 1963). I hope to do for the population of U.S. firms what Cyert and March accomplished for an individual organization.

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For Online Publication: Appendices

A: Existence and Instability of Nash Equilibria

Relaxing the functional forms of §2, each agent has preferences for income, I, and leisure, Λ, with more of each preferred to less. Agent i's income is monotone non-decreasing in its effort level e_i as well as that of the other agents in the group, $E_{\sim i}$. Its leisure is a non-decreasing function of ω_i - e_i. The agent's utility is thus $U_i(e_i; E_i) = U_i(I(e_i; E_{\sim i}), \Lambda(\omega_i - e_i))$, with $\partial U_i/\partial I > 0$, $\partial U_i/\partial \Lambda > 0$, and $\partial I(e_i; E_{\sim i})/\partial e_i > 0$, $\partial \Lambda(e_i)/\partial e_i < 0$. Furthermore, assuming $U_i(I = 0, \cdot) = U_i(\cdot, \Lambda = 0) = 0$, U is single-peaked. Each agent selects the effort that maximizes its utility. The first-order condition is straightforward. From the inverse function theorem there exists a solution to this equation of the form $e_i^* = \max\,[0, \zeta(E_{\sim i})]$. From the implicit function theorem both ζ and e_i^* are continuous, non-increasing functions of $E_{\sim i}$.

Team effort equilibrium corresponds to each agent contributing its e_i^*, and that the other agents are doing so as well, i.e., substituting $E_{\sim i}^*$ for $E_{\sim i}$. Since each e_i^* is a continuous function of $E_{\sim i}$ so is the vector of optimal efforts, $e^* \in [0, \omega]^N$, a compact, convex set. By the Leray-Schauder-Tychonoff theorem an effort fixed point exists. Such a solution constitutes a Nash equilibrium, which is Pareto-dominated by effort vectors having larger amounts of effort for all agents.

An upper bound on size exists for effort adjustments $e_i(t+1) = h_i(E_{\sim i}(t))$, s.t.

$$\frac{dh_i\left(E_{\sim i}\right)}{dE_{\sim i}} = \frac{\partial h_i\left(E_{\sim i}\right)}{\partial e_j} \leq 0\,, \tag{A.1}$$

for all $j \neq i$. Under these circumstances the Jacobian matrix retains the structure described in § 2.3, where each row contains N-1 identical entries and a 0 on the diagonal. The bounds on the dominant eigenvalue derived in §2.3 guarantee that there exists an upper bound on the stable group size, as long as (A.1) is a strict inequality, thus establishing the onset of instability above some critical size.

B: Example: Graphical Depiction of the Solution Space

Consider two agents with $\theta = 0.5$ and $\omega = 1$. Solving (2) for e^* with $E_{\sim i} = e^*$ and $a = b = 1$ yields $e^* = 0.4215$, corresponding to utility level 0.6704. Effort deviations by either agent alone are Pareto dominated by the Nash equilibrium, e.g., decreasing the first agent's effort to e_1 = 0.4000, with e_2 at the Nash level yields utility levels of 0.6700 and 0.6579, respectively. An effort increase to e_1 = 0.4400 with e_2 unchanged produces utility levels of 0.6701 and 0.6811, respectively, a loss for the first agent while the second gains. If both agents decrease their effort from the Nash level their utilities fall, while joint increases in effort are welfare-improving. There exist symmetric Pareto optimal efforts of 0.6080 and utility of 0.7267. However, efforts exceeding Nash levels are not individually rational—each agent gains by putting in less effort. Figure 2 plots iso-utility contours for these agents as a function of effort. The 'U' shaped lines are for the first agent, utility increasing upwards.

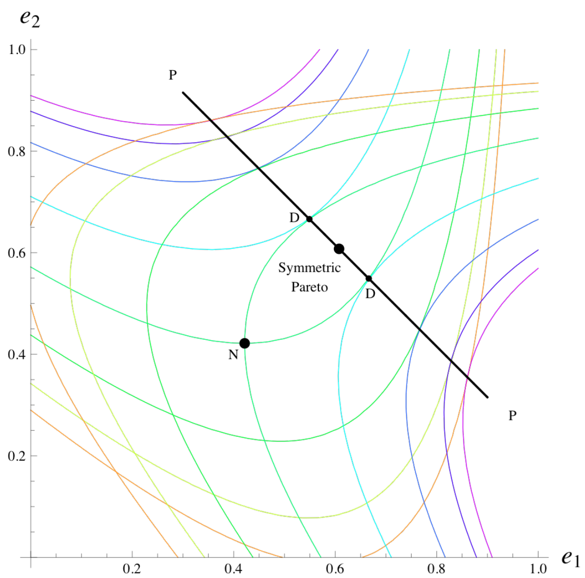

Figure A: Effort level space for two agents with $\theta = 0.5$ and $a = b = \omega = 1$; colored lines are iso-utility contours, 'N' designates the Nash equilibrium, the heavy line from *P-P* are the Pareto optima, and the segment *D-D* represents the Pareto optima that dominate the Nash equilibrium

