Game Theory

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Web Resources

- My homepage:

  `https://mail.sssup.it/~fagiolo/welcome.html`

- To be checked out:

  `http://www.gametheory.net`

Summary

1. Why Game Theory?
2. Cooperative vs. Noncooperative Games
3. Description of a Game
4. Rationality and Information Structure
5. Simultaneous-Move (SM) vs. Dynamic Games
6. Analysis
7. Examples
8. Problems and Suggested Solutions

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Why Game Theory? (1/2)

- Game Theory and Economics: What for?
  - Modeling direct interactions among economic agents
  - Strategic interdependence in multi-person contexts: expectations become crucial!
  - Benchmark: Competitive Market Model (no interactions at all)
  - Externalities: No equilibrium, inefficiency
- Plenty of Examples
  - Any consumption choice (despite economic theory does not tell you so!)
  - Any market exchange which is not mediated only through prices (bargaining, auctions)
  - Oligopoly markets where any firm must take into account what its competitors do
  - Almost all within-firm decisions and strategies (e.g., technological adoptions)
  - Public goods choices (contributing or not to a public good, cooperation)
  - Socially-shaped individual decisions involving norms and conventions (languages, currencies, car driving directions, accounting standards)
  - ... you name it ... in fact, a direct interaction is the rule not the exception in social sciences

Why Game Theory? (2/2)

- Goals of game theory
  - Modeling setups where the choice of any single economic agent directly depends on what s/he expects the others will be doing
  - Introducing a mathematical description of such situations
  - Figuring out what the outcomes of this interaction will be
  - Discussing the efficiency properties of this outcomes
- Rationality of economic agents becomes crucial
  - Agents are fully-rational
  - Agents know that the others are like them
  - Agents know all they need to know about the game
- Some basic definitions...
  - Game: Strategic situation involving a bunch of economic agents
  - Player: An economic agent in a game
  - Action, strategy, decision: What the player does in the game

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Cooperative vs. Noncooperative Games

- A "**cooperative**" game is a game where players can make binding commitments, as opposed to a "**noncooperative**" game, where they cannot

- In "**cooperative**" games, the units of analysis (primitives) are "groups". They are supposed to be able to reach cooperatively some desirable outcomes (for the group and each player) through binding commitments

- In "**noncooperative**" games, the unit is the player or the agent. Each player is a self-interested agent and cares about maximizing its utility in presence of strategic interdependences.
  - This does not mean that agents playing noncooperative games are not able to reach an outcome involving some cooperation! If they do so, however, cooperation is the outcome of competition among self-interested (egoistic) agents...

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Example: The Prisoner Dilemma Game

- Informal description of the game
  - Two individuals suspected to have committed a crime got arrested
  - The police does not have sufficient evidence against them and needs a confession
  - The two are kept apart and put in two separate rooms
  - Everyone is separately told that:
    * If no one confesses, they must stay in jail for 1 month (they are accused of a lighter crime)
    * If they both confess, they must stay in jail for 6 months
    * If only one confesses, the one who confesses is freed, the other must stay in jail for 9 months (6 months for the crime, 3 for not having confessed)
- The game from the point of view of the two suspected
  - Each one must independently choose whether to confess or not by maximizing his/her utility, on the base of what he/she expects the other will do
- The outcome of the game
  - What will the two suspected do? Do they confess (defect) or not (cooperate)?

Description of a Game (1/3)

- **Player**
  - Agent who makes decisions, who is involved in the game:

$$i \in I = \{1, 2, ..., N\}$$

  - Players maximize their utility by choosing actions under strategic interdependence

- **Nature**
  - Non-player who takes actions at specified points in the game according to specified *pdf*

- **Action**
  - An action is a choice that players can make
  - Action Set

$$i \to A_i$$

  - Typically: $A_i = \{a_i^1, ..., a_i^K\}, A_i = R_+$
  - Action combination or profile

$$\{a_1, a_2, ..., a_N\}$$

where for all $i \in I$

$$a_i \in A_i$$

Description of a Game (2/3)

