Advanced Microeconomics

General Equilibrium Theory (GET)

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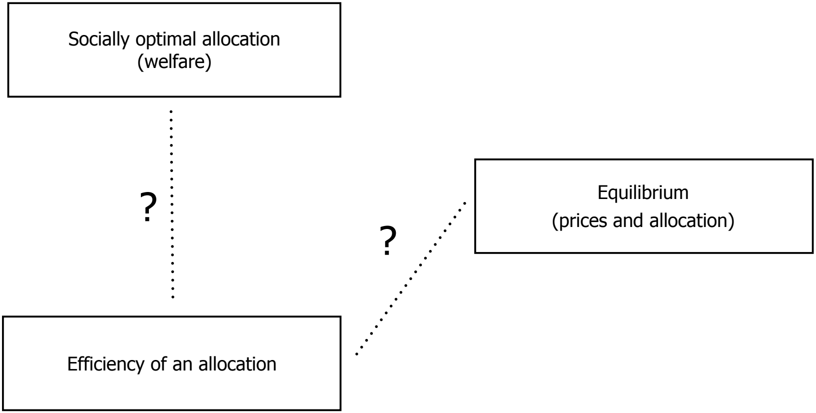
Part 1

This (and next) lecture(s)

- **Introducing the GE model of competitive markets**
  - Sketching GE model with consumers and firms
  - Defining efficiency, social welfare and equilibrium concepts
  - Understanding relation between efficiency and social welfare
- **Main concepts in a nutshell**
  - Feasible allocations
  - Pareto optimal (PO) allocation
  - Social-welfare function (SWF)
  - Competitive (Walrasian) equilibrium
  - Price normalization
  - Market clearing
  - Maximizing a SWF implies finding a PO allocation
  - Any PO allocations can be obtained as the result of the maximization of a SWF

Efficiency, welfare and equilibrium



The Model

Consider a model of a decentralized economy E with the following ingredients:

- Consumers: $i = 1, ..., I$
- Firms: $j = 1, ..., J$
- Commodities: $l = 1, ..., L$

Let us also assume that:

- Each consumer i has preferences $(\succeq_i, X_i)$ and an associated (continuous) utility function u_i which maps consumption bundles $x_i = (x_{1i}, ..., x_{Li}) \in X_i$ into $\mathbb{R}$ and holds an initial endowment vector $\omega_i \in \mathbb{R}^L_+$; let $(\omega_1, ..., \omega_I)$ and $\hat{\omega} = \sum_i \omega_i \in \mathbb{R}^L_+$
- Each firm j holds a technology $Y_j \subseteq \mathbb{R}^L$ and define $y_{\ell j}$ as the netput of firm j for commodity ℓ; hence, the "net amount" available for good ℓ will be: $\hat{\omega}_\ell + \sum_j y_{\ell j}$

The Model

Moreover, assume that:

- Private Ownership: Consumers own firms. Each consumer holds a share $\theta_{ij} \geq 0$ of firm j, s.t. $\sum_i \theta_{ij} = 1$, all j, i.e. $\theta_i \in [0, 1]^J$. Profits are entirely redistributed to consumers accordingly to shares.
- Markets are complete (there exist L markets for the L commodities).
- Commodities are undifferentiated (homogeneous). No firms have then advantage whatsoever in selling them and consumers cannot discriminate between commodities sold by different firms.
- There is perfect information about prices across agents. Consumers and firms are perfectly rational (maximizers) and act as price takers.
- All actual exchanges take place simultaneously at a single price vector, after the latter has been quoted. If a firm sells at lower prices, she will undercut competitors. Reselling is not allowed. Pricing is linear (every unit of commodities is sold at the same unit price).

As a result, the economy will be completely defined by:

$$E = \left(\{X_i, u_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}; \{\omega_i, \theta_i\}_{i=1}^{I} \right)$$

Allocation Concepts (Pre-Institutional)

Definition (Allocation)

An **allocation** is an $L \times (I + J)$ matrix $[(x_i)_{i=1}^{I}, (y_j)_{j=1}^{J}] \in \mathbb{R}_+^{I \cdot L} \times \mathbb{R}^{J \cdot L}$.

