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Decidability in complex social choices

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Decidability in complex social choices

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Abstract

In this paper we develop on a geometric model of social choice among bundles of interdependent elements (objects). Social choice can be seen as a process of search for optima in a complex multi-dimensional space and objects determine a decomposition of such a space into subspaces. We present a series of numerical and probabilistic results which show that such decompositions in objects can greatly increase decidability, as new kind of optima (called local and u-local) are very likely to appear also in cases in which no generalized Condorcet winner exists in the original search space.

Keywords:
Social choice, object construction,
hyperplane arrangement, probability, tournament, algorithm.

MSC (2010): 05C20, 52C35, 91B10, 91B12, 91B14.

JEL Classification: D03, D71, D72.

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1 Introduction

Social choice theory usually assumes that agents are confronted with a set of exogenously given and mutually exclusive alternatives. These alternatives are given in the sense that the pre-choice process through which they are constructed is not analyzed. Moreover, these alternatives are "simple", in the sense that they are one-dimensional objects or, even when they are multidimensional, they are simply points in some portion of the homogeneous $\mathbb{R}^n$ space and they lack any internal structure that limits the set of possible alternatives.

Many choices in real life situations depart substantially from this simple setting. Choices are often made among bundles of interdependent elements. Those bundles may be formed in a variety of ways, which in turn affect the selection process of a social outcome. Let us consider, for instance, a typical textbook example of social choice, i.e. the case of a group of friends deciding how to spend the evening by democratic and sincere pairwise majority voting. The textbook would start from a given and predefined choice set as $X = \{A, B, C, D, \ldots\}$ where $A, B, C, D, \ldots$ could stand for *movie, concert, restaurant, dinner at home,* etc. At closer scrutiny, these alternatives are neither primitive nor exogenously given. Going to the movies or to a restaurant are labels for bundles of *elements* (e.g. with whom, where, when, movie genre, director, type of food, etc.) and everyone's preference is unlikely to be expressed before these labels are specified in their constituting elements.

Moving on, to more serious examples, candidates and parties in political elections stand for complex bundles of interdependent policies and personality traits. Committees and boards are called to decide upon packages of policies, e.g. a recruitment package that a university governing board has to approve. In principle, any combination of elements (subject to a budget or some other feasibility constraint) could be considered and compared (e.g. through majority voting) with any other, but in reality only a relatively small number of packages undergo examination. Typically, the bundling of elements serves the purpose of reducing the number of alternatives to be examined, by decomposing the whole space of alternatives into smaller subspaces.

In Marengo and Pasquali (2011), Marengo and Settepanella (2012), Amendola and Settepanella (2012) some of us have developed present a model of social choice among bundles of elements, called *objects*. This model is independent from the aggregation method, as it applies equally to any social choice rule, and introduces two non-standard features that objects are likely to have. First, generally objects are not simply aggregations of primitive components but have an internal structure that is likely to determine interdependencies and non-separabilities in individual preferences. In the *"what shall we do tonight?"* choice setting, my preferences on the *with whom* element is likely to be highly interdependent with the other elements, as I may well find a given person a perfect companion for an evening at the movies but relatively dislike her or his company if we finally decide to go to a restaurant. On the same token I may prefer Italian food as instantiation of the *type of food* item if *dinner at home* is chosen but French cuisine if we opt for *going to the restaurant* and the *where* item takes the value "Paris".

Second, objects provide structure to the choice problem. Consider again the *"what shall we do tonight?"* case. A possible reply to their point on bundles would be that the choice set X is underspecified and that we should start from a choice set formed by all possible combinations of the elements, i.e. that the set X should be properly built in such a way as to include the exhaustive list of all mutually exclusive alternatives. However, for obvious combinatorial arguments, this set, even in this simple example, would be so large that any exhaustive choice procedure, e.g. pairwise majority voting, could not be completed in a feasible time span. Indeed, objects decompose the search space into quasi-separable subspaces (see also Simon (1982)) and simplify the computational task of collective choice, making decisions possible.