The 'C' shaped curves refer to the second agent, utility growing to the right. The point labeled 'N' is the Nash equilibrium. The 'core' shaped region extending above and to the right of 'N' is the set of efforts that Pareto-dominate Nash. The set of efforts from 'P' to 'P' are Pareto optimal, with the subset from ‘D’ to ‘D’ being Nash dominant.

For two agents with different preferences the qualitative structure of the effort space shown in figure A is preserved, but the symmetry is lost. Increasing returns insures the existence of solutions that Pareto-dominate the Nash equilibrium. For more than two agents the Nash equilibrium and Pareto optimal efforts continue to be distinct.

C: Example: Nash Equilibrium with Free Entry and Exit

Four agents having θs of {0.6, 0.7, 0.8, 0.9} work together with $a = b = \omega_i = 1$. Equilibrium, from (2), has agents working with efforts {0.15, 0.45, 0.68, 0.86}, respectively, producing 6.74 units of output. The corresponding utilities are {1.28, 1.20, 1.21, 1.32}. If these agents worked alone they would, by (3), put in efforts {0.68, 0.77, 0.85, 0.93}, generating outputs of {1.14, 1.36, 1.58, 1.80} and total output of 6.07. Their utilities would be {0.69, 0.80, 0.98, 1.30}. Working together they put in less effort and receive greater reward. This is the essence of team production. Now say a $\theta = 0.75$ agent joins the team. The four original members adjust their effort to {0.05, 0.39, 0.64, 0.84}—i.e., all work less—while total output rises to 8.41. Their utilities increase to {1.34, 1.24, 1.23, 1.33}. The new agent works with effort 0.52, receiving utility of 1.23, above its singleton utility of 0.80. If another agent having $\theta = 0.75$ joins the new equilibrium efforts of the original group members are {0.00, 0.33, 0.61, 0.83}, while the two newest agents contribute 0.48. The total output rises to 10.09 with utilities {1.37, 1.28, 1.26, 1.34} for the original agents and 1.26 for each of the twins. Overall, even though the new agent induces free riding, the net effect is a Pareto improvement. Next, an agent with $\theta = 0.55$ (or less) joins. Such an agent will free ride and not affect the effort or output levels, so efforts of the extant group members will not change. However, since output must be shared with one additional agent, all utilities fall. For the 4 originals these become {1.25, 1.15, 1.11, 1.17}. For the twins their utility falls to 1.12 and that of the $\theta = 0.9$ agent is now below what it can get working alone (1.17 vs 1.30). Since agents may exit the group freely, it is rational for this agent to do so, causing further adjustment: the three original agents work with efforts {0.10, 0.42, 0.66}, while the twins adds 0.55 and the newest agent free rides. Output is 7.52, yielding utility of

{1.10, 0.99, 0.96} for the original three, 0.97 for the twins, and 1.13 for the free rider. Unfortunately for the group, the θ = 0.8 agent now can do better by working alone—utility of 0.98 versus 0.96, inducing further adjustments: the original two work with efforts 0.21 and 0.49, respectively, the twins put in effort of 0.61, and the θ = 0.55 agent rises out of free-riding to work at the 0.04 level; output drops to 5.80. The utilities of the originals are now 0.99 and 0.90, 0.88 for the twins, and 1.07 for the newest agent. Now the θ = 0.75 agents are indifferent to staying or starting new singleton teams.

D: Example: Unstable Nash Equilibrium

Consider a homogeneous group of agents having θ = 0.7, with $a = b = \omega = 1$. From (8) the maximum stable group size is 6. Consider how instability arises as the group grows. For an agent working alone the optimal effort, from (3), is 0.770, utility is 0.799. Now imagine two agents working together. From (4) the Nash efforts are 0.646 and utility increases to 0.964. Each element of the Jacobian (5) is identical; call this k. For n = 2, k = -0.188 = λ_0. For n = 3 utility is higher, and λ_0 = -0.368. The same qualitative results hold for group sizes 4 and 5, with λ_0 approaching -1. At n = 6 efforts again decline but each agent's utility is lower. For n = 7 λ_0 is -1.082: the group is unstable—any perturbation of the Nash equilibrium creates dynamics that do not settle down. This is summarized in table A.1.

