Duopoly Games

- Strategic situation where two firms operate in a non-competitive market
- Both firms know that they may have some “market power” unlike in “competitive markets”

**Competitive Markets (Short-Run)**

- Possibly infinite number of firms with cost function $c(q) = c \cdot q$
- Market price is taken as given
- Demand curve:
  \[ p(Q) = a - Q \]
- Profit maximization:
  \[ \max_q (p - c) \cdot q \quad \Rightarrow \quad p = c \]
- Optimal production:
  \[ p(Q^*) = c \quad \Rightarrow \quad Q^* = a - c \]
- Optimal profits:
  \[ \pi^* = 0 \]
Duopoly Games: Cournot (1/2)

- Cournot Games: The two firms can strategically set the **quantity** in such a way to max profits
- Both firms know that their choice is directly affected by what the other will be doing
- Each firm has a cost function \( c(q) = c \cdot q \) and faces a demand \( p(Q) = a - Q \), with \( c < a, Q < a \)

**Game Description**

- **Players:** \( I = \{1,2\} \)
- **Nature:** Not present
- **Action:** A level of production \( q_i \in R_+ \) for both \( i=1,2 \) \( (q_i < a) \)
- **Order of play:** Firm \( i \) sees \( j \)'s action only after \( j \) has moved (simultaneous)
- **Information Sets:** Nobody sees other’s choice when moving (simultaneous)
- **Strategies:** \( S_1 = S_2 = R_+ \)
- **Payoffs:** Profits \( \pi_i = p(Q) \cdot q_i - c \cdot q_i = p(q_1 + q_2) \cdot q_i - c \cdot q_i \)
  \[ = [a - (q_1 + q_2) - c] \cdot q_i \]
Duopoly Games: Cournot (2/2)

- Analysis of the game: Obtaining firms’ best-replies and finding Nash equilibrium
- Symmetry allows us to solve only the problem of one firm (say firm 1)

Firm 1 \[ \max_{q_1} \left[ a - (q_1 + q_2) - c \right] \cdot q_1 \Rightarrow BR_1(q_2) = \frac{a-c}{2} - \frac{q_2}{2} \]

Firm 2 \[ \ldots \] \[ BR_2(q_1) = \frac{a-c}{2} - \frac{q_1}{2} \]

\[ q^*_1 = q^*_2 = \frac{a-c}{3} \Rightarrow Q_C = \frac{2(a-c)}{3} \]

Thus: \( Q(\text{Cournot}) < Q(\text{competitive market}) \)\( p(\text{Cournot}) > p(\text{competitive market}) \)

Inefficient outcome!

Monopoly: \( Q_M = \frac{a-c}{2} < Q_C < Q^* = a-c \)
Duopoly Games: Bertrand (1/5)

- Bertrand Games: The two firms can strategically set the **price** in such a way to max profits
- Each firm has a cost function $c(q) = c \cdot q$ and faces a demand $D(p)$ decreasing in $p$
- Both firms know that
  - if they set the lowest price ($p^\text{\lowercase{l}}$) they can get the entire demand $D(p^\text{\lowercase{l}})$
  - If both set the same price ($p^\text{\lowercase{h}}$) they equally share the demand (each gets $D(p^\text{\lowercase{h}})/2$)

**Game Description**
- Players: $I = \{1, 2\}$
- Nature: Not present
- Action: A level of price $p_i \in \mathbb{R}^+$ for both $i = 1, 2$ (strictly positive prices)
- Order of play: Firm $i$ sees $j$’s action only after $j$ has moved (simultaneous)
- Information Sets: Nobody sees other’s choice when moving (simultaneous)
- Strategies: $S_1 = S_2 = \mathbb{R}^+$ (strictly positive prices)
- Payoffs: Profits $\pi_i = (p_i - c) X_i(p_i|p_j)$
  
  $X_i(p_i|p_j) = q$.ty sold by $i$ when she chooses $p_i$ and $j$ chooses $p_j$
Duopoly Games: Bertrand (2/5)

Game Description (Cont’d)

\[ X_i(p_i \mid p_j) = \begin{cases} 
D(p_i) & \text{se } p_i < p_j \\
D(p_i)/2 & \text{se } p_i = p_j \\
0 & \text{se } p_i > p_j
\end{cases} \]

- Behavior: Each firm maximizes profits given what she thinks the others will do

Solution of Bertrand’s Duopoly Game

Proposition: There exists a unique NE in pure strategies \((p_1^*, p_2^*)\) wherein both firms set prices \(p_1^* = p_2^* = c\) (marginal cost)