Definition (Conceivable Allocation)

An allocation is **conceivable** if and only if $x_i \in X_i$ and $y_j \in Y_j$ all i, j.

Definition (Feasible Allocation)

An allocation is **feasible** if and only if it is conceivable and $\sum_i x_{\ell i} \leq \hat{\omega}_\ell + \sum_j y_{\ell j}$, all $\ell = 1, ..., L$.

Definition (Pareto Optimal (Efficient) Allocation)

A feasible allocation is **(strongly) Pareto Optimal** (or **efficient**) if and only if there does not exist another feasible allocation s.t. at least one consumer is strictly better off, while all other consumers are not worse off. More formally: A feasible $(x_i)_{i=1}^{I}, (y_j)_{j=1}^{J}$ is **(strongly) Pareto Optimal** iff $\nexists$ a feasible $(x_i')_{i=1}^{I}, (y_j')_{j=1}^{J}$ s.t.

$$u_i(x_i^{'}) \geq u_i(x_i) \forall i \quad \text{and} \quad u_h(x_h^{'}) > u_h(x_h) \text{ some } h \in \{1, ..., I\} \qquad (*)$$

Pareto Optimality: Remarks

Remark

A feasible allocation $(x_i)_{i=1}^{I}, (y_j)_{j=1}^{J}$ *is* ***Weakly Pareto Optimal*** *(or* ***efficient****) if and only if* $\nexists$ *a feasible* $(x_i')_{i=1}^{I}, (y_j')_{j=1}^{J}$ *s.t.* $u_i(x_i') > u_i(x_i)\ \forall i$. *Prove that if* $(x_i)_{i=1}^{I}, (y_j)_{j=1}^{J}$ *is* ***(strongly) Pareto Optimal*** *then* $(x_i)_{i=1}^{I}, (y_j)_{j=1}^{J}$ *is* ***Weakly Pareto Optimal****.*

Remark

The concept of Pareto Optimality has no equity meaning, that is it does not say anything about how utility levels are distributed across consumers.

Social Welfare Function and Utility Possibility Set

Definition (Social Welfare Function)

A **Social Welfare Function** (SWF) is a function $W : \mathbb{R}^I \to \mathbb{R}$ that maps consumers' utility level vectors $(u_1, ..., u_I)$ into a social utility value $W(u_1, ..., u_I)$. Since we want social welfare be not decreasing in any individual utility level, we assume that: $\nabla W \gg 0$ everywhere.

Remark

In the following we will make use of the simplest ***Social Welfare Function****, i.e. the linear SWF (no cross effects are involved) defined as:*

$$W(u_1, ..., u_I; \lambda) = \sum_i \lambda_i u_i = \lambda u, \ \ \textit{with } \lambda \gg 0.$$

Definition (Utility Possibility Set)

The **Utility Possibility Set** U is the set of all attainable vectors of utility levels in the economy E, that is:

$$U = \{(u_1, ..., u_I) \in \mathbb{R}^I : \exists \text{ feasible } (x_i)_{i=1}^I, (y_j)_{j=1}^J : u_i \leq u_i(x_i) \ \forall i\}$$

Social Welfare Function and Utility Possibility Set

Remark

We assume throughout that U is closed and convex.

Definition (Utility Possibility Frontier)

The UPF of U is the set:

$$UPF = \{(u_1, ..., u_I) \in U : \nexists (u'_1, ..., u'_I) \in U : u'_i \geq u_i \forall i \text{ and } u'_h > u_h \text{ for some h } \}.$$

It is straightforward to show that:

$$\text{A feasible } (x_i)_{i=1}^{I}, (y_j)_{j=1}^{J} \text{ is PO } \Leftrightarrow (u_i(x_i))_{i=1}^{I} \in UPF.$$

Remark

Since U is closed, the UPF coincides with the boundary of U.

