Nonparametric Time Series Analysis 1:

Measures of Complexity from Chaos Theory

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Part I

Reconstructing dynamical systems from time series

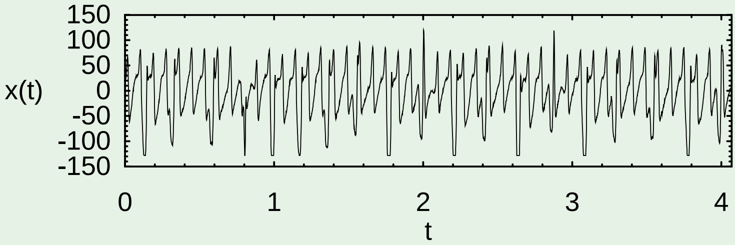
1. Time series
   - Introduction
2. Nonlinear dynamics and chaos
   - Dynamical systems
   - Logistic map
   - Sensitive dependence
   - Hénon
   - Lorenz
   - Rössler
3. Attractor reconstruction from time series
   - Takens theorem

Time series

Sequence of measurements over time, for example

- Historic temperature records
- Electrical activity on the heart (ECG)
- Stock prices

Example: Atrial fibrillation (Koefib217)



Time series analysis

Main goals of time series analysis

- Modelling & prediction
- Characterisation & classification

Approaches

- Linear versus nonlinear
- Parametric versus nonparametric

Time series methods from chaos theory

Correlation integrals

- Originally used for estimation of dynamic invariants (deterministic dynamics)
- Measure of complexity (Hoekstra et al.)
- Tests for independence (BDS test, 1987, 1996)
- Distance measures between distributions (Diks et al.)
- Information theoretic dependence measures

Nonlinear dynamics and chaos

Dynamical system

- State space Ω
- Time variable $t \in \mathbb{Z}$ (discrete) or $t \in \mathbb{R}$ (continuous)
- Evolution operator $\phi(\boldsymbol{x}, t)$, defines a map

$$\phi^t : \Omega \to \Omega, \qquad \phi^t(\boldsymbol{x}) = \phi(\boldsymbol{x}, t)$$

  (flow over time t)
- (semi-)group properties

$$\phi^0 = \mathrm{Id}, \qquad \phi^s \circ \phi^t = \phi^{s+t}$$

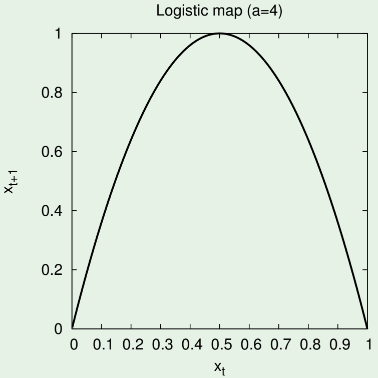
  $s, t \geq 0$ ($\forall s, t \in \mathbb{Z}$ resp. $\mathbb{R}$ for invertible dynamics)

The logistic map

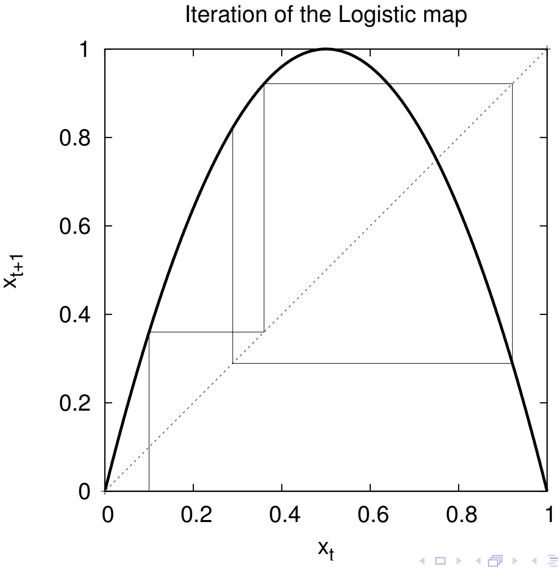
Example: logistic map

Map:

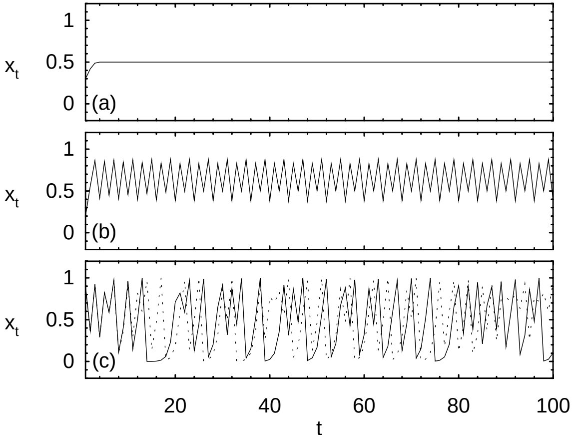
$$x_t = f(x_{t-1}) = a x_{t-1}(1 - x_{t-1}), \qquad 0 < a < 4$$



