Advanced Microeconomics

Partial and General Equilibrium

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Part 4

Recap

So far: We have been studying reductions of the full-fledged GE model to simplify the analysis

1. Reducing the number of markets to only one, while keeping many consumers and firms: partial-equilibrium analysis
2. Reducing the number of firms to zero, while keeping many markets and consumers: pure-exchange analysis

Now: Analyzing the full-fledged GE model, but starting with yet another simplification

3. Allowing for 2 markets, 1 consumer and 1 firm (Robinson Crusoe's economy)

The Full-Fledged Model

Consider a model of a decentralized economy E with the following ingredients:

- Consumers: $i = 1, ..., I$
- Firms: $j = 1, ..., J$
- Commodities: $l = 1, ..., L$

Let us also assume that:

- Each consumer i has preferences $(\succeq_i, X_i)$ and an associated (continuous) utility function u_i which maps consumption bundles $x_i = (x_{1i}, ..., x_{Li}) \in X_i$ into $\Re$; and holds an initial endowment vector $\omega_i \in \Re^L_+$; ; let $\Omega = (\omega_1, ..., \omega_I)$ and $\omega = \sum_i \omega_i \in \Re^L_+$
- Each firm j holds a technology $Y_j \subseteq \Re^L$ and define y_{lj} as the netput of firm j for commodity l; hence, the "net amount" available for good l will be: $\omega_l + \sum_j y_{lj}$

The Full-Fledged Model

Moreover, assume that:

- Private Ownership: Consumers own firms. Each consumer holds a share $\theta_{ij} \geq 0$ of firm j, s.t. $\sum_i \theta_{ij} = 1$, all j, i.e. $\theta_i \in [0, 1]^J$. Profits are entirely redistributed to consumers accordingly to shares.
- Markets are complete (there exist L markets for the L commodities).
- Commodities are undifferentiated (homogeneous). No firms have then advantage whatsoever in selling them and consumers cannot discriminate between commodities sold by different firms.
- There is perfect information about prices across agents. Consumers and firms are perfectly rational (maximizers) and act as price takers.
- All actual exchanges take place simultaneously at a single price vector, after the latter has been quoted. If a firm sells at lower prices, she will undercut competitors. Reselling is not allowed. Pricing is linear (every unit of commodities is sold at the same unit price).

As a result, the economy will be completely defined by:

$$E = \left(\{X_i, u_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}; \{\omega_i, \theta_i\}_{i=1}^{I} \right)$$

Competitive (Walrasian) Equilibrium

Definition (Competitive (Walrasian) Equilibrium)

An allocation $(x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J$ and a price vector $p^* \in \Re_{++}^L$ is a **Competitive (or Walrasian) Equilibrium (CE)** for the economy E iff the following three conditions are satisfied:

1. (Profit Maximization): Given p^*, then $y_j^* = \arg\max(p^* y_j)$, s.t. $y_j \in Y_j$, $\forall j = 1, ..., J$
2. (Utility Maximization): Given Condition 1 and p^*, then $x_i^* = \arg\max u_i(x_i)$, s.t. $p^* x_i \leq p^* \omega_i + \sum_j \theta_{ij} p^* y_j^*$, $x_i \in X_i$, $\forall i = 1, ..., I$
3. (Market Clearing): $\sum_i x_{li}^* \leq \sum_i \omega_{li} + \sum_j y_{lj}^*$, $\forall l = 1, ..., L$

A Robinson Crusoe's Economy

Consider a general-equilibrium model of an economy composed of a single producer and a single consumer. Ingredients:

- 1 price-taker consumers ($I = 1$)
- 1 price-taker firm ($J = 1$)
- Two goods ($L = 2$): Labor/leisure (input) and consumption good (output)
- Labor is used to produce output consumption good
- Endowments: $\omega_L = \bar{L}$ (hours), $\omega_c = 0$ (must be produced)
- Preferences: $u(x_1, x_2) = u(L, x)$, where L=leisure ($\bar{L}$-labor), x=consumption good
- Technology: $f : \mathbb{R}_+ \to \mathbb{R}_+$, where $y = f(z)$=consumption good efficiently produced with z units of labor
- Prices (p, w), p=consumption good price, w=labor price (unit wage)

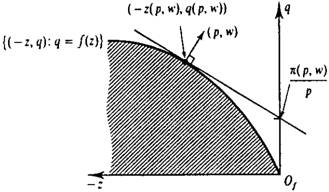
Therefore: RC owns the firm, works for the firm, gets wage and buys the consumption good that the firm produces (. . . a rather weird institutional setting!)