Competitive (Walrasian) Equilibrium

Definition (Competitive (Walrasian) Equilibrium)

An allocation $(x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J$ and a price vector $p^* \in \mathbb{R}_{++}^L$ is a **Competitive (or Walrasian) Equilibrium (CE)** for the economy E iff the following three conditions are satisfied:

1. (Profit Maximization): Given p^*, then $y_j^* = \arg\max(p^* y_j)$, s.t. $y_j \in Y_j$, $\forall j = 1, ..., J$
2. (Utility Maximization): Given Condition 1 and p^*, then $x_i^* = \arg\max u_i(x_i)$, s.t. $p^* x_i \leq p^* \omega_i + \sum_j \theta_{ij} p^* y_j^*$, $x_i \in X_i$, $\forall i = 1, ..., I$
3. (Market Clearing): $\sum_i x_{\ell i}^* \leq \sum_i \omega_{\ell i} + \sum_j y_{\ell j}^*$, $\forall \ell = 1, ..., L$

Remarks on Competitive (Walrasian) Equilibrium

Remark

The candidated allocation $(x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J$ *is not required in advance to be feasible (neither conceivable). However if* $\{(x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J, p^*\}$ *is a WE then by conditions* 1. *and* 2. *the allocation must be conceivable and by condition* 3. *must be feasible.*

Remark

Market-clearing condition holds as equality if local non-satiation is satisfied. Suppose not (i.e. local non-satiation implies (3) with $<$ *sign for some* l*). Then for some consumer* i *and good* l *by slightly increasing commodity-*l *consumption, consumer* i*, will be strictly better off while the new consumption bundle is still affordable. This violates condition 2.*

Remark

We suppose that preferences and technologies are such that any CE implies $p^* \gg 0$*. More on that below.*

Price Normalization

Proposition (Price Normalization)

If $(p^*, (x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J)$ is a CE and $p^* \gg 0 \Rightarrow$ For any scalar $\alpha > 0$, $(\alpha p^*, (x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J)$ is a CE.

Proof.

Let $p^{**} = \alpha p^*$. We claim that $(p^{**}, (x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J)$ is a CE. Condition 3 is still satisfied by $(p^{**}, (x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J)$ because it does not involve prices. We know that net supply function $y(p)$ is homo(0). This implies that $y(p^*) = y(p^{**})$ and that Condition 1 is satisfied by $(p^{**}, (y_j^*)_{j=1}^J)$. Also, demand functions $x_i(p)$ are homo(0) and hence $x_i(p^*) = x_i(p^{**})$, so that $(p^{**}, (x_i^*)_{i=1}^I)$ satisfies (2). □

Remark

Given the last result, it turns out that one can find equilibrium prices up to a normalization (i.e. one can find up to $L - 1$ independent entries in the CE price vector). Two normalizations are often employed: (1) We let $p^* \in Simplex(L) :: \sum_{\ell=1}^L p_\ell^* = 1$; or (2) We let $p^{**} = (p_1^*/p_L^*, ..., p_{L-1}^*/p_L^*, 1)$.

Market Clearing Condition

Proposition (Mkt. Clearing Condition)

Suppose for a given price $p \gg 0$ (not necessarily an equilibrium!) and for a conceivable allocation $(x_i)_{i=1}^{I}$, $(y_j)_{j=1}^{J}$, the following conditions are satisfied:

1. *Markets clear for all but one commodity, i.e.* $\sum_i x_{\ell i} = \sum_i \omega_{\ell i} + \sum_j y_{\ell j}, \forall \ell \neq k$;
2. *All consumers budget constraints hold with equality, i.e.* $px_i = p\omega_i + \sum_j \theta_{ij} py_j$, $\forall i = 1, ..., I$

Then commodity-k market also clears: $\sum_i x_{ki} = \sum_i \omega_{ki} + \sum_j y_{kj}$.

