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New Results on Betting Strategies, Market Selection, and the Role of Luck

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New Results on Betting Strategies, Market Selection, and the Role of Luck

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Abstract

We consider a repeated betting market populated by two agents who wage on a binary event according to generic betting strategies. We derive new simple criteria to establish the relative wealth of the two agents in the long run, only based on the odds they believe fair and how much they would bet when the odds are equal to the ones the other agent believes fair. Using our criteria, we show that for a large class of betting strategies it is generically possible that the ultimate winner is only decided by luck. As an example, we apply our conditions to the case of CRRA betting.

Keywords: Bounded Rationality, Betting Strategies, Market Selection, Recurrent Processes, Luck.

JEL Classification: C60, D53, G11, G12

1 Introduction

Consider a repeated market for betting where two agents wage on the outcomes of a binary event. Agent behavior is described by generic betting strategies that depend upon prevailing odds. If the odds are fixed, the strategy that guarantees optimal wealth growth is the Kelly rule (Kelly, 1956; Breiman, 1961). If the odds depend on agents' bets via the parimutuel procedure, Beygelzimer et al. (2012) show that in a population of Kelly bettors, the one with the most accurate beliefs accrues all the wealth and asymptotically *dominates* the market. This is a particular case of the result derived by Blume and Easley (1992) and Blume and Easley (2009) in the equivalent setting of inter-temporal general equilibrium models with short lived securities. In a similar framework, Evstigneev et al. (2002) clarify that an agent adopting the Kelly strategy and having correct beliefs will dominate the economy irrespective of the strategies adopted by other agents. The global dominance of the Kelly strategy with perfect information being established, it remains to understand what happens when agents do not bet according to Kelly and/or are not perfectly informed. Kets et al. (2014) consider the case of fractional Kelly and CRRA bettors, providing a few tentative results based on numerical simulations. Bottazzi and Giachini (2016) derive sufficient and, apart from hairline cases, necessary conditions for strategy dominance or survival in a market of fractional Kelly bettors. In the present paper we propose more general criteria that can be applied to generic strategies depending on prevailing market odds. The new criteria are simple and abstract from strategy specific details. It is sufficient to know the odds bettors consider fair and the amount each bettor is willing to bet when the odds are equal to the ones the other bettor believes fair to understand if one bettor will eventually dominate the market or, conversely, if the two bettors will asymptotically retain a finite, and fluctuating, amount of wealth. Interestingly, by considering generic strategies, one recovers the traditional role of luck in the game of chance, largely neglected in the previously mentioned studies. Indeed, in our general setting, it is generically possible that the ultimate fate of a bettor is not only decided by the adopted strategy, but also by the specific realized sequence of binary events. As an example, we apply the new criteria to the case of CRRA bettors.

2 Model

Consider two agents who make a sequence of bets against each other on binary events. The outcome of the event $s_t \in \{0, 1\}$ is an independent Bernoulli trial with success probability π^*: $s_t = 1$ means that the event occurs while $s_t = 0$ that it does not. In each round, agent $i \in \{1, 2\}$ has to choose the fraction of wealth to be wagered b^i_t and the side of the bet $\sigma^i_t \in \{0, 1\}$, where 1 means betting on the occurrence of the event while 0 betting against it. We assume the amount bet is redistributed among the winners according to the parimutuel procedure, that is, proportionally to how much they have bet, without any house-take. Let p_t be the prevailing inverse odd ratio at round t for the occurrence of the event. Thus if $s_t = 1$ the agent betting on the occurrence of the event receives $1/p_t$ times the amount bet while if $s_t = 0$ the agent betting against the occurrence of the event receives $1/(1 - p_t)$ times the amount bet. Agents' betting strategies are based on prevailing odds and they try to maximize their gain by increasing their bet when they perceive favorable opportunities. Following Kets et al. (2014), we assume that for each agent i there exists a "fair" inverse odd $\bar{p}^i \in (0,1)$ and a continuous function $b^i \in [0,1)$ such that $\sigma_t^i = 1$ if $p_t < \bar{p}^i$, $\sigma_t^i = 0$ if $p_t > \bar{p}^i$, $b^i(\bar{p}^i) = 0$ and $b^i(p_t) > 0$ when $p_t \neq \bar{p}^i$.[1] Without loss of generality we set $\bar{p}^1 < \bar{p}^2$. Thus, if w_{t-1}^i is the wealth of agent $i \in \{1,2\}$ before the event at time t is realized, the prevailing inverse odd p_t is set by the equation