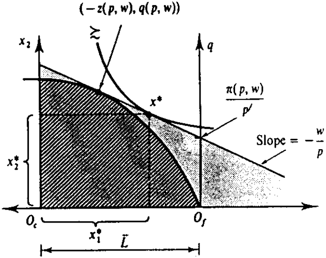
A Robinson Crusoe's Economy

- **RC as a firm**
  - Solves $\max_z pf(z) - wz$
  - Labor demand: $z(p, w)$
  - Output function: $q(p, w)$
  - Profit function: $\pi(p, w)$
- **RC as a consumer**
  - Solves $\max_{L,x} u(L, x)$, s.t. $px + wL \leq w\bar{L} + \pi(p, w)$
  - Leisure: $L(p, w)$
  - Labor: $\bar{L} - L(p, w)$
  - Consumption good: $x(p, w)$
- **Assumptions**
  - Strictly convex, strictly monotone preferences
  - Strictly convex technologies
- **General equilibrium: It is a p^*/w^* ratio such that**
  1. $x(p^*, w^*) = q(p^*, w^*)$
  2. $\bar{L} - L(p^*, w^*) = z(p^*, w^*)$

A Robinson Crusoe's Economy

- **FOCs for RC as a firm**
  - From $\max_z pf(z) - wz$ one gets:
  - FOCs: $f'(z(p,w)) = w/p$
- **FOCs for RC as a consumer**
  - From $\max_{L,x} u(L,x)$, s.t. $px + wL \leq w\bar{L} + \pi(p,w)$ one gets:
  - $\mathcal{L} = u(L,x) - \lambda[px + wl - w\bar{L} + \pi(p,w)]$
  - $\partial u/\partial L = \lambda w$
  - $\partial u/\partial x = \lambda p$
  - FOCs: $u_L/u_x = w/p$

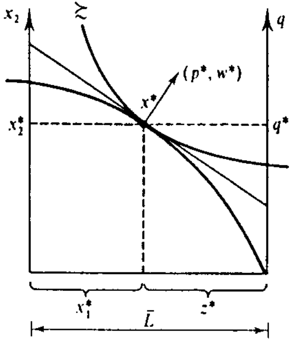




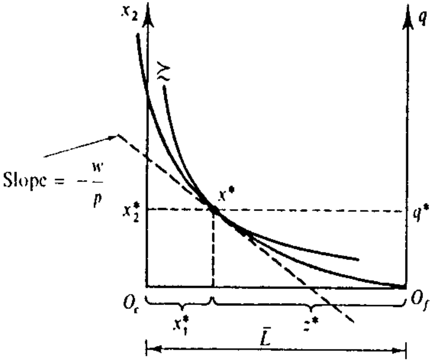
A Robinson Crusoe's Economy: Equilibrium

$$f'(z(p^*, w^*)) = \frac{w^*}{p^*} = \frac{u_L}{u_x}$$
$$x(p^*, w^*) = q(p^*, w^*)$$

If (p^*, w^*) is a WE, then RC maximizes utility while satisfying technological and resource constraints. Therefore the resulting allocation is PO (1st welfare theorem revisited). It is easy to see that also the 2nd WT holds (given convexity assumptions).

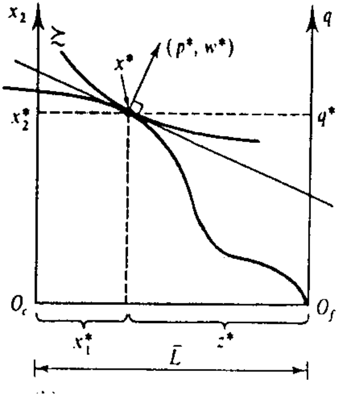


A Robinson Crusoe's Economy: Non-Convex Technology



The case of non-convex technologies. In x^* the consumer is maximizing his utility and its welfare and the allocation is PO. However, at the unique price ratio that supports this allocation the firm is not maximizing her profits since her technology exhibits increasing returns to scale (she would prefer to use more labor). Therefore the 2nd WT fails because there is no price vector supporting the PO allocation x^*.

A Robinson Crusoe's Economy: Non-Convex Technology



Even if technology presents some non-convexities the 1st WT applies. The allocation and price vector (x^*, p^*, w^*) is a WE and the consumer is maximizing her welfare.

