Advanced Microeconomics

Partial and General Equilibrium

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Part 3

This (and next) lectures. . .

- **Pure-exchange economies**
  - GET simultaneously focuses on production, exchange and consumption in all markets
  - Here we restrict our attention on pure-exchange economies
- **Basic Assumptions**
  - Production is not possible
  - Consumers are given endowments of L goods
  - They trade commodities for their mutual advantage
  - Consumers maximize their utility given budget constraint
- **Goals**
  - Does a price vector exist such that all consumers maximize given their budget constraints and all markets clear?
  - Are the two welfare theorems true also in this case?

The Model

Primitives of the model

- L goods $\ell \in \mathcal{L} \equiv \{1, \ldots, L\}$
- I agents $i \in \mathcal{I} \equiv \{1, \ldots, I\}$
  - Endowments $e^i \in \mathbb{R}^L_+$; agents do not have monetary wealth, but rather an endowment of goods which they can trade or consume
  - Preferences represented by utility function $u^i : \mathbb{R}^L_+ \to \mathbb{R}$
- Endogenous prices $p \in \mathbb{R}^L_+$, taken as given by each agent

Each agent i solves

$$\max_{x^i \in \mathbb{R}^L_+} u^i(x^i) \text{ such that } p \cdot x^i \leq p \cdot e^i \equiv \max_{x^i \in B^i(p)} u^i(x^i)$$

where $B^i(p) \equiv \{x^i \in \mathbb{R}^L_+ : p \cdot x^i \leq p \cdot e^i\}$ is the budget set for i

PO Allocations

Definition (feasible allocation)

Allocations $(x^i)_{i \in \mathcal{I}} \in \mathbb{R}_+^{I \cdot L}$ are feasible iff for all $\ell \in \mathcal{L}$,

$$\sum_{i \in \mathcal{I}} x_\ell^i \leq \sum_{i \in \mathcal{I}} e_\ell^i.$$

Definition (Pareto optimality)

Allocations $x \equiv (x^i)_{i \in \mathcal{I}}$ are Pareto optimal iff

1. x is feasible, and
2. There is no other feasible allocation $\hat{x}$ such that $u^i(\hat{x}^i) \geq u^i(x^i)$ for all $i \in \mathcal{I}$ with strict inequality for some i.

Walrasian Equilibrium

Definition (Walrasian equilibrium)

Prices p and quantities $(x^i)_{i\in\mathcal{I}}$ are a Walrasian equilibrium iff

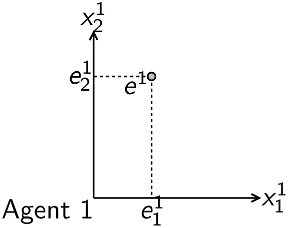
1. All agents maximize their utilities; i.e., for all $i \in \mathcal{I}$,

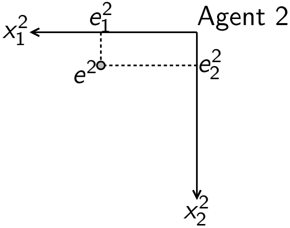
$$x^i \in \operatorname*{argmax}_{x\in B^i(p)} u^i(x);$$

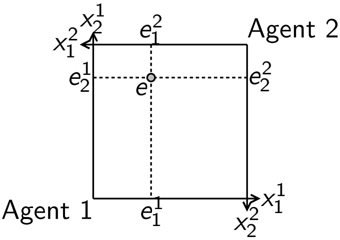
2. Markets clear; i.e., for all $\ell \in \mathcal{L}$,