There is also another way in which objects can contribute to making the determination of a social outcome easier. By appropriate object construction, intransitive cycles that often characterize social decisions can almost always be eliminated (Marengo and Settepanella, 2012). In general, coarse objects, i.e. those made of many elements, tend to produce many cycles, whereas fine objects made of one or few elements do not. However finer objects do so by increasing the number of locally stable social optima and thereby making the social outcome more manipulable through the control of object construction, initial conditions and agendas.

In the classical social choice model, whereby choice takes place among an exhaustive set of unstructured and primitive alternatives by searching for a generalized Condorcet winner, decidability tends to be low, both because the likelihood of encountering an intransitive cycle is very high and because an exhaustive comparison among alternatives may take too long. In our framework instead social choice is among structured bundles of elements (called objects). The space of alternatives is decomposed and many social optima (called local and u-local optima) are generated, while the likelihood of intransitive cycles is sharply reduced. Thus decidability increases sharply, but also the possibility to manipulate the social outcome grows in parallel.

In this paper, we provide a more general set of results. We exploit a probabilistic approach and we develop some numerical calculations that allow us to show that the introduction of objects strictly increases the probability to get a social optimum and that an authority who has the power to construct objects (and choose how elements are described) may obtain a desired outcome even when the latter is freely chosen in a democratic process.

The paper is organized as follows. In section 2 we briefly discuss the similarities and differences between our approach and those already existing in the literature. In section 3 we provide a simplified algebraic and geometric version of the the mathematical model introduced in Marengo and Settepanella (2012), Amendola and Settepanella (2012) that we also illustrate by means of a series of examples.

Sections 4 and 5 present our novel results obtained, respectively, with numerical and probabilistic approaches. Notably, section 4 introduces a series of numerical simulations that show how decidability is greatly enhanced in our model, as new kind of social optima (that we call local and u-local) tend to appear also in case in which no Condorcet winner exists in the classical model.

Section 5, presents a further elaboration by means of probabilistic tools. Here we prove that the probability to obtain at least a local optimum when each element (called feature) has two possible outcomes (the yes/no or 0/1 case) is always greater than 60%, i.e. the decidability in this case is always very high.

From the methodological point of view, we believe that an important contribution of our paper is to show how algebraic (graphs), geometric (hyperplanes arrangements), numerical (combinatorial) and probabilistic approaches can converge in a general framework.

Finally, in Section 6 we draw some conclusions.

2 Relation to Literature

To our knowledge, the issue of object construction has not been dealt with by economic models before the recent contributions of Marengo and Pasquali (2011), Marengo and Settepanella (2012), Amendola and Settepanella (2012). In the first paper the notion of object construction power is presented and discussed by way of examples and agent based simulations. In the second one a mathematical model is given. In the third one the problem is tackled using tournament theory and an efficient algorithm that finds local optima is presented.

The literature on multidimensional voting models (Kramer, 1972; Shepsle, 1979; Denzau and Mackay, 1981; Enelow and Hinich, 1983) is relatively close to the perspective of this paper. In particular, Shepsle (1979) presents a model of majority voting in which institutions play a similar role to the one objects have in the model considered in this paper, i.e. that of limiting the set of outcomes that undergo examination. Two institutional mechanisms are analyzed: jurisdictional restrictions – especially those induced by decentralization and division of labour among decision making units – and agenda limitations with respect to the possible amendments to the *status quo*. Both limit the set of attainable outcomes and equilibria (called structure-induced equilibria) and can rule out cycles. There are at least two important differences between this stream of literature and our work. First, the problem tackled by all these papers is essentially the one arising from the sequential interdependency of voting: how we settle an issue today may change how we prefer to settle a related issue tomorrow. Enelow and Hinich (1983) also consider a similar multi-issue case in which each issue is voted sequentially and the agenda induces path-dependency, which might be mitigated by the agents' forecasting abilities. In our approach, we instead focus on interdependencies generated by how elements interact within the particular objects we are deliberating upon. Second, in Shepsle (1979), restrictions on attainable outcomes are placed by legal and organizational rules, that limit the set of allowed amendments. Instead, in our approach restrictions are placed by the object construction process exerted by some agent or institution: once an object has been defined, all its instances are always admissible and compared.