n	***e****	***U(e*)***	***k***	**$\lambda_0 = (n-1)k$**
1	0.770	0.799	not applicable	not applicable
2	0.646	0.964	-0.188	-0.188
3	0.558	1.036	-0.184	-0.368
4	0.492	1.065	-0.182	-0.547
5	0.441	1.069	-0.181	-0.726
6	0.399	1.061	-0.181	-0.904
7	0.364	1.045	-0.180	-1.082

Table A.1: Onset of instability in a group having θ = 0.7; Nash eq. in groups larger than 6 are unstable

Groups of greater size are also unstable in this sense. For lesser θ instability occurs at smaller sizes, while groups having higher θ can support larger numbers.

E: Summary of Empirical Data Utilized

Table A.2 summarizes the firm data to which the model outputs are compared. Data that are conceptually similar are colored similarly.

	Datum or data compared	Source	In text
1	Size of the U.S. workforce: 120 million	U.S. Census	Table 2
2	Number of firms with employees: 6 million	U.S. Census	Figure 7
3	Number of new firms monthly: 100 thousand	Kauffman Foundation	Figure 7
4	Number of exiting firms monthly: 100 thousand	Kauffman Foundation	Figure 7
5	Variance higher for exiting firms to new firms	Davis, Haltiwanger and Schuh	Figure 7
6	Average firm size: 20 employees/firm	U.S. Census	Figure 8
7	Maximum firm size: 1 million employees	Forbes 500	Figure 8
8	Number of job-to-job changes monthly: 3-4 million	Fallick and Fleischman	Figure 10
9	Number of jobs created monthly: 2 million	Davis, Haltiwanger and Schuh	Figure 10
10	Number of jobs destroyed monthly: 2 million	Davis, Haltiwanger and Schuh	Figure 10
11	Variance higher for jobs destroyed than jobs created	Davis, Haltiwanger and Schuh	Figure 10
12	Firm size distribution (employees): Zipf	U.S. Census	Figure 11a
13	Firm size distribution (output): Zipf	U.S. Census	Figure 11a
14	Aggregate returns to scale: constant	Basu and Fernald	Figure 12
15	Productivity distribution: exponential	various	Figure 13
16	Firm age distribution: exponential with mean 18 years	Bureau of Labor Statistics	Figure 14
17	Joint dist. of firms, size and age: linear in age, log size	various	Figure 15
18	Average firm size vs age: increasing linearly in age	Bureau of Labor Statistics	Figure 16ab
19	Avg. firm age vs size: increasing linearly in log size	Bureau of Labor Statistics	Figure 16cd
20	Firm survival probability: increasing with age	Bureau of Labor Statistics	Figure 17
21	Log firm growth rate distribution: heavy-tailed	Stanley *et al.* [1996]	Figure 18
22	Mean log firm growth rate: 0.0	Stanley *et al.* [1996]	Figure 18
23	Mean log firm growth rate vs size: sensitive to def'n	Dixon and Rollin	Figure 19a
24	Std. dev. log firm growth rate vs firm size: exp = 0.14	Stanley *et al.*	Figure 19b
25	Mean log firm growth rate vs firm age: decreasing	Dixon and Rollin	Figure 20
26	Std. dev. log firm growth rate vs firm age: decreasing	Dixon and Rollin	Figure 20
27	Income distribution: exponential	Yakovenko	Figure 21
28	Job tenure dist.: exponential with mean 80 months	Bureau of Labor Statistics	Figure 22
29	Employment vs age: exp. with mean 1000 employees	Bureau of Labor Statistics	Figure 23
30	Florence median: 500 employees	U.S. Census	
31	Large firm vs workforce size: increasing sublinearly	historical Forbes 500	Figure 26
32	Simultaneous hiring and separation	Davis, Faberman and Haltiwanger	Figure 27
33	Degree distribution of the labor flow network	Guerrero and Axtell (2013)	Figure 24a
34	Edge weight distribution of the labor flow network	Guerrero and Axtell (2013)	Figure 24b
35	Clustering coefficient of the labor flow network	Guerrero and Axtell (2013)	Figure 24c
36	Assortativity of the labor flow network	Guerrero and Axtell (2013)	Figure 24d

Table A.2: Empirical data to which the model is compared

F: Sensitivity to Bounded Rationality Specifications

So far, agents have adjusted their effort levels to anywhere within the feasible range [0, ω]. A different behavioral model involves agents making only small changes from their current effort level each time they are activated. Think of this as a kind of prevailing *work ethic* within the group or *individual habit* that constrains the agents to keep doing what they have been, with small changes.