Market Clearing Condition (Cont'd)

Proof.

By summing up budget constraints in (2), and noting that $\sum_i \theta_{ij} = 1$, one gets:

$$\sum_i px_i = \sum_i p\omega_i + \sum_i \sum_j \theta_{ij} py_j$$

$$\sum_i px_i = \sum_i p\omega_i + \sum_j (\sum_i \theta_{ij}) py_j$$

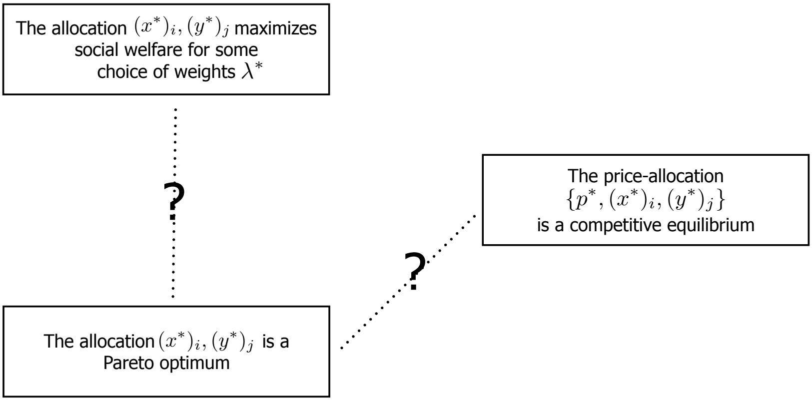
$$\sum_i px_i = \sum_i p\omega_i + \sum_j py_j$$

$$\sum_\ell p_\ell (\sum_i x_{\ell i} - \sum_i \omega_{\ell i} - \sum_j y_{\ell j}) = 0$$

$$\sum_{\ell \neq k} p_\ell (\sum_i x_{\ell i} - \sum_i \omega_{\ell i} - \sum_j y_{\ell j}) = -p_k (\sum_i x_{ki} - \sum_i \omega_{ki} - \sum_j y_{kj})$$

As the LHS of the last line is 0 by cond. 1, then as $p_k > 0$, it must be that $\sum_i x_{ki} = \sum_i \omega_{ki} + \sum_j y_{kj}$. □

Efficiency, welfare and equilibrium



From Social Welfare to Pareto Optimality

We can prove that if the SWF is linear, then all allocations associated to some utility levels vectors maximizing the SWF are PO. More formally:

Proposition (SWF Maximizers → PO Allocations)

If $u^* = (u_1^*, ..., u_I^*) = (u_1(x_1^*), ..., u_I(x_I^*))$ *solves the problem*

$$\max W(u_1, ..., u_I; \lambda) = \lambda \cdot u, \quad u \in U$$

and $\lambda \gg 0$*, then* $u^* \in UPF$*, i.e. the associated feasible allocation* $(x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J$ *is PO.*

Proof.

Suppose $(x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J$ is not PO. Then $\exists\ (u_1', ..., u_I') \in U : u_i' \geq u_i^*\ \forall i$ and $u_h' > u_h^*$ some h. As $\lambda \gg 0$, then $W(u'; \lambda) = \lambda u' > \lambda u^* = W(u^*; \lambda)$, contradicting the hypothesis that u^* solves the problem. □

From Pareto Optimality to Social Welfare

Conversely, we can show that if the UPS is convex and the SWF is linear, then any point in the UPF, associated to which we already know there is a PO allocation, maximizes the SWF for some particular choice of non-negative weigths λ.