Our model paper presents some instances of a wide family of aggregation paradoxes in voting. Saari and Sieberg (2001) discuss the links between aggregation paradoxes in voting and similar aggregation paradoxes arising in statistics such as the so-called Simpson's paradox. Logrolling models

(Buchanan and Tullock, 1962) discuss some of these paradoxes which are similar to those analyzed in the present paper. Bernholz (1974) shows that logrolling implies cycles, therefore our result proving that cycles may be broken or created by appropriate object construction also extends to logrolling. Let us remark that our approach is strongly related to the existence of cycles, that is the existence of aggregation paradoxes in the multidimensional voting. But it does not analyze the aggregation paradoxes themselves since it assumes that an aggregation rule, i.e. a social decision rule, is given and the results of aggregation are already known.

Brams, Kilgour, and Zwicker (1998) analyze the aggregation paradox that occurs in the context of multiple elections, in which voters may not know the result of one election before they vote in another. In particular they study the case when votes are aggregated separately for each proposition (proposition aggregation), and they compare it to the case when votes are aggregated by combination (combination aggregation). An interesting connection between theirs and our model could be to study what happens when an aggregation is employed that puts together more than one but less than all the propositions, i.e. to generalize their results to all possible proposition aggregations, analogously to what our object construction does.

Our work is also closely related to Lang (2007). Indeed the sequential voting rules introduced by Lang corresponds to the model considered here in the particular case where each object is made of only one feature. In this context not only our model is a generalization of Lang's approach to the problem of voting in combinatorial domains, but it also casts some new light on the results presented in his paper and it answers some questions left open in that model. In the appendix we give a purely combinatorial description of the model described in section 3. This description should help understand the relation between our paper, Lang (2007) and the related literature on tournaments.

Indeed, even if the model in Marengo and Settepanella (2012) and Amendola and Settepanella (2012) is more complete, since it is described from both an algebraic (and hence, graphic and combinatorial) and a geometric (and hence topological) point of view, it can be easily reduced to any of these languages. This is a strong point of this approach: many different mathematical tools are involved that can be used not only for a better understanding of the phenomenon, but also for a deeper study.

Our paper is also related to recent literature that has begun to analyze decision-making when agents group states of the world into coarse categories

(Mullainathan, 2000; Fryer and Jackson, 2008). These papers show, among other things, that in these circumstances agents may be persuaded, meaning that uninformative messages may influence their decisions (Mullainathan, Schwartzstein, and Shleifer, 2008). Our perspective is different and complementary: our objects are not categories based on similarities among the states of the world, but are bundles of different and separate elements with an internal structure of interdependencies and not sets of states of the world that agents cannot distinguish from each other.

Context-dependent voting has also been analyzed by some papers (as, for instance, in Callander and Wilson (2006)). In these papers context-dependency refers to the violation of the axiom of Independence of Irrelevant Alternatives (IIA), i.e. the assumption that the preference expressed by an agent between two outcomes x and y does not depend on the presence or absence of other outcomes in the choice set. Psychologists and marketing scholars have observed systematic violations of IIA (Kahneman and Tversky, 2000). In the model presented in this paper, authors assume a different form of context dependency, meaning that preferences between two instantiations of an element (feature in our terminology) in general depend on the value taken by other traits.

Our results are also related to the literature which addresses the question of designing some principles for selecting a set of *best* alternatives when choosing from a tournament. These tournament solutions include the top cycle set (Schwartz, 1972; Miller, 1977), the uncovered set (Miller, 1980), the Banks set (Banks, 1985), the minimal covering set (Dutta, 1988), the tournament equilibrium set (Schwartz, 1990), and others. In Fey (2008) and Scott and Fey (2012) it is proven that, with probability approaching one, the top cycle set, the uncovered set, the Banks set and the minimal covering set are the entire set of alternatives in a randomly chosen large tournament. That is to say, each of these tournament solutions almost never rule out any of the alternatives under consideration. On the contrary, our results show that if we define the set of best alternatives as the local optima set then this set is always a proper subset of the set of alternatives. The problem is that this set could be empty, but the results in section 5 show that, if we consider the yes/no case, then, in a randomly chosen large tournament, the set of best alternatives is not empty with high probability (greater than t0.6). Hence, differently from all other sets of best alternatives, it is a proper non-empty subset of the set of alternatives with high probability.