Experiments have been conducted for each agent searching over a range of 0.10 around its current effort level: an agent working with effort e_i picks its new effort from the range $[e_L, e_H]$, where $e_L = \max(0, e_i - 0.05)$ and $e_H = \min(e_i + 0.05, 1)$. This slows down the dynamics somewhat, yielding larger firms. This is because as large firms tend toward non-cooperation, sticky effort adjustment dampens the downhill spiral to free riding. I have also experimented with agents who 'grope' for welfare gains by randomly perturbing current effort levels.

G: Stabilizing Effect of Agent Loyalty to its Firm

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H: Hiring

One aspect of the base model is very unrealistic: that agents can join whatever firms they want, as if there is no barrier to getting hired by any firm. The model can be made more realistic by instituting local hiring policies.

Let us say that one agent in each firm does all hiring, perhaps the agent who founded the firm or the one with the most seniority. We will call this agent the 'boss'. A simple hiring policy has the boss compare current productivity to what would be generated by the addition of a new worker, *assuming that no agents adjust their effort levels*. The boss computes the minimum effort, $\phi E/n$, for a new hire to raise productivity as a function of *a, b,* β, *E* and *n*, where ϕ is a fraction:

$$\frac{aE + bE^{\beta}}{n} < \frac{a\left(E + \phi\frac{E}{n}\right) + b\left(E + \phi\frac{E}{n}\right)^{\beta}}{n+1} = \frac{aE\left(1 + \frac{\phi}{n}\right) + bE^{\beta}\left(1 + \frac{\phi}{n}\right)^{\beta}}{n+1}. \quad (13)$$

For $\beta = 2$ this can be solved explicitly for the minimum ϕ necessary

$$\phi_{*} = \frac{-n(a + 2bE) + \sqrt{n^{2}(a + 2bE)^{2} + 4bEn(a + bE)}}{2bE}.$$

For all values of ϕ_{*} exceeding this level it makes sense to hire the prospective worker. For the case of $a = 0$, (13) can be solved for any value of β: $\phi_{*} = n\left(\frac{n+1}{n}\right)^{1/\beta} - n$; this is independent of *b* and *E*. Numerical values for ϕ_{*} as a function of β and *n* are show in Table 3.

$n \backslash \beta$	**1.0**	**1.5**	**2.0**	**2.5**
1	1.0	0.59	0.41	0.32
2	1.0	0.62	0.45	0.35
5	1.0	0.65	0.48	0.38
10	1.0	0.66	0.49	0.39
100	1.0	0.67	0.50	0.40

Table 3: Dependence of the minimum fraction of average effort on firm size and increasing returns

As *n* increases for a given β, ϕ_{*} increases. In the limit of large *n*, ϕ_{*} equals $1/\beta$. So with sufficient increasing returns the boss will hire just about any agent who wants a job! These results can be generalized to hiring multiple workers.

Adding this functionality to the computational model changes the behavior of individual firms and the life trajectories of individual agents but does not substantially alter the overall macrostatistics of the artificial economy.

I: Effort Monitoring and Worker Termination

In the base model, shirking goes completely undetected and unpunished. Effort level monitoring is important in real firms, and a large literature has grown up studying it; see Olson (1965), the models of mutual monitoring of Varian (1990), Bowles and Gintis (1998), and Dong and Dow (1993), the effect of free exit (Dong and Dow 1993), and endowment effects (Legros and Newman 1996); Ostrom (1990) describes mutual monitoring in institutions of self-governance.

It is possible to *perfectly* monitor workers in our model and fire the shirkers, but this breaks the model by pushing it toward static equilibrium. All real firms suffer from imperfect monitoring. Indeed, many real-world compensation systems can be interpreted as ways to manage incentive problems by substituting reward for supervision, from efficiency wages to profit-sharing (Bowles and Gintis 1996). Indeed, if incentive problems in team production were perfectly handled by monitoring there would be no need for corporate law (Blair and Stout 1999).

To introduce involuntary separations, say the residual claimant knows the effort of each agent and can thus determine if the firm would be better off if the least hard working one were let go. Analogous to hiring we have:

$$\frac{aE + bE^{\beta}}{n} < \frac{a\left(E - \phi\frac{E}{n}\right) + b\left(E - \phi\frac{E}{n}\right)^{\beta}}{n-1} = \frac{aE\left(1 - \frac{\phi}{n}\right) + bE^{\beta}\left(1 - \frac{\phi}{n}\right)^{\beta}}{n-1}$$

Introducing this logic into the code there results unemployment: agents are terminated and do not immediately find another firm to join. Experiments with terminations and unemployment have been undertaken and many new issues are raised, so we leave full investigation of this for future work.

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