$$\sum_{i\in\mathcal{I}} x^i_\ell = \sum_{i\in\mathcal{I}} e^i_\ell.$$

Endowments in the Edgeworth Box

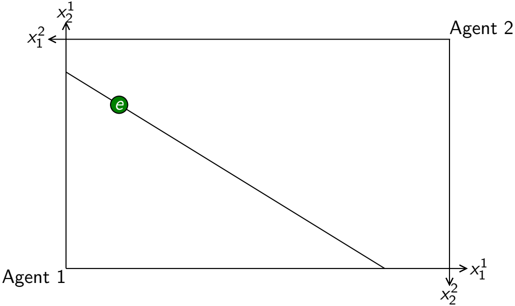






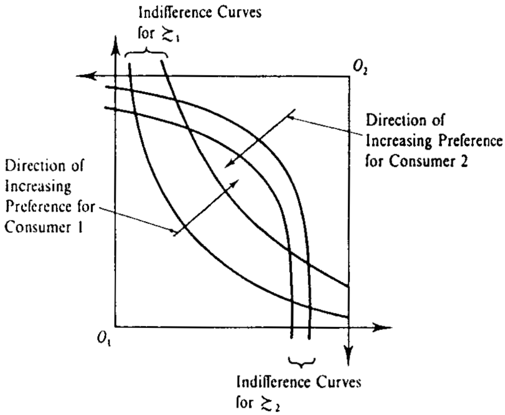
Budget Constraints in the Edgeworth Box

$$p \cdot x^1 = p \cdot e^1 \text{ coincides with } p \cdot x^2 = p \cdot e^2$$



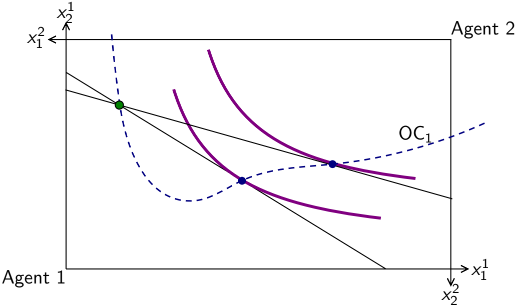
- **Remarks**
  - Size of the box is equal to initial endowments
  - All budget lines pass through *e* and differ by steepness (price ratio)

Utility Functions in the Edgeworth Box



- **Intuition**
  - Equilibrium: a single point in the EB (thus satisfying market clearing) where both consumers are simultaneously maximizing given price ratio
  - Pareto optimal allocation: a single point in the EB where there is no way to strictly improve consumer *i* utility while making the other no worse off

Offer Curves in the Edgeworth Box

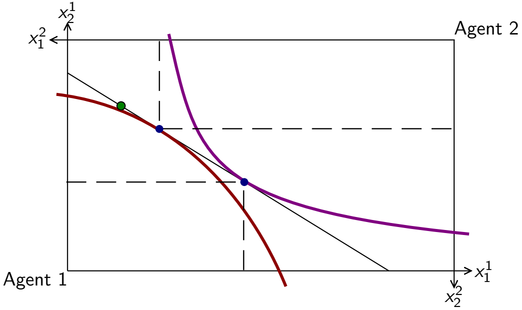


- Offer curve: locus of all bundles that maximize consumer utility given budget constraint as prices vary
- Remark 1: The curve must pass through the endowment point e.
- Remark 2: Since e is affordable at every p, then every point on the OC must be at least as good as e (otherwise at the price vector associated to any point in the OC the consumer would have chosen e). Thus all points on the OC must lie in the upper contour set passing through e. If utility function is smooth, the OC must be tangent to the upper contour set passing through e.

Cobb-Douglas Utilities

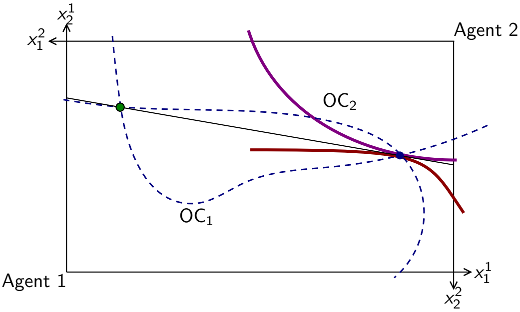
See Exercise!