$$w_{t-1}^1 b^1(p_t) = w_{t-1}^2 b^2(p_t) \tag{1}$$

being always $\sigma_t^1 = 0$ and $\sigma_t^2 = 1$. We require that the functions b^i are such that (1) admits one and only one solution. This is for instance the case if they are monotonic, strictly concave or strictly convex on the set of attainable prices. The amount of wealth that is not bet is invested in a risk-less asset that pays no interest. Hence, after the event at round t is realized, the wealth of agents is updated according to

$$w_t^i = \left(1 - b^i(p_t)\right) w_{t-1}^i + \delta_{s_t,i-1} w_{t-1}^i b^i(p_t) \left(\frac{\delta_{i,1}}{1-p_t} + \frac{\delta_{i,2}}{p_t} \right) \tag{2}$$

where $\delta_{a,b}$ is the Kronecker delta. Since the house takes no fee, the aggregate wealth is constant and we set $w_t = w_t^1 + w_t^2 = 1$ such that $p_t \in [\bar{p}^1, \bar{p}^2]$ and $p_t = \bar{p}^i$ if and only if $w_t^i = 1$.

3 Long-run Selection

The dynamics of wealth described by (2) can lead to two different outcomes: either a single agent accrues all the wealth and dominates the market or both agents indefinitely survive, each with a positive, and fluctuating, fraction of wealth. In general, the fate of an agent could depend on the specific sequence of realizations of the random variable s_t. Let $\sigma = \{s_1, s_s, \dots\}$ denote a realization of the Bernoulli process and let $w_t^i(\sigma)$ be the associated sequence of agent i's wealth. The long-term outcomes of the repeated betting are formalized in the following.

Definition 3.1. Agent i (asymptotically) *dominates* on σ if $\lim_{t\to\infty} w_t^i(\sigma) = 1$. Agent i (asymptotically) *survives* on σ if $\limsup_{t\to\infty} w_t^i(\sigma) > 0$.

Agent i (asymptotically) *dominates* if $\lim_{t\to\infty} w_t^i(\sigma) = 1$ for almost all σ. Agent i (asymptotically) *survives* if $\limsup_{t\to\infty} w_t^i > 0$ for almost all σ.

Notice that dominance implies survival. If one agent dominates, the other cannot survive and we say that it *vanishes.* At the same time, if one agent survives the other cannot dominate.

[1] We rule out the possibility that agents bet all their wealth as this would lead them to wealth zero almost surely

We will show that in order to decide survival or dominance of agents, it is not generically necessary to know all the details of the investment strategies, but simply the Bernoulli probability π^*, the inverse odds considered fair by the two agents, $\bar{p}^i$, and two positive numbers, $b^1(\bar{p}^2)$ and $b^2(\bar{p}^1)$, representing the fraction of wealth one agent bets if the odds are equal to those the other agent would consider fair.

Proposition 3.1. *Consider the quantities*

$$\mu^1 = \pi^* \log \frac{\bar{p}^1 + (1-\bar{p}^1) b^2(\bar{p}^1)}{\bar{p}^1} + (1-\pi^*) \log(1 - b^2(\bar{p}^1)) \tag{3}$$

and

$$\mu^2 = -\pi^* \log(1 - b^1(\bar{p}^2)) - (1-\pi^*) \log \frac{1 - \bar{p}^2 + \bar{p}^2 b^1(\bar{p}^2)}{1-\bar{p}^2} \ . \tag{4}$$

If agents' betting strategies satisfy the requirements of Section 2, then

i) if $\mu^1 > 0$ and $\mu^2 > 0$ agent 2 dominates and $\lim_{t\to\infty} p_t = \bar{p}^2$ almost surely;

ii) if $\mu^1 < 0$ and $\mu^2 < 0$ agent 1 dominates and $\lim_{t\to\infty} p_t = \bar{p}^1$ almost surely;

iii) if $\mu^2 < 0$ and $\mu^1 > 0$ both agents survive;

iv) if $\mu^2 > 0$ and $\mu^1 < 0$ either agent 1 dominates or agent 2 dominates depending on the realization of the Bernoulli process.