Iterating the logistic map

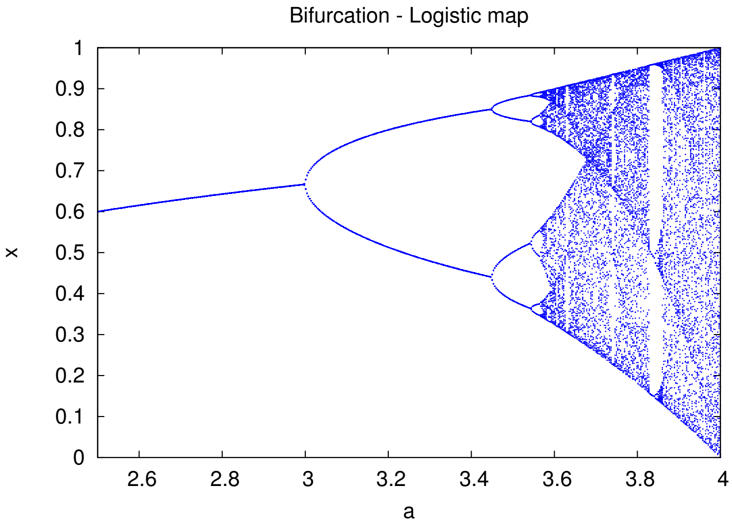


Qualitative behaviour depends on a



(a) $a = 2$, (b) $a = 3.5$, (c) $a = 4$

Bifurcation diagram for logistic map



Sensitive dependence on initial conditions

Near states may either diverge or converge in the long-run

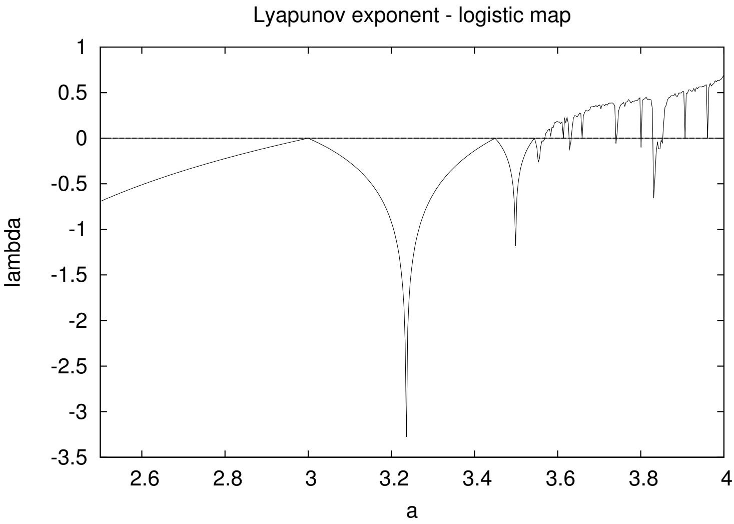
Average growth rate of near trajectories characterized by the Lyapunov exponent

Lyapunov exponent (for 1-dimensional maps)

$$\lambda = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \log \left| f'(x_t) \right| , \qquad \text{with } f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x}$$

If the system has a unique invariant measure, μ say, λ is the same for μ-almost all initial states $x_0 \sim \mu$

Lyapunov exponents



The Hénon map

Example: The Hénon map

$$\begin{aligned} x_{t+1} &= 1 - a x_t^2 + y_t \\ y_{t+1} &= b x_t \end{aligned}$$

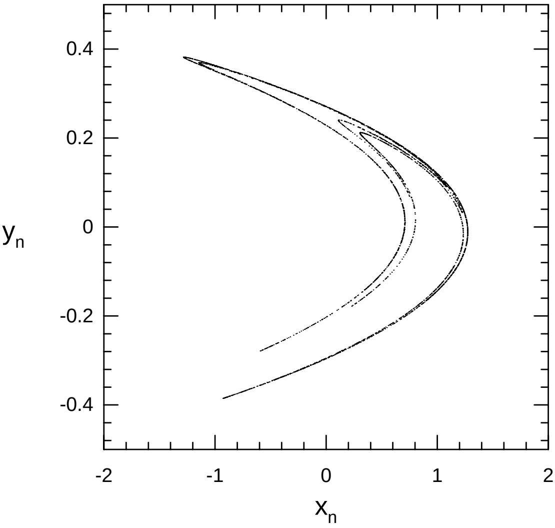
Default values: $a = 1.4$, $b = 0.3$

Jacobian

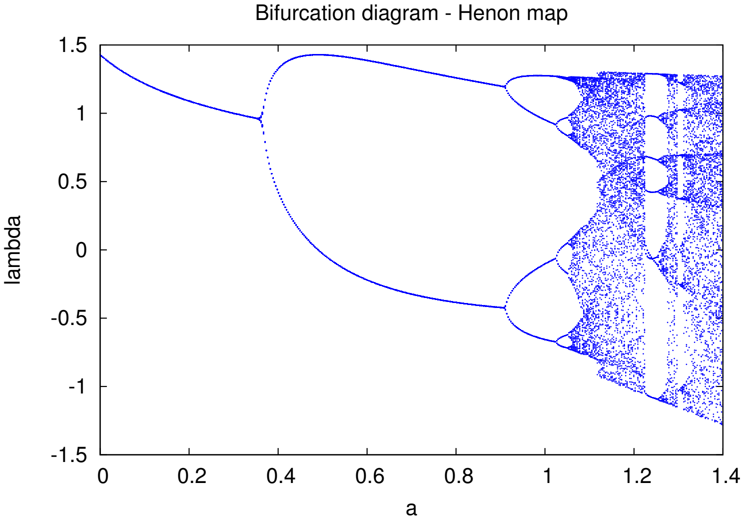
$$J = \begin{pmatrix} -2a x_n & 1 \\ b & 0 \end{pmatrix} \qquad \det J = -b$$