- **Order of Play**
  - It specifies when actions in A_i becomes available to player i
- **Information Set**
  - What each player knows (e.g., values of variables) at each point in time
  - A value for a given variable enters the information set of a player if the player thinks that value may be attained by that variable at that point in time (can be distinguished by him)
- **Strategy**
  - It is a rule s_i that tells the player which action to choose at each point in time, given its information set
  - Strategies specify how the player **may** act in every possible distinguishable circumstance in which the player can find itself ("instruction book")
  - Agents hold a long list of strategies contained in the set of strategies:

$$i \to S_i$$

  - Strategy combination or profile

$$\{s_1, s_2, ..., s_N\}$$

  - In simplest games: strategies $\equiv$ actions

Description of a Game (3/3)

- **Payoffs**
  - They describe players' preferences over strategies

$$\pi_i : S_1 \times \cdots \times S_N \to R$$

where

$$\pi_i(s_1, s_2, ..., s_N)$$

can be interpreted as the utility player i **receives** (**expects**) after all players and Nature picked their strategies and the game has been played out.

  - Define the **outcome** of the game as a "set of interesting elements that the modeller picks from actions, payoffs, and other variables, *after the game has been played out*" (e.g., action profiles). Payoffs can be then mapped from the set of strategies to the set of outcomes wlog.

Example: The Prisoner Dilemma Game

Players	$I = \{1, 2\}$
Nature	Not present
Action	$A_i = A = \{C, NC\}$ ={"CONFESS","DO NOT CONFESS"}
Order of Play	Players move simultaneously
Information Sets	Nobody sees other's choice when moving
Strategies	$S_1: \quad s_1^1 = \{\text{C}\},\ s_1^2 = \{\text{NC}\}$ $S_2: \quad s_2^1 = \{\text{C}\},\ s_1^2 = \{\text{NC}\}$
Payoffs	Outcome (C,C): 1 gets -6; 2 gets -6 Outcome (C,NC): 1 gets 0; 2 gets -9 Outcome (NC,C): 1 gets -9; 2 gets 0 Outcome (NC,NC): 1 gets -1; 2 gets -1

Description of a Game: Extensive Form

- Description in terms of a "game tree"
  - who moves when
  - what actions each player can take at each point in time
  - what players know when they move
  - what the payoff are
- Example I: Matching Pennies 1.0

Players	$I = \{1, 2\}$
Nature	Not present
Action	$A_i = A = \{H, T\}$ ={"HEAD","TAIL"}
Order of Play	Player 1 moves first
Information Sets	2 sees 1's choice when it moves
Strategies	S_1 : $s_1^1 = \{\text{H}\}$, $s_1^2 = \{\text{T}\}$ S_2 : $s_2^1 = \{\text{H}\vert\text{H,H}\vert\text{T}\}$, $s_2^2 = \{\text{H}\vert\text{H,T}\vert\text{T}\}$, $s_2^3 = \{\text{T}\vert\text{H,H}\vert\text{T}\}$, $s_2^4 = \{\text{T}\vert\text{H,T}\vert\text{T}\}$
Payoffs	Outcomes (H,H),(T,T): 1 gets -1; 2 gets $+1$ Outcomes (H,T),(T,H): 1 gets $+1$; 2 gets -1