Proof. Consider the process $\{z_t = \log(w_t^2/w_t^1)\}$ and notice that $\mu^1 = \lim_{z\to-\infty} E[z_{t+1} - z_t | z_t = z]$ and $\mu^2 = \lim_{z\to+\infty} E[z_{t+1} - z_t | z_t = z]$. Define the (conditional) increment $g(p,s) = z_{t+1} - z_t$ when $p_t = p$ and $s_{t+1} = s$. From (2) remembering that, by hypothesis, $b^1(p)$ and $b^2(p)$ cannot be both zero for the same p and are continuous, it is immediate to see that

$$\log \frac{1 - B^2}{1 + B^1 \bar{p}^2/(1-\bar{p}^2)} < g(p,0) < 0 < g(p,1) < \log \frac{1 + B^2(1-\bar{p}^1)/\bar{p}^1}{1 - B^1} \ ,$$

where $B^i = \max\{b^i(x) | \bar{p}^1 \leq x \leq \bar{p}^2\}$. Thus the increments g are finite and bounded and Theorems 2.2, 3.1 and 3.2 of Bottazzi and Dindo (2015) can be applied to the process $\{z_t\}$. If $\mu^1 > 0$ and $\mu^2 > 0$ then $\lim_{t\to\infty} z_t = +\infty$, whence *i)*. If $\mu^1 < 0$ and $\mu^2 < 0$ then $\lim_{t\to\infty} z_t = -\infty$, whence *ii)*. If $\mu^2 < 0$ and $\mu^1 > 0$ then there exists a finite interval A such that $z_t \in A$ almost surely for any t, whence *iii)*. If $\mu^2 > 0$ and $\mu^1 < 0$ then on any Bernoulli sequence either $\lim_{t\to\infty} z_t = +\infty$ or $\lim_{t\to\infty} z_t = -\infty$, whence *iv)* □

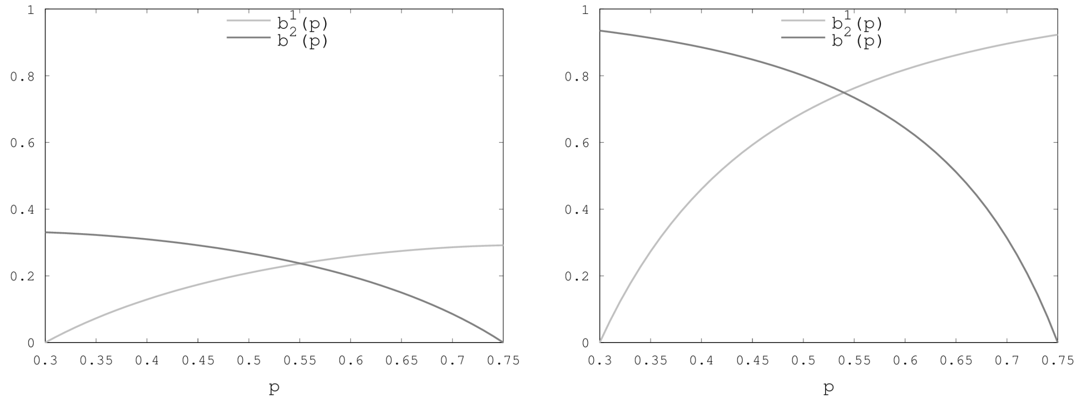

Figure 1: Agents' betting strategies. In both panels we set $\bar{p}^1 = 0.3$ and $\bar{p}^2 = 0.75$, in the first plot we have $\gamma^1 = \gamma^2 = 2$ while in the second it is $\gamma^1 = \gamma^2 = 0.5$.

It is immediate to see that if $\pi^* > \bar{p}^2$ we are in case $i)$ while if $\pi^* < \bar{p}^1$ we are in case $ii)$, recovering a result in Kets et al. (2014).[2] Notice that the dominance of one agent in case $iv)$ is not (only) realized on specific zero-measure sequences, like the sequence of 1's or the sequence of 0's, but on sets of sequences with finite probability. This is where the luck enters into the picture: both agents might dominate, but only the blind goodness will decide who.

4 Example with CRRA Bettors

The betting strategies introduced in Section 2 are flexible enough to accommodate several behavioral prescriptions. As an illustrative example, we consider the case in which agents bet to maximize the expected utility of wealth using a power utility function with Constant Relative Risk Aversion (CRRA). Call $\gamma^i > 0$ the relative risk aversion coefficient of agent i and π^i the subjective probability (belief) that agent i assigns to the realization of the event, which is precisely the inverse odd that agent i would consider fair. Assuming $\pi^1 < \pi^2$, for $p_t \in [\pi^1, \pi^2]$ agent 1 bets against the occurrence of the event a fraction of wealth b^1 which maximizes $\pi^1(1-b^1)^{1-\gamma^1} + (1-\pi^1)(1-b^1 p_t/(1-p_t))^{1-\gamma^1}$ to obtain

$$b^1(p_t) = \frac{(p_t(1-\pi^1))^{\frac{1}{\gamma^1}} - (\pi^1(1-p_t))^{\frac{1}{\gamma^1}}}{(p_t(1-\pi^1))^{\frac{1}{\gamma^1}} + p_t\,(\pi^1)^{\frac{1}{\gamma^1}}\,(1-p_t)^{\frac{1-\gamma^1}{\gamma^1}}}\ . \tag{5}$$