$\Rightarrow$ volume contracting for $|b| \leq 1$

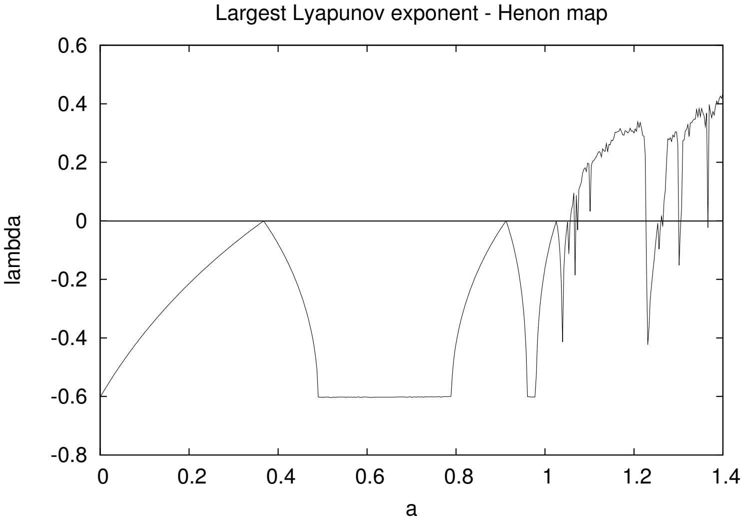
Phase plot – Hénon attractor



Bifurcation plot – Hénon map



Largest Lyapunov exponent – Hénon map



The Lorenz attractor

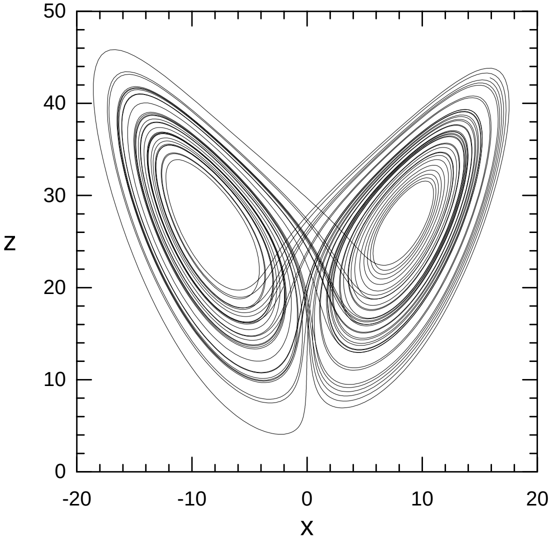
Lorenz system

$$
\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= (r - z)x - y \\
\dot{z} &= xy - bz
\end{aligned}
$$

Default parameter values

$$
\begin{aligned}
b &= 8/3, & & \\
\sigma &= 10, & & \text{(Prandtl number)} \\
r &= 28 & & \text{(Rayleigh number)}
\end{aligned}
$$

Lorenz butterfly



The Rössler dynamical system

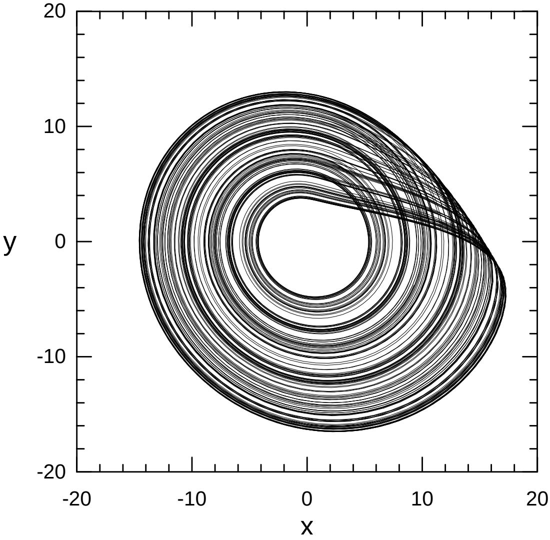
Rössler system

$$
\begin{aligned}
\dot{x} &= -y - z \\
\dot{y} &= x + ay \\
\dot{z} &= b + z(x - c)
\end{aligned}
$$

Standard parameter values

$$
\begin{aligned}
a &= 0.15, \\
b &= 0.2, \\
c &= 10
\end{aligned}
$$

Rössler attractor



Takens reconstruction theorem

The state space can be reconstructed from an observed scalar time series.

Main idea: starting from a given state, a unique time series pattern arises

Use consecutive data $(X_t, X_{t+\tau}, \ldots, X_{t+(m-1)\tau})$ to fix the state at time t

Under which conditions does this work? If it does, how should m and τ be chosen?

Takens reconstruction theorem

Assumptions (continuous time case):

- Unknown dynamical system

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{y}_t), \qquad \boldsymbol{y} \in \Omega$$

- Dynamics confined to finite dimensional compact subspace $M \subset \Omega$
- Observations are generated as $x_t = h(\boldsymbol{y}_t)$, at regularly spaced times $t = t_0 + k\tau$, (k integer) where h is a continuous measurement function

$$h : M \to \mathbb{R}, \quad \boldsymbol{y} \mapsto h(\boldsymbol{y})$$

- For flows: some requirements on the time interval τ (e.g. no periodic orbits with period $k\tau$)

Takens reconstruction theorem

Continuous time: flow over time interval τ is described by map

$$\phi^\tau : \Omega \to \Omega, \qquad \phi^t(\boldsymbol{y}) = \phi(\boldsymbol{y}, t)$$

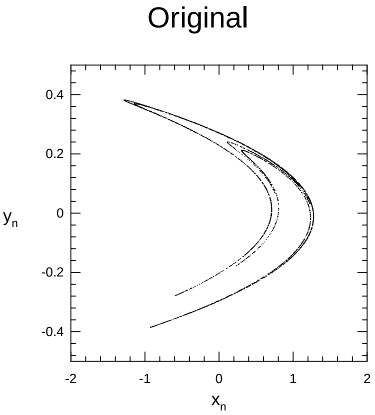
From now, use same notation for discrete and continuous time, with map $F : \Omega \to \Omega$ describing the flow over a time lag τ

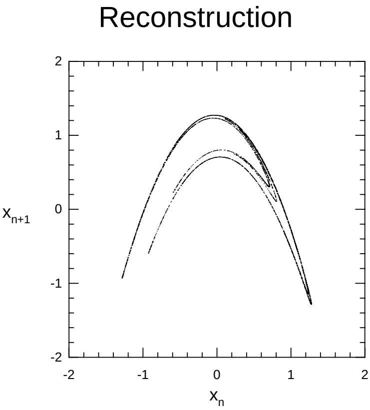
Note:

$$\boldsymbol{\Phi}_2 : \boldsymbol{y} \mapsto (h(\boldsymbol{y}), h(F(\boldsymbol{y})))$$

is a map from $M \subset \Omega$ to the plane, $\mathbb{R}^2$

Reconstruction of the Hénon attractor





Takens reconstruction theorem

Generalisation:

$$\boldsymbol{\Phi}_m : \boldsymbol{y} \mapsto (h(\boldsymbol{y}), h(F(\boldsymbol{y})), \ldots, h(F^{m-1}(\boldsymbol{y})))$$

is a map from $M \subset \Omega$ to $\mathbb{R}^m$

Takens reconstruction theorem

For smooth measurement functions h, and for m sufficiently large, $\boldsymbol{\Phi}_m$ generically is a smooth invertible map from M to $\mathbb{R}^m$ with a smooth inverse (diffeomorphism)