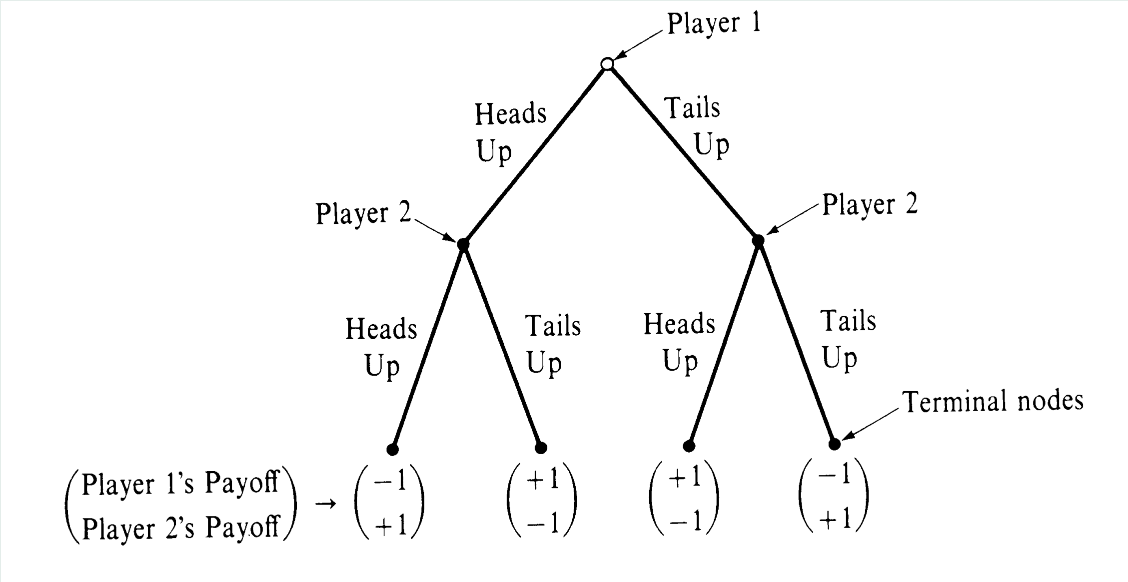

Figure 1: Matching Pennies V 1.0

- Example II: Matching Pennies 2.0

Players	$I = \{1, 2\}$
Nature	Not present
Action	$A_i = A = \{H, T\}$ ={"HEAD","TAIL"}
Order of Play	2 sees 1's choice **only after** it has moved
Information Sets	Nobody sees other's choice when moving (equivalent to simultaneous moves)
Strategies	$S_1: \quad s_1^1 = \{\text{H}\},\ s_1^2 = \{\text{T}\}$ $S_2: \quad s_2^1 = \{\text{H}\},\ s_2^2 = \{\text{T}\}$
Payoffs	Outcomes (H,H),(T,T): 1 gets -1; 2 gets $+1$ Outcomes (H,T),(T,H): 1 gets $+1$; 2 gets -1