3 Choices among bundle of elements

3.1 Social decision rules and Tournaments

Social decision rules Consider a population of ν *agents*. Each agent i is characterized by a *system of transitive preferences* $\succeq_i$ over the set of social outcomes X. The set of systems of transitive preferences $\succeq$ is denoted by $\mathcal{P}$. A *social decision rule* $\mathcal{R}$ is a function:

$$\begin{array}{cccc} \mathcal{R}: & \mathcal{P}^\nu & \longrightarrow & \overline{\mathcal{P}} \\ & (\succeq_1, \ldots, \succeq_\nu) & \longmapsto & \succeq_{\mathcal{R}(\succeq_1,\ldots,\succeq_\nu)} \end{array}$$

which determines a *system of social preferences* or *social rule* $\succeq_{\mathcal{R}(\succeq_1,\ldots,\succeq_\nu)}$ from the preferences of ν individual agents. With $\overline{\mathcal{P}}$ we denote the set of systems of social preferences. As well known, the social rule $\succeq_{\mathcal{R}(\succeq_1,\ldots,\succeq_\nu)}$ is not, in general, transitive, even when all individuals have indeed transitive preferences.

If Δ is the diagonal of the cartesian product $X \times X$, the element $\succeq_{\mathcal{R}} \in \overline{\mathcal{P}}$ defines a subset

$$\mathcal{Y}_{1,\succeq_{\mathcal{R}}} = \{(x, y) \in X \times X \setminus \Delta \mid x \succeq_{\mathcal{R}} y\}$$

and the set of *relevant* social outcomes

$$\mathcal{Y}_{0,\succeq_{\mathcal{R}}} = \{x \in X \mid \forall y \in X,\ (x, y) \in \mathcal{Y}_{1,\succeq_{\mathcal{R}}} \text{ or } (y, x) \in \mathcal{Y}_{1,\succeq_{\mathcal{R}}}\}.$$

If $\mathcal{Y}_{0,\succeq_{\mathcal{R}}}$ is the whole X, the social rule is said to be *complete*. If the two conditions $x \succeq_{\mathcal{R}} y$ and $y \succeq_{\mathcal{R}} x$ are mutually exclusive, the social rule is said to be *strict*. For the sake of simplicity *we will focus on complete and strict social preferences* $\succ$, generalization to weak preferences is almost always straightforward.

Graphs and tournaments The sets $\mathcal{Y}_{0,\succ}$ and $\mathcal{Y}_{1,\succ}$ are, respectively, the sets of nodes and arcs of a oriented graph $\mathcal{Y}_\succ = (\mathcal{Y}_{0,\succ}, \mathcal{Y}_{1,\succ})$. Two nodes x and y in $\mathcal{Y}_{0,\succ}$ are connected by an arc if $(x, y) \in \mathcal{Y}_{1,\succ}$ or $(y, x) \in \mathcal{Y}_{1,\succ}$, directed from x to y in the former case and from y to x in the latter. Notice that the assumption on preferences (completeness and strictness) guarantees that we will deal only with complete directed graphs, that is with *tournaments*. For the sake of simplicity, we will use letters such as x for the nodes of $\mathcal{Y}_\succ$ and pairs (x, y) for its arcs.

We recall here very few basic notions on tournament theory, for a more complete discussion we refer the reader to Chartrand and Lesniak (2005) and Moon (1968).