Reconstructed attractor

Corollary of the reconstruction theorem

Delay vectors $(x_t, x_{t+\tau}, \ldots, x_{t+(m-1)\tau})$ $\Omega \in \mathbb{R}^m$ lie on a faithful image of the attractor (for m large enough)

In case there is a finite dimensional attractor, the delay vectors also lie on a finite dimensional set, the reconstructed attractor

Takens: $m \geq 2\dim M + 1$ suffices

This bound can be improved further (Sauer et al., 1991): $m > 2$ boxdim A (box-counting dimension of the attractor)

For estimation of $\dim A$, only $m \geq \dim A$ is required

Pespective

- Control theory: Takens' theorem is an observability result (Aeyels)
- Generalisation of Whitney's embedding theorem (to a restricted set of maps from M to $\mathbb{R}^m$)

Part II

Correlation integrals

4. Correlation integrals
    - Correlation integrals

5. Dynamic invariants
    - GP algorithm

6. Correlation integrals for noisy data
    - Observational vs dynamic noise
    - Gaussian kernel correlation integrals

Characterising the reconstructed attractor

- Delay vectors $(X_t, X_{t+\tau}, \ldots, X_{t+(m-1)\tau})$ for stationary time series have a well-defined long-run distribution
- Reconstruction has an associated measure $\mu_{m,\tau}$, called reconstruction measure
- The dynamics has some properties that don't depend on the representation of the dynamics (dynamic invariants)
- For instance, correlation dimension and correlation entropy are independent on the representation

Correlation integrals

Definition of the correlation integral

The correlation integral of the reconstruction measure for embedding dimension m is

$$C_m(r) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \Theta(r - \|\mathbf{x} - \mathbf{y}\|)\, \mu(\mathrm{d}\mathbf{y})\, \mu(\mathrm{d}\mathbf{x})$$

with

$$\Theta(s) = \begin{cases} 0, & s < 0, \\ 1, & s \geq 0. \end{cases}$$

(Heaviside function)

Compact notation

$$C_m(r) = \int_{\mathbb{R}^m} \int_{\mathcal{B}_r(\mathbf{x})} \mu(\mathrm{d}\mathbf{y})\, \mu(\mathrm{d}\mathbf{x}) := \int_{\mathbb{R}^m} \mu\left(\mathcal{B}_r(\mathbf{x})\right)\, \mu(\mathrm{d}\mathbf{x})$$

Scaling law

For finite dimensional attractors: scaling relation

$$C_m(r) \sim e^{-mrK_2} r^{D_2}$$

- D_2 is called the correlation dimension. Geometric measure of complexity
- K_2 is known as the correlation entropy. Dynamical measure of complexity

Renyi spectrum

Renyi family of correlation integrals indexed by order q

$$\begin{aligned} C_{q,m}(r) &= \left(\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} \Theta(r - \|\boldsymbol{x} - \boldsymbol{y}\|)\, \mu(\mathrm{d}\boldsymbol{y}) \right)^{q-1} \mu(\mathrm{d}\boldsymbol{x}) \right)^{\frac{1}{q-1}} \\ &= \left(\int_{\mathbb{R}^m} \left(\mu\left(\mathcal{B}_r(\boldsymbol{x})\right)\right)^{q-1} \mu(\mathrm{d}\boldsymbol{x}) \right)^{\frac{1}{q-1}} \end{aligned}$$

Scaling law

$$C_{q,m}(r) \propto e^{-mrK_q} r^{D_q}$$

Directly generalises D_2 and K_2

Sufficient condition for chaos

Definition of chaos: $K_1 > 0$
Estimation of inner integral

$$\left(\int_{\mathbb{R}^m} \Theta(r - \|\boldsymbol{x} - \boldsymbol{y}\|) \, \mu(\mathrm{d}\boldsymbol{y}) \right)^{q-1}$$

problematic for $q = 1$. For $q \to 1$, one finds (l'Hopital)

$$C_{1,m}(r) := \lim_{q \to 1} C_{q,m}(r) = \int \ln \left(\int_{\mathcal{B}_r(\boldsymbol{x})} \mu(\mathrm{d}\boldsymbol{y}) \right) \mu(\mathrm{d}\boldsymbol{x})$$

$K_2 > 0$ implies K_1, hence chaos

Estimating correlation integrals

empirical reconstruction measure

The set of delay vectors with equal mass $\frac{1}{n}$ associated to each point (also called empirical delay vector distribution)

Empirical correlation integral

$$\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Theta(r-\|\boldsymbol{x}_i-\boldsymbol{x}_j\|)$$

(V-statistic)

$$\frac{2}{n(n-1)}\sum_{i=2}^{n}\sum_{j=1}^{i}\Theta(r-\|\boldsymbol{x}_i-\boldsymbol{x}_j\|)$$

(U-statistic)

Theiler correction

In practice we use

$$\frac{2}{(n-T)(n-T+1)}\sum_{i=T+1}^{n}\sum_{j=1}^{i-1}\Theta(r-\|\boldsymbol{x}_i-\boldsymbol{x}_j\|)$$

T is called the Theiler correction. $T=1$ corresponds to a U-statistic (no Theiler correction)