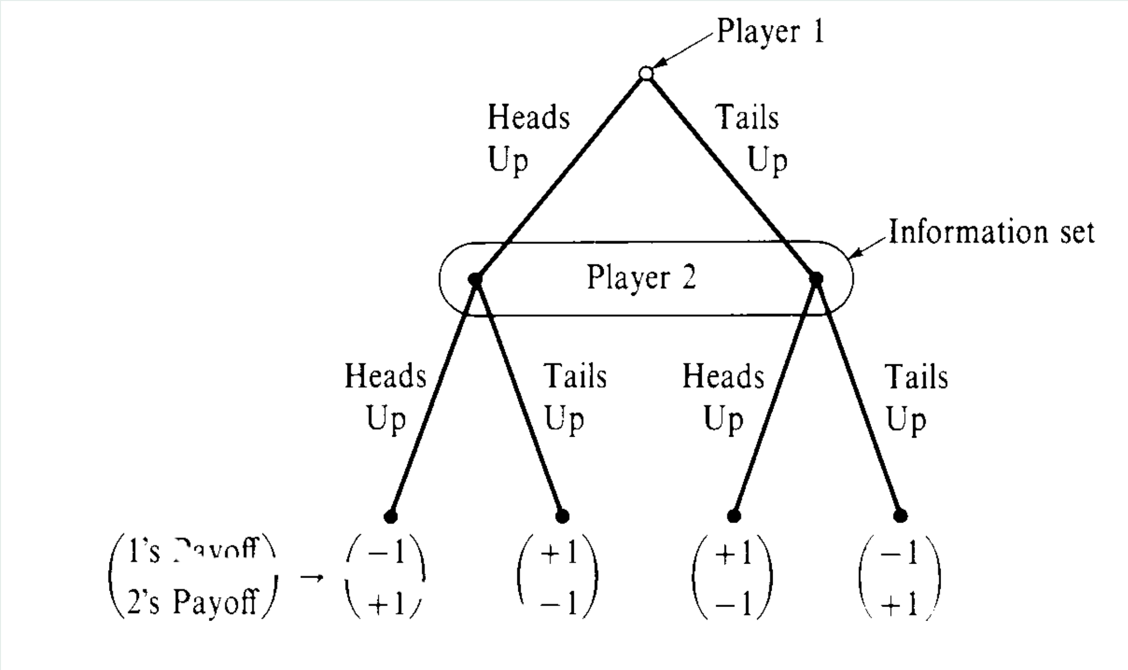

Figure 2: Matching Pennies V 2.0

- Example III: Matching Pennies 3.0

Players	$I = \{1, 2\}$
Nature	Chooses randomly who moves first
Action	$A_i = A = \{H, T\}$ ={“HEAD”,“TAIL”}
Order of Play	Player i moves first with $p = \frac{1}{2}$
Information Sets	$j \neq i$ sees i’s choice when it moves
Strategies	S_i : $s_i^1 = \{\text{H}\}$, $s_i^2 = \{\text{T}\}$ S_j : $s_j^1 = \{\text{H}\|\text{H,H}\|\text{T}\}$, $s_j^2 = \{\text{H}\|\text{H,T}\|\text{T}\}$, $s_j^3 = \{\text{T}\|\text{H,H}\|\text{T}\}$, $s_j^4 = \{\text{T}\|\text{H,T}\|\text{T}\}$
Payoffs	Outcomes (H,H),(T,T): i gets -1; j gets $+1$ Outcomes (H,T),(T,H): i gets $+1$; j gets -1