[2]The definitions of survival and dominance in Kets et al. (2014) are weaker than the ones adopted here. Given the relative simplicity of the considered process, however, their conclusions are still valid under Definition 3.1.

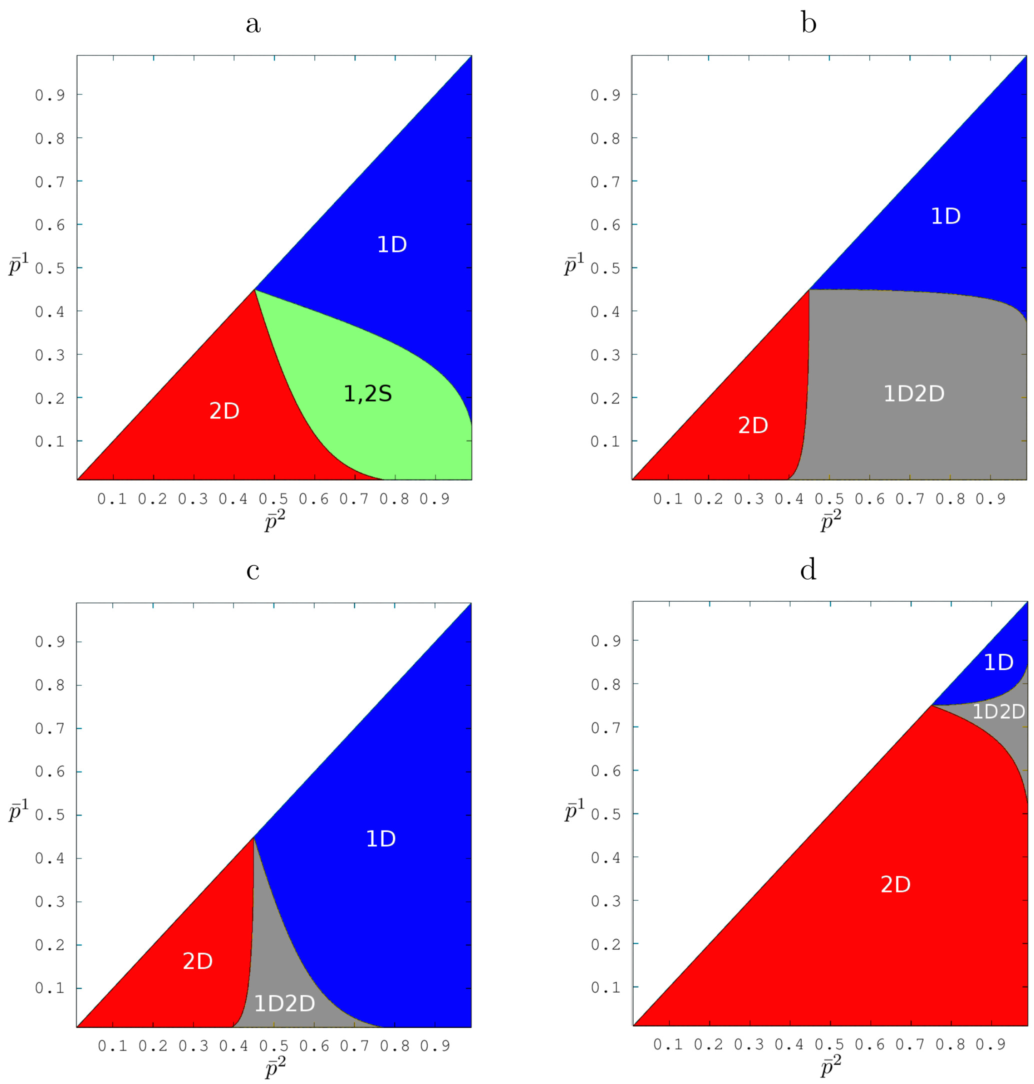

Figure 2: Dominance, survival and vanishing for different combinations of fair inverse odd ratios $\bar{p}^i$. 1D: agent 1 dominates; 2D: agent 2 dominates; 1,2S: both agents survive; 1D2D: either agent 1 or agent 2 dominates. Panel a: $\pi^* = 0.45$, $\gamma^1 = \gamma^2 = 2$; panel b: $\pi^* = 0.45$, $\gamma^1 = \gamma^2 = 0.5$; panel c: $\pi^* = 0.45$, $\gamma^1 = 2$, $\gamma^2 = 0.5$; panel d: $\pi^* = 0.75$, $\gamma^1 = 0.5$, $\gamma^2 = 2$.