A cycle

$$(x_1, x_2), (x_2, x_3), \ldots, (x_h, x_1)$$

in the tournament $\mathcal{Y}_{\succ}$ corresponds to a cycle *à la* Condorcet-Arrow (Condorcet de Caritat, 1785; Arrow, 1951), i.e. to the sequence

$$x_1 \succ x_2 \succ \cdots \succ x_h \succ x_1.$$

The probability $\mathrm{Prob}\,(M)$ that a tournament with M nodes is irreducible, i.e. each pair of nodes is contained in a cycle, can be computed recursively by the formula:

$$\mathrm{Prob}\,(M) = 1 - \sum_{i=1}^{M-1} \binom{M}{i} \frac{\mathrm{Prob}\,(i)}{2^{i(M-i)}}, \text{ with } \mathrm{Prob}\,(1) = 1.$$

The values of $\mathrm{Prob}\,(M)$ for $M \leqslant 16$ are given in Figure 1. In general, $\mathrm{Prob}\,(M) \sim 1 - \frac{M}{2^{M-2}}$ and as M tends to infinity, $\mathrm{Prob}\,(M) \to 1$.

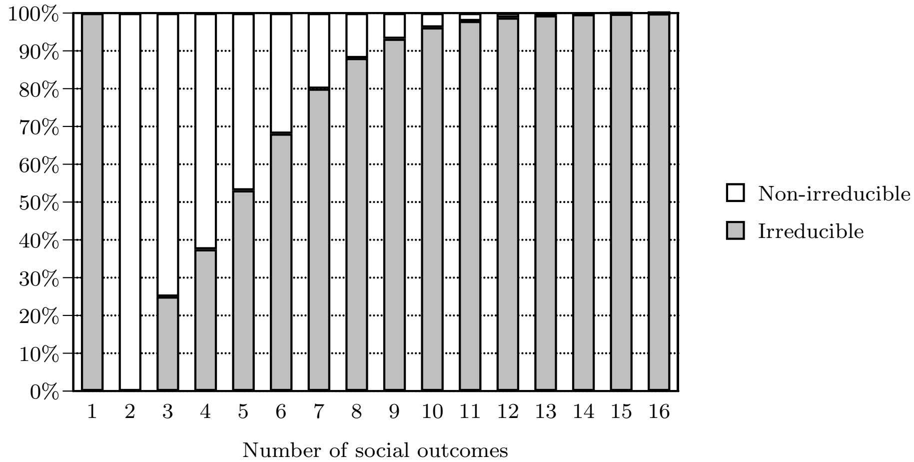

Figure 1: The probability that a tournament be irreducible.

3.2 Features and decisions

Features Let $F = \{f_1, \ldots, f_n\}$ be a bundle of elements, called *features*, the i-th of which takes m_i values, i.e. $\{0, 1, 2, \ldots, m_i - 1\}$, with $i = 1, \ldots, n$. Denote by $m = (m_1, \ldots, m_n)$ the multi-index of the numbers of values of the features. From now on, a *social outcome* (or *configuration*) will be an n-sequence $v_1 \cdots v_n$ of values such that $0 \leqslant v_i < m_i$. The set of all social outcomes will be denoted by X. The cardinality of X is $\prod_{i=1}^{n} m_i$ and will be denoted by M.

The decision process We suppose that the social choice proceeds along the following steps:

- select an ordered set A whose elements are subsets of F such that their union is the set F,
- start from a *status quo* social outcome,
- take the first element in A, i.e. a subset of features, and find the most preferred configuration keeping constant all the values of the features not in this element,
- repeat for all elements in A,
- repeat until either a cycle is encountered or a configuration that is the most preferred of any subset in A (an *optimum*) is reached.

3.3 Two illustrative examples

Let us now go back to the *"what shall we do tonight?"* example. If the friends have simply to decide upon where and when to go, we get, for instance, the following two-dimensional case, where we consider only two features (Where and When) and we code as follows the values that each feature can take:

Where? movie (0), restaurant (1), pub (2) First feature f_1

When? 20:00 (0), 22:00 (1) Second feature f_2

There are $3 \times 2 = 6$ possibilities and each alternative is a bundle of interdependent elements. So, for instance, if "movie at 20:00" is preferred to "pub at 22:00", this preference is denoted by $00 \succ 21$.