Incompatible Demands



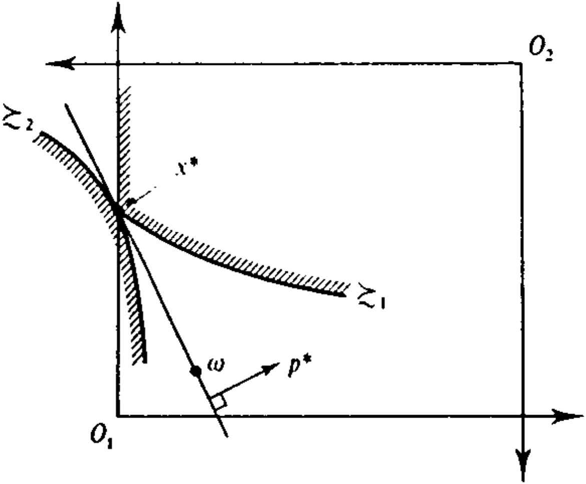
- Given prices p, consumers are both maximizing as both UCS are tangent to the BC. However, market clearing condition is not satisfied: this is not a WE
- In particular: agents are willing to consume a quantity of good 1 that is greater than that available as initial endowments (good 1 is in excess demand), whereas they are demanding strictly less than what is available for good 2 (which is in excess supply)
- Consumer 1 is a net demander of good 1 (she wants to consume more than its own endowments for good 1). Although consumer 2 is a net supplier of good 1 (she wants to consume less than its endowment), he is not willing to supply enough to satisfy consumer 1's needs... prices must change

Walrasian Equilibrium



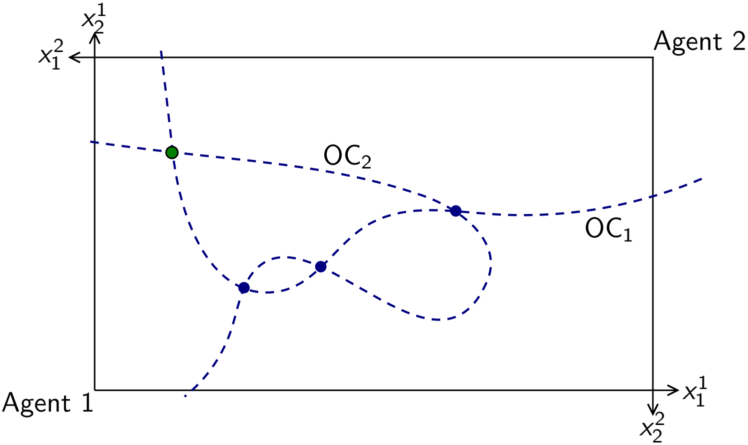
- How can we find an equilibrium?
- Both consumers must maximize given a price ratio: only points in the two OC can be equilibria
- The two points must coincide: only points when the two OCs intersect can be equilibria
- Are all intersections equilibria? No, the endowment point e is not an equilibrium because any other point in OCs are preferred to e.
- Therefore: a point $x^* = (x_1^*, x_2^*) \neq e$ in the EB is a WE for the economy if and only if the two OCs intersect at x^*

Pathologies I



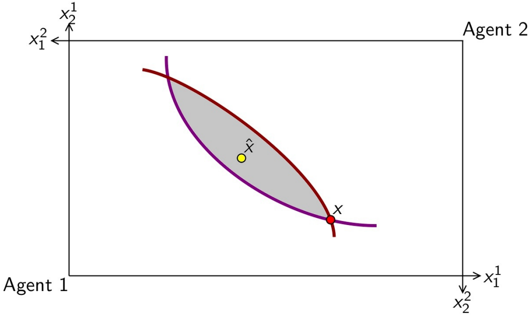
A WE allocation on the boundary of the EB