Statistically, the Theiler correction is a finite sample size correction

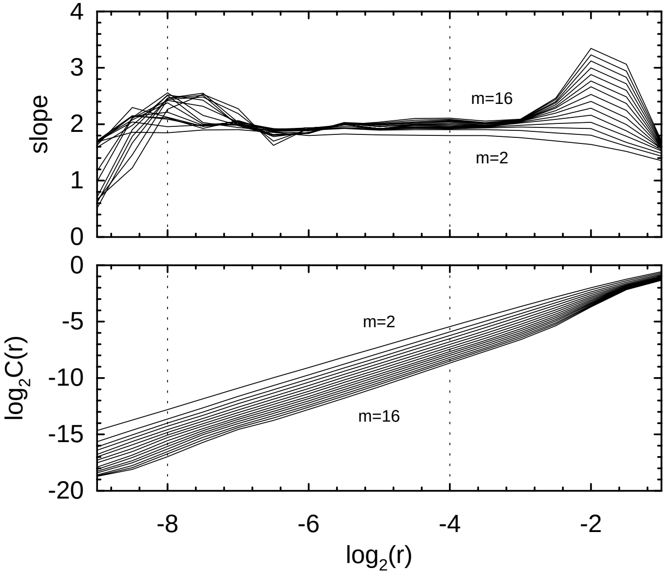
Grassberger-Procaccia method for estimating D_2 and K_2

1. Estimate the correlation integral for $m = 1, \ldots, m_{\max}$

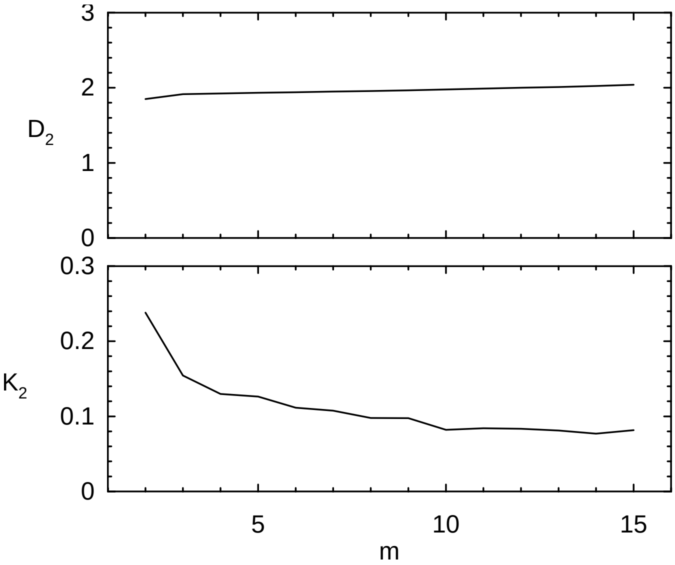
$$\widehat{C}_m(r) = \frac{2}{(n-T)(n-T+1)} \sum_{i=T+1}^{n} \sum_{j=1}^{i-1} \Theta(r - \|\mathbf{x}_i - \mathbf{x}_j\|)$$

   for a range of r-values, typically $r = ca^k$ for integer k
2. Look for a 'scaling region' of r-values
3. Estimate D_2 and K_2 using the estimated correlation integrals from the scaling region
4. Check for convergence with m

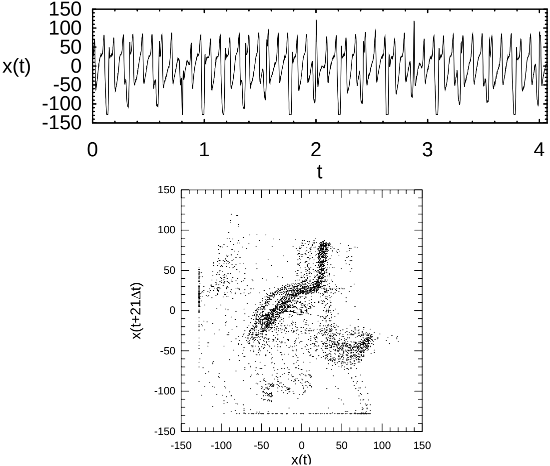
Correlation integrals for the Rössler



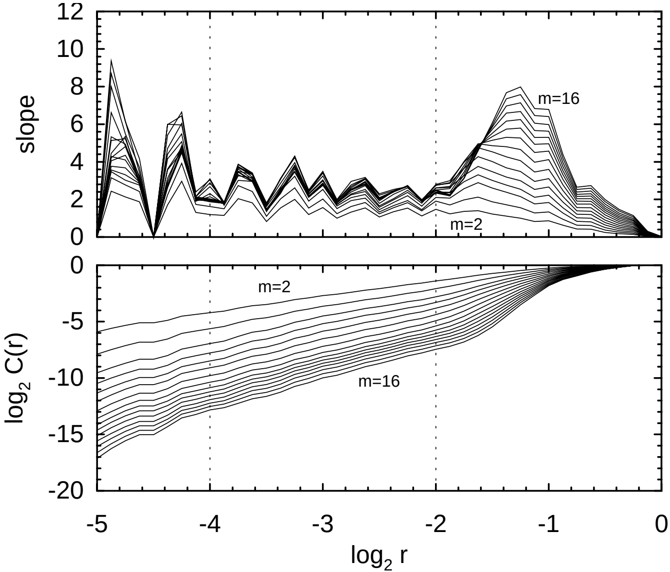
Estimated correlation dimension and entropy