What's Next

So far: Showing that in a RC economy with some peculiar features (convexity throughout) an equilibrium exists and welfare theorems apply

Now: Analyzing the full-fledged GE model with many markets, many consumers, and many firms. We ask two questions:

1. Does an equilibrium always exist? If not, what are sufficient conditions guaranteeing existence of (at least one) equilibrium?
2. If an equilibrium exists, do the 1st and 2nd welfare theorem apply in the most general setting? Under what conditions?

Excess of Demand

The 'production-inclusive excess of demand' (excess demand in what follows) is the function/ correspondence that we obtain by computing the difference between Marshallian demand functions/correspondences (given profit functions/correspondences) and net amount of available commodities (initial endowments plus net supply)

More formally: For any price vector $p \gg 0$, define the 'excess of demand' $\overline{z} : \Re^L_{++} \rightarrow \Re^L$ as

$$\overline{z}(p) = \sum_{i=1}^{I} x_i(p, p\omega_i + \sum_j \theta_{ij}\pi_j(p)) - \left[\sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j(p)\right]$$

where $x_i(\cdot,\cdot)$ is the Marshallian demand function of consumer i; $\pi_j(p)$ and $y_j(p)$ are the profit- and the net-supply functions/correspondences of firm j.

Obviously (but try to prove it formally!) we have that: $p^* \gg 0$ is a price-equilibrium vector if and only if $\overline{z}(p^*) = 0$.

Studying the existence of a CE means studying whether a $p^* \gg 0$ exists which solves the system in L equations and L unknowns defined by $\overline{z}(p) = 0$.

Existence of a Walrasian Equilibrium

Claim

Assume that:

1. *Utility functions u_i are continuous, strictly increasing and concave;*
2. *Technology sets Y_j are closed, convex and bounded from above;*
3. *By using $\omega = \sum_{i=1}^{I} \omega_i$ it is possible to produce a strictly positive consumption bundle $\overline{x} \gg 0$.*

Then there exists a $p^* \gg 0$ such that $\overline{z}(p^*) = 0$.

Proof.

See Mas-Colell, Whinston and Green, Ch. 17.A, 17.B, 17.C. For a historical perspective, see Ingrao and Israel (1990). $\square$

First Welfare Theorem

Theorem (First Welfare Theorem in a General Equilibrium Model with Production)

Suppose that $(x_i^*)_{i=1}^I$, $(y_j^*)_{j=1}^J$ and p^* is a competitive equilibrium for the associated economy E and that utility functions u_i all satisfy local non-satiation. Then the allocation $(x_i^*)_{i=1}^I$, $(y_j^*)_{j=1}^J$ is Pareto Efficient.

Proof.

Suppose $(x_i^*)_{i=1}^I$, $(y_j^*)_{j=1}^J$ and p^* is a competitive equilibrium for E. Define:

$$w_i = p^*\omega_i + \sum_j \theta_{ij} p^* y_j^*$$

Notice that:

$$\sum_i w_i = p^* \sum_i \omega_i + \sum_j p^* y_j^*$$

Equilibrium conditions read: (1) $\forall j : p^* y_j \leq p^* y_j^*$, $\forall y_j \in Y_j$;(2) $\forall i : u_i(x_i) \leq u_i(x_i^*)$, $\forall x_i \in X_i$ such that $p^* x_i \leq w_i$; (3) $\sum_i x_i^* = \sum_i \omega_i + \sum_j y_j^*$.

$\square$

First Welfare Theorem

Proof (Cont'd).

By (2) above, we know that:

$$u_i(x_i) > u_i(x_i^*) \text{ implies that } p^* x_i > w_i.$$

Moreover, by local non-satiation (LNS) we also know that:

$$u_i(x_i) \geq u_i(x_i^*) \text{ implies that } p^* x_i \geq w_i.$$

To see this, suppose that $u_i(x_i) \geq u_i(x_i^*)$ implied that $p^* x_i < w_i$. Then by LNS $\exists\, x_i'$ sufficiently close to x_i which is strictly preferred to x_i but is still strictly affordable, i.e. $p^* x_i' < w_i$. Then $u_i(x_i') > u_i(x_i^*)$ and $p^* x_i' < w_i$ which contradicts the hypothesis that x_i^* is a utility maximizing bundle. □

First Welfare Theorem

Proof (Cont'd).