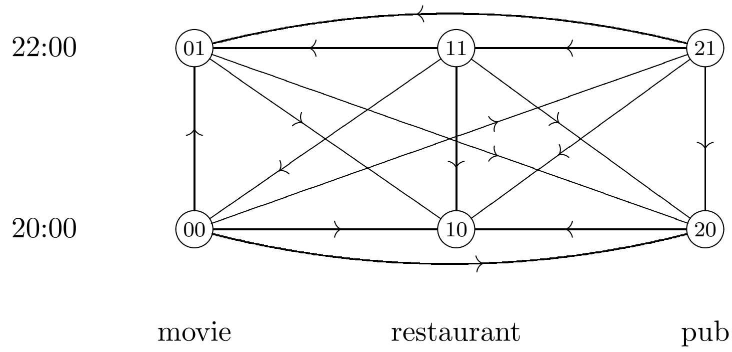

Figure 2: The graph associated to the 2d social rule.

If all preferences are expressed, one obtains the associated graph, which states the aggregated preferences of the group. In Figure 2 an example of such a graph is shown. In order to refer to this example later on, we will call it *the 2d social rule.*

In Figure 3 instead we present an hypothetical graph associated to a social rule with three features, each with two sub-alternatives. In order to make the graph readable, we draw only the main edges, i.e. the ones that determine, in our framework, optima and cycles irrespectively of the direction of the remaining edges. The reader is free to draw the remaining edges as she or he likes (and she or he should do it in order to have a complete oriented graph and hence a complete social rule). We will call this example *the 3d social rule.*

Example 3.1. Consider now the following decision processes in the 2d social rule:

- $10 \overset{f_2}{\to} 11 \overset{f_1}{\to} 21$ stop,
- $10 \overset{f_1}{\to} 00$ stop,
- $01 \overset{f_1}{\to} 21$ stop;

see Figure 4. (In this and the following figures we draw the relevant arcs with two arrows.) In any of the three cases the last configuration cannot be improved anymore by considering each feature separately from the other.

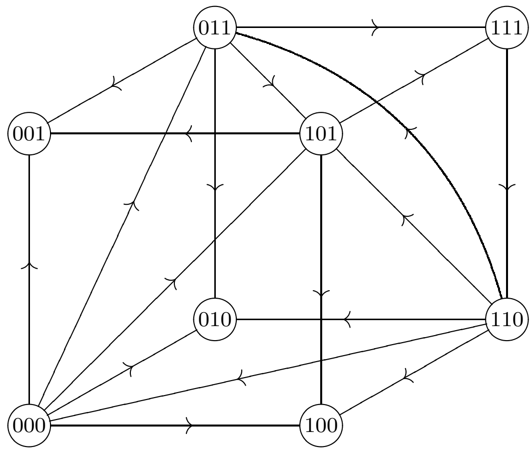

Figure 3: The graph associated to the 3d social rule.

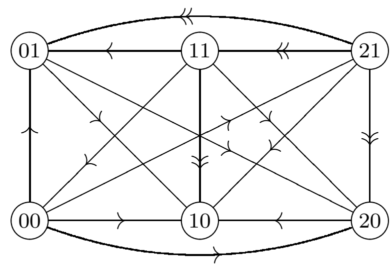

Figure 4: The decision processes in example 3.1.

The classical theory of social choice would correspond in our approach to the case in which only one object is considered which contains all the features. When compared to the standard one, our approach has advantages in terms of decidability, e.g. we can find an "optimum" more often than a generalized Condorcet winner, but in general there will be more and one optimum and therefore the social outcome will be subject to manipulability. An authority who has the power to decide how features are bundled together into objects, the order in which objects are examined, and the status quo which the choice process starts from will be enjoy a vast power to determine the final social outcome.

Example 3.2 (New kinds of optima)**.** In the 2d social rule the social outcomes 00 and 21 are "optima", i.e. there are decision processes that end in

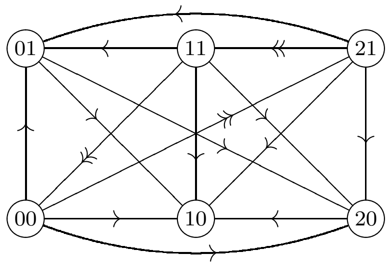

Figure 5: In the 2d social rule the social outcomes 00 and 21 are not generalized Condorcet winners.

them. However, none of them is a generalized Condorcet winner, because they are contained in a cycle, as shown in Figure 5.