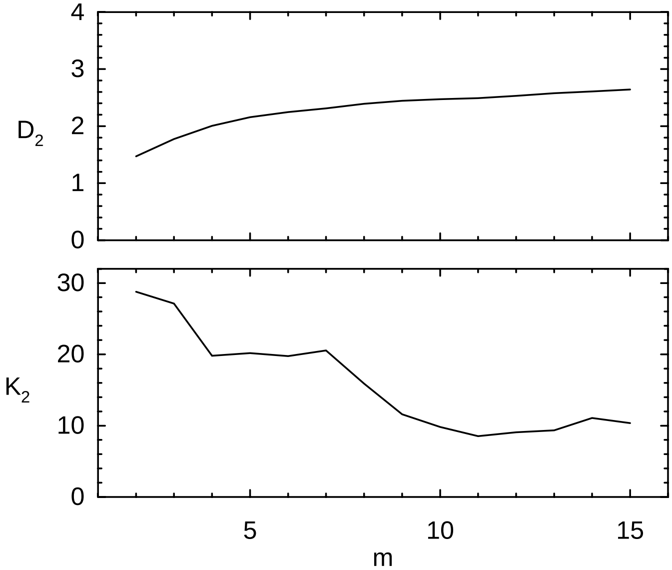
Application to atrial fibrillation



Atrial fibrillation: correlation integrals



Atrial fibrillation: estimated D_2 and K_2



Observational vs dynamic noise

Observational noise

$$\mathbf{Y}_t = F(\mathbf{Y}_{t-1})$$

$$X_t = h(\mathbf{Y}_t) + \varepsilon_t,$$

Dynamic noise

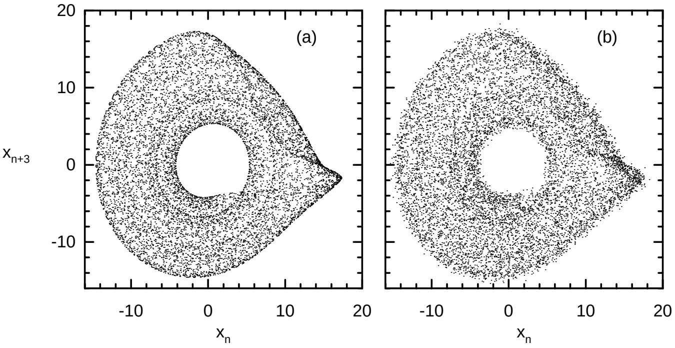
$$\mathbf{Y}_t = F(\mathbf{Y}_{t-1}, \varepsilon_t),$$

For example:

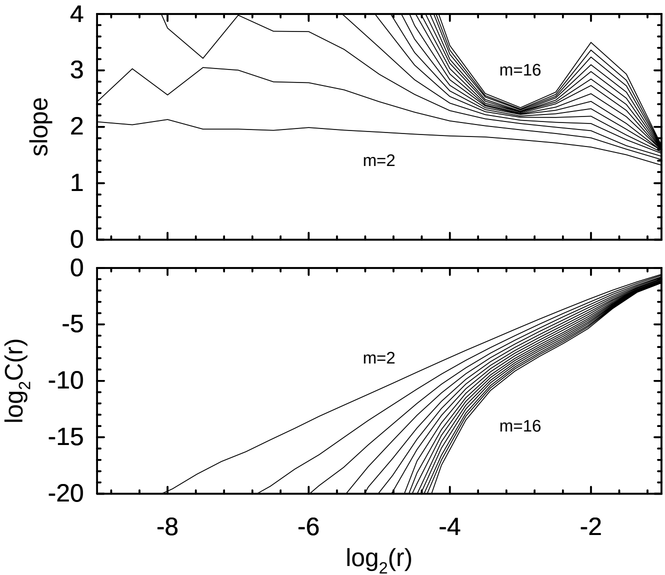
$$\mathbf{Y}_t = F(\mathbf{Y}_{t-1}) + \varepsilon_t,$$

$$X_t = h(\mathbf{Y}_t)$$

Phase plots from clean and noisy (5%, normal) Rössler data



Correlation integrals for noisy Rössler data



Effect of observational noise on reconstruction measure

Delay vector (m-dimensional)

$$\boldsymbol{X}_t^m := (X_{t-m+1}, \ldots, X_t)$$

is replaced by

$$\boldsymbol{X}_t^m := (X_{t-m+1} + \varepsilon_{t-m+1}, \ldots, X_t + \varepsilon_t)$$

$\Rightarrow$ The 'clean' reconstruction measure μ_m^0 is replaced by $\mu_m = \mu_m^0 \circ \nu_m$ (convolution) where ν_m denotes the noise reconstruction measure

Correlation integrals under observational noise

Correlation integral

$$
\begin{aligned}
C_m(r) &= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} I_{[0,r]}(\|\boldsymbol{x} - \boldsymbol{y}\|)\mu(\mathrm{d}\boldsymbol{x})\mu(\mathrm{d}\boldsymbol{y}) \\
&= \int_{\mathbb{R}^m} \cdots \int_{\mathbb{R}^m} I_{[0,r]}(\|\boldsymbol{x} + \boldsymbol{v} - \boldsymbol{y} - \boldsymbol{w}\|)\nu(\mathrm{d}\boldsymbol{v})\nu(\mathrm{d}\boldsymbol{w})\mu^0(\mathrm{d}\boldsymbol{x})\mu^0(\mathrm{d} \\
&= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} I_{[0,r]}(\|\boldsymbol{z} + \boldsymbol{s}\|)\nu(\mathrm{d}\boldsymbol{s})\xi(\mathrm{d}\boldsymbol{z})
\end{aligned}
$$

with

$$\eta(A) = \mathbb{P}[\boldsymbol{V} - \boldsymbol{W} \in A], \quad (\boldsymbol{V}, \boldsymbol{W} \sim \nu \text{ independent})$$

and

$$\xi(A) = \mathbb{P}[\boldsymbol{X} - \boldsymbol{Y} \in A], \quad (\boldsymbol{X}, \boldsymbol{Y} \sim \mu^0 \text{ independent})$$

Correlation integrals under observational noise (ctd)

Clean correlation integral:

$$C_m^0(r) = \int_{\mathbb{R}^m} I_{[0,r]}(\|\mathbf{z}\|)\xi(\mathrm{d}\mathbf{z}) = \int_0^r \mathrm{d}C_m^0(s)$$

Correlation integral in presence of observational noise

$$C_m(r) = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} I_{[0,r]}(\|\mathbf{z}+\mathbf{s}\|)\nu(\mathrm{d}\mathbf{s})\xi(\mathrm{d}\mathbf{z})$$