Proof:
Step 1: Showing that \((c, c)\) is a NE
Step 2: Showing that there cannot exist another NE \(\neq (c, c)\)
Step 1: Must show that if a firm chooses $p_j$ while the other chooses $p_k=c$, then $j$ has no incentive to move away from $p_j=c$

- By symmetry it is sufficient to show it for just one firm

- Consider firm 1 profits
  - If $p_1 < c$: $\pi_1(p_1, c) = (p_1 - c)D(p_1) < 0$
  - If $p_1 = c$: $\pi_1(p_1, c) = 0 \cdot D(c)/2 = 0$
  - If $p_1 > c$: $\pi_1(p_1, c) = (p_1 - c) \cdot 0 = 0$

- Thus firm 1 has no strict improvement in moving unilaterally away from $p_1=c$
Step 2: Must take any other pair \((p_1, p_2)\) in \(\mathbb{R}_+^2\) and show that it is not NE

Remark: By symmetry we can choose only \(p_2 \geq p_1\)

We have 5 cases:

<table>
<thead>
<tr>
<th>Condition</th>
<th>(\pi_1)</th>
<th>(\pi_2)</th>
<th>Then</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1 (p_1 &lt; c), (p_2 &gt; p_1)</td>
<td>(&lt;0)</td>
<td>0</td>
<td>Firm 1 wants to (\uparrow p_1)</td>
</tr>
<tr>
<td>A2 (p_2 &gt; p_1 &gt; c)</td>
<td>(&gt;0)</td>
<td>0</td>
<td>Firm 2 wants to (\downarrow p_2)</td>
</tr>
<tr>
<td>A3 (p_1 = p_2 &lt; c)</td>
<td>(&lt;0)</td>
<td>(&lt;0)</td>
<td>Both firms want to (\uparrow p_1) and (p_2)</td>
</tr>
<tr>
<td>A4 (p_1 = p_2 &gt; c)</td>
<td>(&gt;0)</td>
<td>(&gt;0)</td>
<td>Both firms want to (\downarrow p_1) and (p_2)</td>
</tr>
<tr>
<td>A5 (p_1 = c), (p_2 &gt; p_1)</td>
<td>0</td>
<td>0</td>
<td>Firm 1 wants to (\uparrow p_1) (marginally)</td>
</tr>
</tbody>
</table>

In all other pairs \((p_1, p_2)\) at least one firm wants to unilaterally change her price \(\Rightarrow (c, c)\) is the unique NE (in pure strategies)
Remarks

• Two assumptions are crucial:
  o D is C₀
  o p ∈ R₊ (firms can move smoothly prices)

• Extension to n>2 employs same argument
  o In any NE with n>2 all sales take place at price = MC
  o Multiple equilibria where k≤n firms choose p=c while others choose p>c may exist

• Bertrand Games imply Competitive (Efficient) Outcome. Why?
  o Game is one-shot (i.e. works as a competitive bidding with sealed bids where lowest bid takes all)
  o Each firm faces a perfectly elastic demand: if she sets a price lower than the one set by her competitor, their demand drops to 0

• The unique (c,c) NE involves firms playing Weakly-dominated strategies!
  o (c,c) is WD because there exists another p₁ ≠ c:
    \[ \pi₁(p₁,p₂) ≥ \pi₁(c,p₂) \] for all p₂ (and strictly for some p₂)
  o A WD strategy is “bad” NE because is not robust
  o There can be a way to avoid WD strategies (see below)
Extensions

1. Prices named in discrete units of account of size $\Delta>0$

   o Let $m \in \mathbb{N}$: $(m-1)\Delta \leq c < m\Delta$
   o $m$ is the smallest integer such that $m\Delta$ is strictly greater than MC
   o Then $p_1^* = p_2^* = m\Delta$ is a NE where $\pi_1 = \pi_2 > 0$
   o Moreover $(m\Delta, m\Delta)$ is not WD
   o As $\Delta \downarrow 0$, we recover Bertrand’s model

2. Heterogeneous technologies

   o Suppose $c_1 < c_2$ (Firm 1 has the best technology)
   o Firm 2 cannot set a price strictly lower than $c_2$ (otherwise profits<0)
   o Firm 1 wants to set a price which is smaller (but closer as possible) as $c_2$ because she takes the whole market and makes positive profits
   o The unique NE is the limit for $\varepsilon \downarrow 0$ of $(p_1, p_2) = (c_2 - \varepsilon, c_2)$