We are then ready to show that the allocation $(x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J$ is Pareto Efficient. Suppose not. In this case, there will exist another **feasible** allocation $(x_i)_{i=1}^I, (y_j)_{j=1}^J$ such that:

$$\begin{aligned} u_i(x_i) &\geq u_i(x_i^*) \ \forall i \\ u_h(x_h) &> u_h(x_h^*) \ \text{some } h \end{aligned}$$

As a consequence:

$$\begin{aligned} p^* x_i &\geq w_i \ \forall i \\ p^* x_h &> w_h \ \text{some } h \end{aligned}$$

Summing up over all i's:

$$\sum_i p^* x_i > \sum_i w_i = p^* \sum_i \omega_i + \sum_j p^* y_j^*$$

$\square$

First Welfare Theorem

Proof (Cont'd).

Since y_j^* maximizes j's profits, then $\forall j : p^* y_j^* \geq p^* y_j$. Hence:

$$\sum_i p^* x_i > p^* \sum_i \omega_i + \sum_j p^* y_j^* \geq p^* \sum_i \omega_i + \sum_j p^* y_j$$

so that:

$$p^* \left[\sum_i x_i - \sum_i \omega_i - \sum_j y_j \right] > 0$$

And as $p^* \gg 0$, then $\sum_i x_i - \sum_i \omega_i - \sum_j y_j \neq 0$, which means the the alleged allocation is not feasible. This contradicts the thesis that $(x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J$ is not Pareto Efficient. $\square$

Second Welfare Theorem

Theorem (Second Welfare Theorem in a General Equilibrium Model with Production)

Suppose E is such that: (1) Production sets are convex; (2) Utility functions u_i have convex upper contour sets and satisfy LNS. Then if the allocation $(x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J$ is Pareto Efficient, there exists a redistribution of endowments and profits such that $(x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J$ and p^* is a WE.

Proof.

See Mas-Colell, Whinston and Green, Ch. 16.D. □

The GE Model

- We have shown that, under some restrictions, there are very stringent relationships between concepts as decentralized CE, PO allocations and social welfare optima.
- Suppose that we are given an economy

$$E = \left(\{X_i, u_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}; \{\omega_i, \theta_i\}_{i=1}^{I} \right)$$

where all assumptions made at the beginning apply.

The GE Model

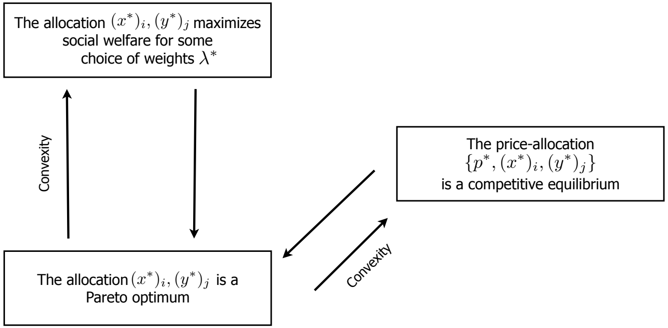
1. If a CE $(p^*, (x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J)$ exists for E, then irrespective of any convexity assumption, the allocation $(x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J$ is a Pareto Optimum. Moreover, if U is convex and the SWF is linear, then there exist weights $\lambda^* \geq 0$ such that the associated utility levels maximize the SWF (i.e. the correspondent allocation would have been chosen by a benevolent planner).

2. Irrespective of any convexity assumption, if the allocation $(x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J$ corresponds to a social welfare optimum chosen by the benevolent planner for some $\lambda^* \geq 0$, then $(x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J$ is PO. Moreover, under some convexity assumptions (see Second Welfare Theorem), there exists a price vector $p^* \gg 0$ and a redistribution of endowments and profits such that $(x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J$ can be sustained by p^* as a CE.

The GE Model

Therefore:

- If a planner maximizes W, s/he reaches a PO allocation, *but only under convexity assumptions this allocation is supported by equilibrium prices*.
- If we let the economy work, then we reach a PO allocation, *but only under convexity assumptions this allocation is socially fair*.



What's Next

So far: We have established that if the economy satisfies certain (very stringent) properties, then at least one WE exists. We are left with other crucial questions to answer. . .

1. Uniqueness: When a WE exists, is it unique? How many WE are there?
2. Stability: Suppose that a unique WE does exist. How does the economy (as distinct from the modeler) "find" the equilibrium, i.e. how does it go there? And if the economy starts far from the equilibrium, is it able to actually go back to the WE?
3. Empirical validity: Can we take the GET to the data and test whether its implications are true? What implications can we test?