Gaussian observational noise already difficult analytically

- Inner integral difficult to evaluate
- Generally, noisy CI is not a functional of the 'clean' CI

Gaussian kernel correlation integral

Although the CI generally is not a functional of the 'clean' CI, it is if $\|\cdot\|$ is the Euclidean norm. In that case we can define:

$$\int_{\mathbb{R}^m} I_{[0,r]}(\|\boldsymbol{z}+\boldsymbol{s}\|)\nu_m(\mathrm{d}\boldsymbol{s}) := g_r(\|\boldsymbol{z}\|),$$

and hence

$$C_m(r) = \int_{\mathbb{R}^m} g_r(\|\boldsymbol{z}\|)\xi(\mathrm{d}\boldsymbol{z}) = \int_0^\infty g_r(s) f^0_{\|\boldsymbol{z}\|}(s)\,\mathrm{d}s = \int_0^\infty g_r(s)\mathrm{d}C^0_m(s)$$

More generally, whenever the kernel function used for calculating the CI depends on the Euclidean distance, the noisy CI is a functional of the noise-free CI

Relations between 'noisy' and 'clean' correlation integrals

The 'classical' correlation integral is of the general form

$$T_m^0(h) = \int_0^\infty K_{m,h}(s) f^0_{\|\mathbf{z}\|}(s)\,\mathrm{d}s$$

with $K_{m,h}(s) = I_{[0,h]}(s)$.

$$T_m(r) = \int_0^\infty g_r(s) f^0_{\|\mathbf{z}\|}(s)\,\mathrm{d}s$$

If $g_r(s)$ is of the same form as the kernel function used for calculating the correlation integral, the noisy and clean CI's then will be automatically simply related.

The Gaussian kernel

For

$$K_{m,h}(s) = e^{-\frac{1}{2}\frac{s^2}{h^2}}$$

$$\begin{aligned} g_r(\|\mathbf{z}\|) &= \int_{\mathbb{R}^m} e^{-\frac{1}{2h^2}\|\mathbf{z}+\mathbf{s}\|^2} \nu_m(\mathrm{d}\mathbf{s}) \\ &= (2\pi\sigma)^{\frac{m}{2}} \int_{\mathbb{R}^m} e^{-\frac{1}{2h^2}\|\mathbf{z}+\mathbf{s}\|^2 - \frac{1}{2\sigma^2}(s_1^2+\ldots+s_m^2)} \mathrm{d}s_1 \cdots \mathrm{d}s_n \\ &= \prod_{i=1}^{m} \int_{\mathbb{R}} e^{-\frac{1}{2h^2}(z_i+s_i)^2 - \frac{1}{2\sigma^2}s_i^2} \mathrm{d}s_1 \cdots \mathrm{d}s_n \end{aligned}$$

Since

$$\int_{\mathbb{R}} e^{-\frac{1}{2h^2}(z+s)^2 - \frac{1}{2\sigma^2}s^2} = \left(\frac{h^2}{h^2+\sigma^2}\right)^{\frac{1}{2}} e^{-\frac{1}{2(h^2+\sigma^2)}} ds$$

one finds

$$g_r(\|\mathbf{z}\|) = \left(\frac{h^2}{h^2+\sigma^2}\right)^{\frac{m}{2}} e^{-\frac{\|\mathbf{z}\|^2}{2(h^2+\sigma^2)}}$$

Modified scaling law in the presence of noise

Gaussian kernel correlation integral in the presence of noise

$$T_m(h) = \left(\frac{h^2}{h^2+\sigma^2}\right)^{\frac{m}{2}} T_m^0(h^2+\sigma^2)$$

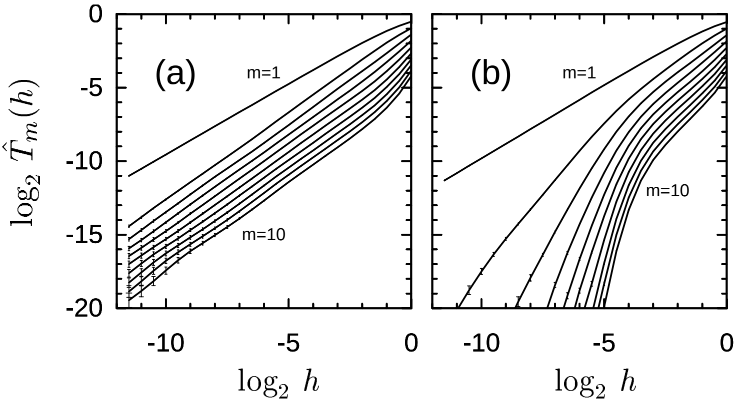
The noise-free gaussian kernel behaves similar to the usual

$$T_m^0(h) \simeq \mathrm{cnst} \times m^{-\frac{D}{2}} e^{-Km} h^D.$$

Modified scaling law

$$T_m(h) \simeq \mathrm{cnst} \times m^{-\frac{D}{2}} \times \left(\frac{h^2}{h^2+\sigma^2}\right)^{\frac{m}{2}} e^{-Km} (h^2+\sigma^2)^{\frac{D}{2}}$$

Noisy and noise-free Hénon CI



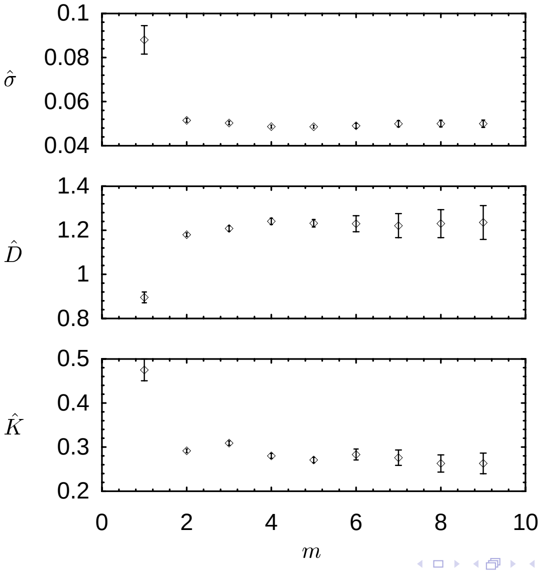
Estimation of D_2 and K_2 in presence of noise

Nonlinear least squares (Levenberg-Marquardt), either

- Weighted least squares. Requires standard errors and possibly covariances of $\widehat{T}_m(h)$.
- We used $\mathrm{Var}\widehat{T}_m(h) \propto \widehat{T}_m(h)(2 - \widehat{T}_m(h))$.