3.4 Hyperplanes and social choice

In this section we present a formalization of the social choice problem outlined above which is based upon hyperplane arrangements.

The hyperplane arrangement In the n-dimensionale space $\mathbb{R}^n$, an *hyperplane* H is a *flat* subset of dimension $n-1$. Any hyperplane can be given in coordinates as the zero locus of a single degree-1 polynomial $\alpha_H \in \mathbb{R}[\lambda_1, \ldots, \lambda_n]$. An *hyperplane arrangement* is a finite set of hyperplanes.

Consider the hyperplane arrangement defined by

$$\mathcal{A}_{n,m} = \left\{ H_{i,j} \mid \alpha_{H_{i,j}} = \lambda_i - j \right\}_{\substack{1 \leqslant i \leqslant n \\ 0 \leqslant j < m_i - 1}} .$$

Note that the hyperplanes $H_{i,*}$ correspond to the i-th feature, and they are one less than the number of values taken by the i-th feature.

The *complement* of an hyperplane arrangement $\mathcal{A}$ is defined as the whole space minus the hyperplanes in $\mathcal{A}$, i.e.

$$\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}} H.$$

The complement of $\mathcal{A}$ is disconnected. It is made up of separate pieces (called *chambers*) each of which may be either bounded or unbounded. There is a correspondence (Marengo and Settepanella, 2012) between the set X of social

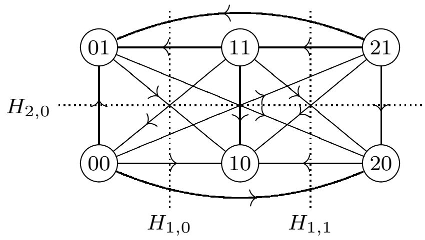

Figure 6: The hyperplane arrangement corresponding to the 2d social rule.

outcomes and the set of the chambers of the hyperplane arrangement $\mathcal{A}_{n,m}$. Namely, $x = v_1 \cdots v_n$ corresponds to the chamber that contains the set

$$\{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \mid v_j - 1 < \lambda_j < v_j,\ j = 1, \ldots, n\}\,.$$

Example 3.3. The hyperplane arrangement

$$\mathcal{A}_{2,(3,2)} = \{H_{1,0},\ H_{1,1},\ H_{2,0}\}$$

corresponding to the 2d social rule is shown in Figure 6. Each chamber corresponds to a vertex (i.e. a social outcome), and vice versa. Moreover, each vertex is connected to any other by an arc that crosses one or more hyperplanes.

Objects schemes Given a non-empty subset $I \subseteq \{1, \ldots, n\}$, the *object* $\mathcal{A}_I$ is the subset

$$\mathcal{A}_I = \{H_{i,j}\}_{\substack{i \in I \\ 0 \leqslant j < m_i - 1}}$$

of the hyperplane arrangement $\mathcal{A}_{n,m}$, i.e. the subset made up of the hyperplanes corresponding to the features belonging to I. The complemental set of a set I in $\{1, \ldots, n\}$ will be denoted by I^c, and corresponds to the complemental hyperplane arrangement $\mathcal{A}_I^c = \mathcal{A}_{n,m} \setminus \mathcal{A}_I$ of the hyperplane arrangement $\mathcal{A}_I$ in $\mathcal{A}_{n,m}$.

An *objects scheme* is a set of objects $A = \{\mathcal{A}_{I_1}, \ldots, \mathcal{A}_{I_k}\}$ such that $\bigcup_{j=1}^k I_j = \{1, \ldots, n\}$. Note that the sets I_j may have non-empty intersections. From now on, unless explicitly stated, A will always denote an objects scheme $\{\mathcal{A}_{I_1}, \ldots, \mathcal{A}_{I_k}\}$.