Pathologies II



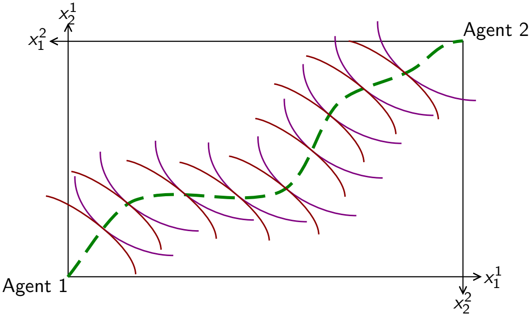
Multiple Equilibria

Pareto Optimality I



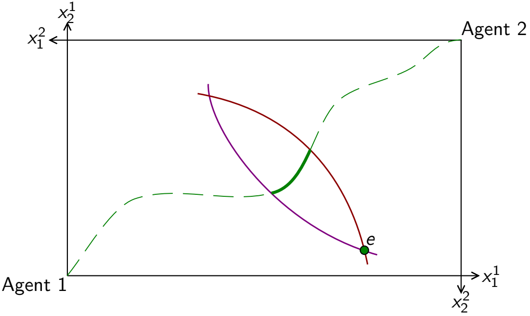
- How can we find PO allocations?
- Start from any given initial endowment point x and draw indifference curves passing through x
- Any allocation $\hat{x}$ inside the gray area makes both consumers better off than at x
- To look for a PO allocation we must travel inside the gray area until we find a point where it is not possible to strictly improve unilaterally anymore (while leaving the other consumer no worse off)
- Under usual smoothness conditions for all u_i, such interior PO allocations must lie where the two indifference curves are tangent

Pareto Optimality II



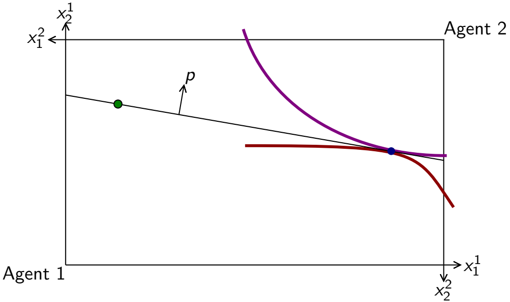
- Pareto Set: locus of all points in the EB where the two indifference curves are tangent (PO allocations)

Pareto Optimality III



- Contract Curve: part of the Pareto set where both consumers do at least as well as they can do with their initial endowments e (bold green curve in the figure above)
- We expect any bargaining between the two consumers to result in an agreement to trade at some point on the contract curve
- These are the only points where both of them do as well as their initial endowments and for which there is no alternative trade that can make both consumers better off

Pareto Optimality and Walrasian Equilibria



- Walrasian equilibrium allocations are Pareto optimal (1st welfare theorem)
- Intuition: At any WE, the corresponding allocation must be such that the two indifference curves are tangent, given equilibrium prices. But if the two indifference curves are tangent, then the equilibrium allocation must be PO

Pareto Optimality and Walrasian Equilibria

Theorem (First Welfare Theorem)

Suppose $u^i(\cdot)$ is increasing (i.e., $u^i(x^{i\prime}) > u^i(x^i)$ for any $x^{i\prime} \gg x^i$) for all $i \in \mathcal{I}$.
If p and $(x^i)_{i\in\mathcal{I}}$ are a Walrasian equilibrium, then the allocations $(x^i)_{i\in\mathcal{I}}$ are Pareto optimal.

Proof.

Suppose in contradiction that $\hat{x}$ Pareto dominates x; i.e.,

1. $\hat{x}$ is feasible,
2. $u^i(\hat{x}^i) \geq u^i(x^i)$ for all $i \in \mathcal{I}$,
3. $u^{i'}(\hat{x}^{i'}) > u^{i'}(x^{i'})$ for some $i' \in \mathcal{I}$.

Pareto Optimality and Walrasian Equilibria

Proof (continued).