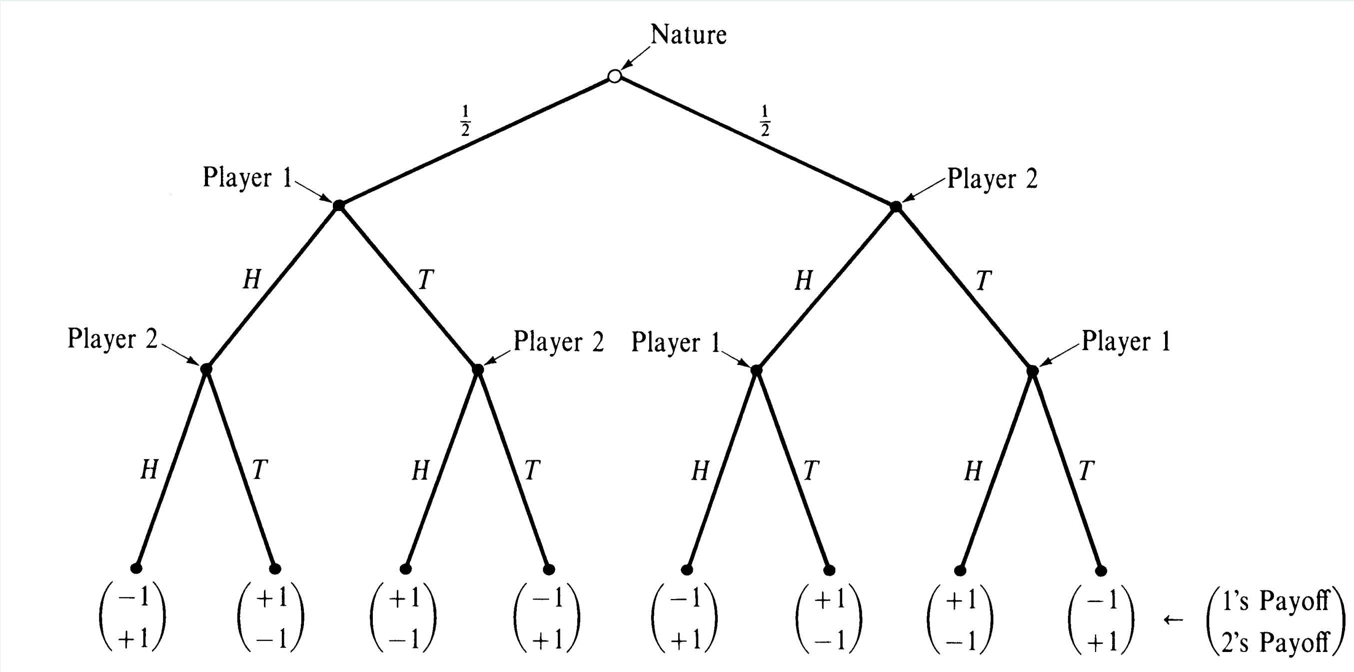

Figure 3: Matching Pennies V 3.0

Example: The Prisoner Dilemma Game

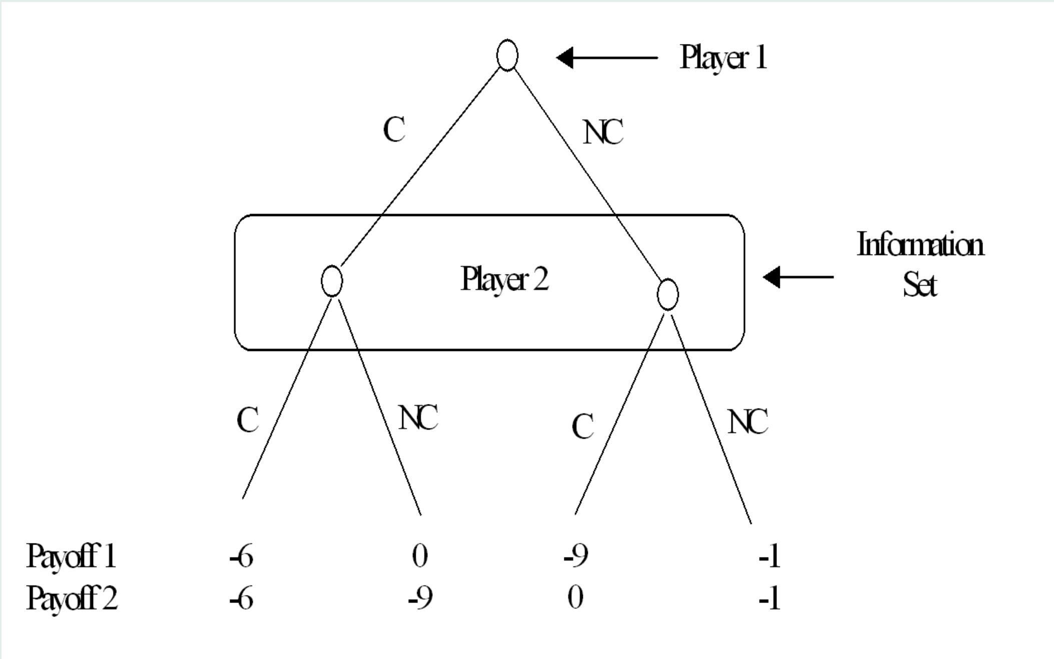

Figure 4: Prisoner Dilemma in Extensive Form

Description of a Game: Normal Form

- Description in terms of a strategy-payoff maps

$$S_1 \times S_2 \times \cdots \times S_N \quad \rightarrow \quad \pi_i, \qquad i = 1, 2, ..., N$$

- Examples:
  - Matching Pennies 1.0

		Player 2			
	(π_1, π_2)	s_2^1	s_2^2	s_2^3	s_2^4
Player 1	s_1^1	$(-1, +1)$	$(-1, +1)$	$(+1, -1)$	$(+1, -1)$
	s_1^2	$(+1, -1)$	$(-1, +1)$	$(+1, -1)$	$(+1, -1)$

  - Matching Pennies 2.0

		Player 2	
	(π_1, π_2)	H	T
Player 1	H	$(-1, +1)$	$(+1, -1)$
	T	$(+1, -1)$	$(-1, +1)$

Example: The Prisoner Dilemma Game

		Player 2	
	(π_1, π_2)	C	NC
Player 1	C	$(-6, -6)$	$(0, -9)$
	NC	$(-9, 0)$	$(-1, -1)$