- Unweighted
- Implicitly also imposes weights (log-log scale, lin-lin scale)

Estimated invariants from noisy Hénon CI



Course-grained quantities

Even if assuptions made so far (determinism + gaussian observational noise) are violated, one may use correlation integrals

Idea is to use behavior of correlation integrals as a measure of complexity:

- Large slope of $\ln C_m(r)$ indicates large number of relevant state variables
- Difference $\ln C_m(r) - \ln C_{m+1}(r)$ is a measure of unpredictability at scale r

Motivates examining course-grained quantities

Definition of course-grained quantities

Course-grained correlation dimension and entropy

Coarse-grained correlation dimension

$$D_2(m, r) = \frac{d \ln C_m(r)}{d \ln r}$$

Course-grained correlation entropy

$$K_2(m, r) = \ln C_m(r) - \ln C_{m+1}(r)$$

Part III

Local linear prediction

7 Prediction by analogy

8 Casdagli method

Prediction by analogy, nearest neighbour method

Reconstruct state space, state vectors $\boldsymbol{X}_s^m$, $1 \le s \le n$
Idea is to exploit

$$X_{t+\ell} = g(\boldsymbol{X}_t^m)$$

To make ℓ-step-ahead forecast from $\boldsymbol{X}_t^m$ we need an approximation (estimate) of g

- find k nearest neighbours $\boldsymbol{X}_{s_i}^m$ among $\boldsymbol{X}_s^m$, $s < t$ or $s \neq t$
- locally constant model leads to prediction by averaging

$$\widehat{X}_{t+\ell} = \sum_{i=1}^{k} X_{s_i+\ell}$$

- alternatively, construct local linear map through linear regression (requires $k > m$). Model:

$$X_{t+\ell} = a_0 + \sum_{j=1}^{m} a_j X_{s_i+1-j} + \varepsilon_{t+\ell}, \quad i = 1, \ldots, k$$

Prediction by analogy, kernel methods

Comparable to k-nearest neighbour method, only using all points with weights determined by distance
Prediction

$$\widehat{X}_{t+\ell} = \sum_s w_{s,t} X_{s+\ell}$$

weights w_i determined by distance in state space

$$w_{s,t} = \frac{K_h(\boldsymbol{X}_s^m - \boldsymbol{X}_t^m)}{\sum_s K_h(\boldsymbol{X}_s^m - \boldsymbol{X}_t^m)}.$$

Prediction by analogy, other methods

- Polynomials (global nonlinear model)
- Neural networks (global nonlinear)
- Radial basis functions (local, linear in coefficients)

$$F(\boldsymbol{x}) = \alpha_0 + \sum_i \alpha_i \Phi(\|\boldsymbol{x} - \boldsymbol{x}_i\|)$$

e.g. $\Phi(\boldsymbol{s}) = 1/(1 + e^{\boldsymbol{bx} - c})$

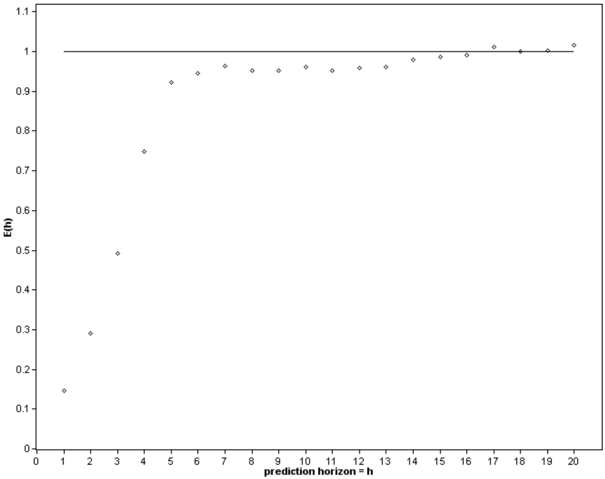
Parameter estimation by error backpropagation (gradient descent)

Casdagli method

- Divide the data in a fitting set $x_1, \ldots, x_{N_f}$ and a testing set $x_{N_f+1}, \ldots, x_{N_f+N_t}$
- Vary k in the nearest neighbour method, for each k:
- Choose a number of random reference points to predict from
- Determine prediction error $e_i(k) = |\widehat{x}_{i+\ell}(k) - x_{i+\ell}|$
- Repeat to determine RMSE

$$E_m(k) = \left(\frac{1}{n} \sum_i e_i^2(k) \right)^{\frac{1}{2}} / \sigma$$

Example Casdagli method (Hénon data)



$m = 3$, $\varepsilon = 0.1$, 200 predictions

- Takens' reconstruction theorem, correlation integrals, fractal dimensions, correlation entropy, estimation of dynamic invariants, local linear prediction
- *Nonparametric tests based on correlation integrals*: Divergences between reconstruction measures, U-statistics estimators, tests for symmetry of multivariate distributions, attractor comparison, testing for reversibility, tests for serial independence and linearity
- *Statistical aspects of nonparametric tests*: asymptotic results for U-statistics in time series context, bootstrap and Monte Carlo tests, consistency, bandwidth selection problem, diagnostic model checking, nuisance parameters, tests based on empirical copulas
- *Granger causality tests*: conditional independence, linear versus nonlinear Granger causality, nonparametric Granger causality tests, testing for Granger causality using correlation integrals, consistency, local measures of conditional dependence