By revealed preference and Walras' law, $p \cdot \hat{x}^i \geq p \cdot x^i$ for all i, and $p \cdot \hat{x}^{i'} > p \cdot x^{i'}$. Thus

$$\sum_{i \in \mathcal{I}} p \cdot \hat{x}^i > \sum_{i \in \mathcal{I}} p \cdot x^i$$
$$\sum_{\ell \in \mathcal{L}} \sum_{i \in \mathcal{I}} p_\ell \hat{x}^i_\ell > \sum_{\ell \in \mathcal{L}} \sum_{i \in \mathcal{I}} p_\ell x^i_\ell.$$

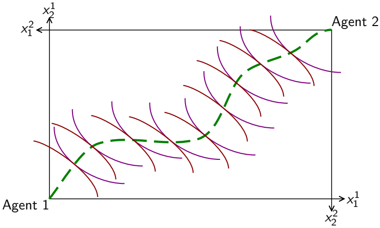
So for some $\tilde{\ell}$ it must be that

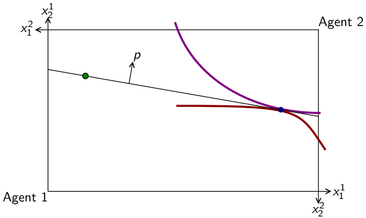
$$\sum_{i \in \mathcal{I}} \hat{x}^i_{\tilde{\ell}} > \sum_{i \in \mathcal{I}} x^i_{\tilde{\ell}} = \sum_{i \in \mathcal{I}} e^i_{\tilde{\ell}},$$

so $\hat{x}$ cannot be feasible. $\square$

Pareto Optimality and Walrasian Equilibria

- Is the converse true as well?
- Is that true (and, if so, under which conditions) that a planner can achieve any desired PO allocation by appropriately redistributing wealth (endowments) among consumers in a lump-sum fashion and the letting the market work?





- Can PO allocations be always supported as WE with prices that separate agents' upper contour sets?
- Yes, but only if a series of stringent convexity hypotheses (not required by the 1st WT) are satisfied. In particular we need all u_i to be continuous, strictly increasing and concave (i.e. UCS convex). Furthermore, we require agents to have a non-zero wealth (i.e. positive initial endowments).

Definition of Walrasian Equilibrium Once Again

Definition (Walrasian equilibrium)

Prices p and quantities $(x^i)_{i\in\mathcal{I}}$ are a Walrasian equilibrium iff

1. All agents maximize their utilities; i.e., for all $i \in \mathcal{I}$,

$$x^i \in \operatorname*{argmax}_{x \in B^i(p)} u^i(x);$$

2. Markets clear; i.e., for all $\ell \in \mathcal{L}$,

$$\sum_{i\in\mathcal{I}} x^i_\ell = \sum_{i\in\mathcal{I}} e^i_\ell.$$

Do Walrasian Equilibria Exist for Every Economy?

Theorem

Suppose for all $i \in \mathcal{I}$,

1. $u^i(\cdot)$ *is continuous;*
2. $u^i(\cdot)$ *is increasing; i.e.,* $u^i(x') > u^i(x)$ *for any* $x' \gg x$*;*
3. $u^i(\cdot)$ *is concave; and*
4. $e^i \gg \mathbf{0}$*; i.e., every agent has at least a little bit of every good.*

There exist prices $p \in \mathbb{R}^l_+$ *and allocations* $(x^i)_{i \in \mathcal{I}}$ *such that* p *and* x *are a Walrasian equilibrium.*

Excess Demand

Definition (excess demand)

The excess demand of agent i is

$$z^i(p) \equiv x^i(p, p \cdot e^i) - e^i,$$

where $x^i(p, w)$ is i's Walrasian demand correspondence.

Aggregate excess demand is

$$z(p) \equiv \sum_{i \in \mathcal{I}} z^i(p).$$

If $p \in \mathbb{R}_+^L$ satisfies $z(p) = \mathbf{0}$, then p and $\left(x^i(p, p \cdot e^i)\right)_{i \in \mathcal{I}}$ are a Walrasian equilibrium

Properties of Excess Demand I

$$z(p) \equiv \sum_{i \in \mathcal{I}} x^i(p, p \cdot e^i) - \sum_{i \in \mathcal{I}} e^i$$

Under the assumptions of our existence theorem ($u^i(\cdot)$ is continuous, increasing, and concave, and $e^i \gg \mathbf{0}$ for all i):