Proposition (PO Allocations $\rightarrow$ SWF Maximizers)

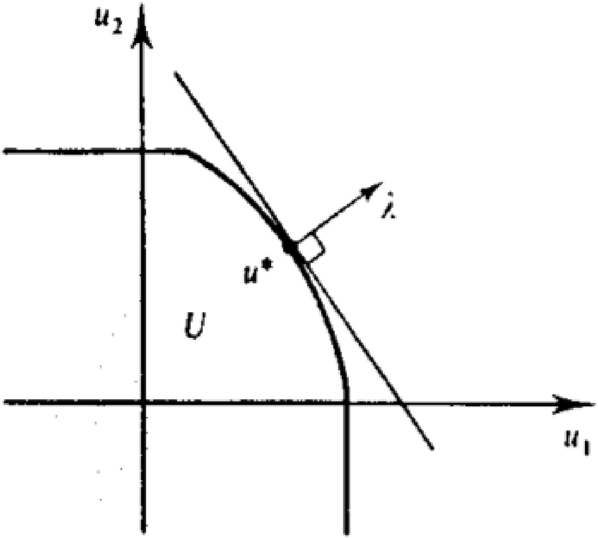
If U is convex and W is linear, then for any $\widetilde{u} \in UPF$, there exists a $\widetilde{\lambda} \geq 0$, $\widetilde{\lambda} \neq 0$, such that $\widetilde{u}$ solves

$$\max W(u_1, ..., u_I; \widetilde{\lambda}) = \lambda \cdot u, \quad u \in U$$

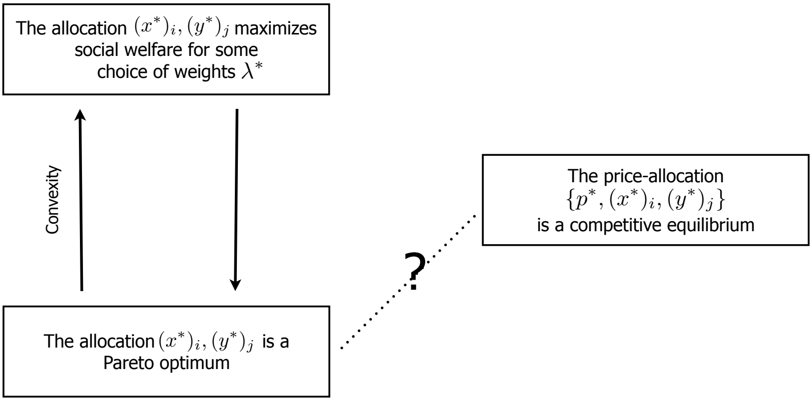
Proof.

Since $\widetilde{u} \in UPF$, U is convex and closed, then we can apply the Supporting Hyperplane Theorem. The theorem says that there will exist a $\widetilde{\lambda} \neq 0 : \widetilde{\lambda}\widetilde{u} \geq \widetilde{\lambda}u, \forall u \in U$. This implies that $\widetilde{u}$ solves the problem: $\max W(u_1, ..., u_I; \widetilde{\lambda}), \ \ u \in U$. We are left with proving that actually $\widetilde{\lambda} \geq 0$. Suppose not, e.g. that $\widetilde{\lambda}_i < 0$. Since any $u \leq \widetilde{u}$ still belongs to U, then take some u with an arbitrarily small $u_i \ll 0$. For that point: $\widetilde{\lambda}\widetilde{u} < \widetilde{\lambda}u$, contradicting the result that $\widetilde{u}$ solves the problem: $\max \lambda \cdot u, \ \ u \in U$. □

Efficiency, welfare and equilibrium



Efficiency, welfare and equilibrium



Efficiency, welfare and equilibrium

- **What's next...**
  - Studying relationships between competitive equilibria and Pareto optimality
  - Do competitive equilibria reach Pareto optimal allocations?
  - Can any Pareto optimal allocation be reached as an equilibrium outcome, i.e. does a price vector sustaining that PO allocation ever exist?

- **Three scenarios...**
  - Partial equilibrium: studying a market in isolation (given all other market prices)
  - General equilibrium in the pure-exchange case: all prices are endogenously set but goods cannot be produced, they come in fixed endowments that consumers exchange among them
  - General equilibrium with production: all prices are endogenously set, goods are produced and consumed