Extensive vs. Normal Form Representations

- Extensive form (EF) allows one to appreciate the timeline and information structure
- Normal form (NF) projects the game in a "timeless" game setting
- From any EF there is only one NF representation
- Any NF can be compatible with many EF representations !
- For complicated games, the NF omits some details (not complete)
- For games studied here: EF and NF equivalent

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Rationality and Information Structure (1/2)

- Basic Assumptions
  - **Full Rationality**: Players
    - Max (expected) utility given strategic interdependence.
    - Hold unbounded computational skills.
  - **Common Knowledge**: Players

    H1 Know the structure of the game (players, actions, order of moves, strategies, information sets, **their own** payoffs)

    H2 Know others' payoffs

    H3 Know that also other players are fully rational, know that their opponents know that they know, ... and so on *ad infinitum*
  - **Perfect Recall**: Players do not forget what they once knew

Rationality and Information Structure (2/2)

Information Category	*Meaning*
Perfect	Each information set is a singleton
Symmetric	No player has information *structurally* different from other players
Complete	Nature does not move first

- Remarks
  - A game of perfect information has no simultaneous moves
  - Incomplete information implies imperfect information
    - Some players' information sets contain more than one node
  - Asymmetric information implies imperfect information
    - Info sets that differ across players cannot be singletons

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Simultaneous-Move (SM) vs. Dynamic Games

- Simultaneous-move games are games where:

  i. Nature *does not* move first (complete information)

  ii. Info sets *are not* singletons (imperfect information)

- A non-simultaneous move game is called a **dynamic game** because

  i. If nature moves first, no simultaneous moves can occur

  ii. If info is perfect, players act "knowing what happened before"

  iii. Credibility issues become crucial...

- In what follows: Focus on **Simultaneous-Move (Static) Games**
  - Players move only once and at the same time: Strategies and actions coincide
  - We employ normal-form representation: $\Gamma_N = [I, \{S_i\}, \{\pi_i\}]$

Example: The Adoption Game

- Informal description of the game
  - Two friends (1,2) must adopt one of the two existing technological standards
  - Example: Two competing operating systems for computers (X,Y)
  - They get utility from exchanging their files fast
  - It is known that X is more efficient than Y (e.g., in protecting against web viruses)
- The game from the point of view of the two friends
  - If they both end up with the same standard, they get a utility level higher than that they would get if they adopt different OS (say, -1)
  - If they both end up with X, they get 2; if they both end up adopting Y, they get 1.
- The outcome of the game
  - Which standards will the two friends adopt?

- Adoption Game 1.0

Players	$I = \{1, 2\}$
Nature	Not present
Action	$A_i = A = \{X, Y\}$
Order of Play	Simultaneous Moves
Information Sets	Nobody sees other's choice when moving
Strategies	$S_1 : \quad s_1^1 = \{\text{X}\},\ s_1^2 = \{\text{Y}\}$ $S_2 : \quad s_2^1 = \{\text{X}\},\ s_2^2 = \{\text{Y}\}$
Payoffs	Outcome (X,X): 1 gets $+2$; 2 gets $+2$ Outcome (Y,Y): 1 gets $+1$; 2 gets $+1$ Outcomes (X,Y),(Y,X): 1 gets -1; 2 gets -1

	(π_1, π_2)	Player 2 X	Y
Player 1	X	$(+2, +2)$	$(-1, -1)$
	Y	$(-1, -1)$	$(+1, +1)$

The Adoption Game: Extensions

- Adoption Game 2.0: Now agent 1 moves first and agent 2 sees his choice when moving
  - Describe the game in extensive and normal form
- Adoption Game 3.0: Go back to v1.0. Now, if and only if both agents choose the most efficient OS, the two friends join a group.
  - With probability 0.8 the group is made of X-users, which increases the payoffs of friends 1 and 2 to 10.
  - With probability 0.2 the group is made of Y-users, which decreases the payoffs of friends 1 and 2 to 0.
  - Describe the game in extensive and normal form