- $z(\cdot)$ is continuous
  - Continuity of u^i implies continuity of x^i

Properties of Excess Demand II

$$z(p) \equiv \sum_{i \in \mathcal{I}} x^i(p, p \cdot e^i) - \sum_{i \in \mathcal{I}} e^i$$

- $z(\cdot)$ is homogeneous of degree zero
  - $x^i(p, w^i)$ is homogeneous of degree zero
  - $x^i(p, p \cdot e^i)$ is homogeneous of degree zero in p
  - $z^i(p) \equiv x^i(p, p \cdot e^i) - e^i$ is homogeneous of degree zero
  - $z(p) \equiv \sum_i z^i(p)$ is homogeneous of degree zero

This implies we can normalize one price

Properties of Excess Demand III

$$z(p) \equiv \sum_{i \in \mathcal{I}} x^i(p, p \cdot e^i) - \sum_{i \in \mathcal{I}} e^i$$

- $p \cdot z(p) = 0$ for all p (Walras' Law for excess demand)
  - By Walras' Law, $p \cdot x^i(p, w^i) = w^i$
  - $p \cdot x^i(p, p \cdot e^i) = p \cdot e^i$
  - $p \cdot z^i(p) \equiv p \cdot \left(x^i(p, p \cdot e^i) - e^i\right) = 0$
  - $p \cdot z(p) \equiv p \cdot \sum_i z^i(p) = 0$

Suppose all but one market clear; i.e., $z_2(p) = \cdots = z_L(p) = 0$

$$p \cdot z(p) = p_1 z_1(p) + \underbrace{p_2 z_2(p) + \cdots + p_L z_L(p)}_{=0} = 0$$

by Walras' Law; hence $z_1(p) = 0$ as long as $p_1 > 0$

Thus if all but one market clear, the final market must also clear

Finding W.E. means solving $Z(p) = 0$

Consider a two-good economy

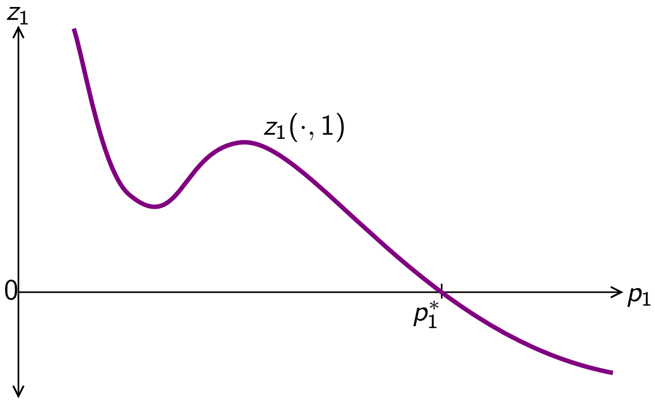
- Normalize $p_2 = 1$ by homogeneity of degree zero of $z(\cdot)$
- As long as the good one market clears, the good two market will as well (by Walras' Law)

We can find a W.E. whenever $z_1(p_1, 1) = 0$

- $z_1(\cdot, 1)$ is continuous
- As $p_1 \to 0$, excess demand for good one must go to infinity since preferences are increasing and $e_2^i > 0$ for all i
- As $p_1 \to \infty$, excess demand for good one must be negative since preferences are increasing and $e_1^i > 0$ for all i

Finding W.E. means solving $Z(p) = 0$

By an intermediate value theorem, there is at least one W.E.



Conclusions

- **Pure-exchange model**
  - Focusing on economies where goods cannot be produced and agents consume and exchange their endowments
  - Many results (existence of equilibrium, optimality, etc.) can be well understood in a simple two-consumer setups
- **Other issues still to be addressed**
  - Uniqueness: How many Walrasian equilibria are there?
  - Stability: How does an economy (as distinct from an economist) "find" equilibria?
  - Empirical validation: Can we take the Walrasian model to the data? What are empirically-testable implication of GET?
- **The road ahead**
  - Equilibrium with production: some simple examples
  - Addressing uniqueness, stability, and empirical validation (using the pure-exchange model again)