Conversely agent 2 bets in favor of the realization of the event a fraction of wealth b^2 which maximizes $\pi^2(1+b^2(1-p_t)/p_t)^{1-\gamma^2} + (1-\pi^2)(1-b^2)^{1-\gamma^2}$ to obtain

$$b^2(p_t) = \frac{(\pi^2(1-p_t))^{\frac{1}{\gamma^2}} - (p_t(1-\pi^2)^{\frac{1}{\gamma^2}}}{(\pi^2(1-p_t))^{\frac{1}{\gamma^2}} + (1-p_t)\,(1-\pi^2)^{\frac{1}{\gamma^2}}\,(p_t)^{\frac{1-\gamma^2}{\gamma^2}}} \,. \tag{6}$$

The positive risk aversion implies that agents never bet the totality of their wealth. Figure 1 provides two examples of how agents' betting strategies vary depending on the inverse odd ratio. In the effective price support, betting strategies are always continuous and strictly concave. Figure 2 reports the long-run selection outcomes inferred using the conditions from Proposition 3.1. Depending on agents' risk aversion and beliefs any case of Proposition 3.1 may generically occur. Notice how low risk aversion and asymmetric beliefs enhance the role of luck in deciding the ultimate winner.

5 Conclusion

In this paper we consider a market for bets where two agents repeatedly wage on an uncertain event with two possible outcomes using generic betting strategies. We propose simple criteria to decide about the asymptotic amount of wealth of the two bettors, based on a few quantities: their fair odds and the amount they are willing to bet at the fair odds of the opponent. When generic betting strategies are considered, three outcomes are possible in the long-run: 1) one bettor accrues all the wealth with probability 1; 2) both bettors survive with a positive and fluctuating amount of wealth or 3) one of the two bettor eventually accrues all the wealth with finite probability. In the third case, luck recovers the role of ultimate arbiter, traditionally attributed to it in games of chance. Notice that if one confines the analysis to specific families of strategies, like Kelly or fractional Kelly strategies, the third outcome becomes non-generic or disappear (Bottazzi and Giachini, 2016). This explains why it was largely unobserved or not satisfactorily discussed in previous studies.

References

Beygelzimer, A., J. Langford, and D. M. Pennock (2012). Learning performance of prediction markets with kelly bettors. In *Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems-Volume 3*, pp. 1317–1318. International Foundation for Autonomous Agents and Multiagent Systems.

Blume, L. and D. Easley (1992). Evolution and market behavior. *Journal of Economic Theory 58*, 9–40.

Blume, L. and D. Easley (2009). The market organism: Long-run survival in markets with heterogenous traders. *Journal of Economic Dynamics and Control 33*, 1023–1035.

Bottazzi, G. and P. Dindo (2015). Drift criteria for persistence of discrete stochastic processes on the line. LEM Papers Series 2015/26, Laboratory of Economics and Management (LEM), Sant'Anna School of Advanced Studies, Pisa, Italy.

Bottazzi, G. and D. Giachini (2016). Far from the madding crowd: Collective wisdom in prediction markets. LEM Papers Series 2016/14, Laboratory of Economics and Management (LEM), Sant'Anna School of Advanced Studies, Pisa, Italy.

Breiman, L. (1961). Optimal gambling systems for favorable games. *Proceedings of the 4th Berkley Symposium on Mathematical Statistics and Probability 1*, 63–68.

Evstigneev, I., T. Hens, and K. Schenk-Hoppé (2002). Market selection of financial trading strategies: Global stability. *Mathematical Finance 12*, 329–339.

Kelly, J. (1956). A new interpretation of information rates. *Bell System Technical Journal 35*, 917–926.

Kets, W., D. M. Pennock, R. Sethi, and N. Shah (2014). Betting strategies, market selection, and the wisdom of crowds. In *Twenty-Eighth AAAI Conference on Artificial Intelligence.*