Example 3.4. In the 2d social rule there are three objects (see Figure 6):

$$\mathcal{A}_{\{1\}} = \{H_{1,0}, H_{1,1}\}, \qquad \mathcal{A}_{\{2\}} = \{H_{2,0}\}, \qquad \mathcal{A}_{\{1,2\}} = \{H_{1,0}, H_{1,1}, H_{2,0}\}.$$

The sets

$$\{\mathcal{A}_{\{1\}},\ \mathcal{A}_{\{2\}}\}, \qquad \{\mathcal{A}_{\{1,2\}}\} \qquad \text{and} \qquad \{\mathcal{A}_{\{1\}},\ \mathcal{A}_{\{1,2\}}\}$$

are three different objects schemes.

Agenda An *agenda* α *on an objects scheme* A is an ordered t-uple of indices $(h_1, \ldots, h_t)$ with $t \geq k$ such that $\{h_1, \ldots, h_t\} = \{1, \ldots, k\}$. An agenda α states the order in which the objects $\mathcal{A}_{I_i}$ are decided upon. The ordered t-uple of objects $(\mathcal{A}_{I_{h_1}}, \ldots, \mathcal{A}_{I_{h_t}})$ is denoted by A_α. The set of all possible agendas of A is denoted by $\Lambda(A)$. Note that repetitions, in general, are allowed.

Domination path A *domination path* $DP(x, y, A)$ through an objects scheme A is a sequence of social outcomes

$$x = x_0 \to x_1 \to \cdots \to x_s = y$$

such that x_i is the optimum among the social outcomes that lie on the same side of the hyperplanes in $\mathcal{A}_{I^c_{h_i}}$ as x_{i-1}. The social outcome x_i is called the *best neighbor* of x_{i-1} with respect to the object $\mathcal{A}_{I_{h_i}}$. Note that a social outcome x_{i-1} can be the best neighbor of itself, i.e. it is the preferred choice among the social outcomes that lie on the same side of itself when we consider the hyperplanes in the complement of the object $\mathcal{A}_{I_{h_i}}$. The domination path is said to *end in* x_s if it can be indefinitely extended to

$$x_0 \to x_1 \to \cdots \to x_s \to \cdots \to x_s$$

by considering all the objects in $\mathcal{A}$ at least once, or equivalently if x_s is the best neighbor of itself with respect to each object in $\mathcal{A}$. Note that no assumption on the order of the objects $\mathcal{A}_{I_{h*}}$ is made.

Example 3.5. In the 2d social rule consider the objects scheme $A = \{\mathcal{A}_{\{1\}},\ \mathcal{A}_{\{2\}}\}$. The sequence of social outcomes

$$10 \to 11 \to 21$$

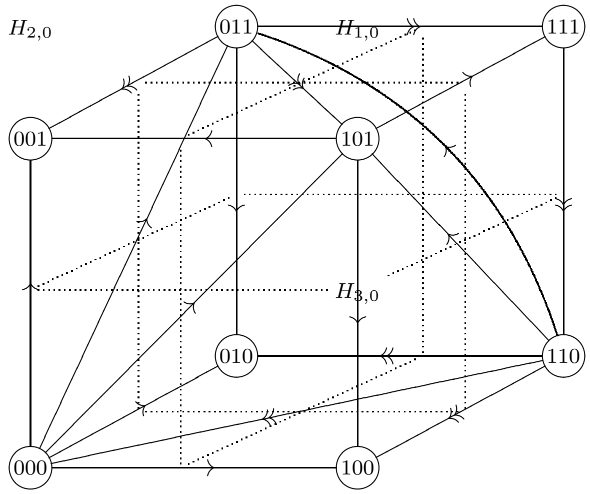

Figure 7: The arcs yielding the domination path $000 \to 110 \to 111 \to 011$ for the 3d social rule.

is a domination path through A. Indeed, the social outcome 11 is the best neighbor of 10 with respect to the object $\mathcal{A}_{\{2\}}$ (because 11 dominates 10), and the social outcome 21 is the best neighbor of 11 with respect to the object $\mathcal{A}_{\{1\}}$, because 21 dominates 01 and 11 (see Figure 4). Moreover, this domination path ends in the social outcome 21.

Example 3.6. In the 3d social rule consider the objects scheme $A = \{\mathcal{A}_{\{1,2\}},\ \mathcal{A}_{\{2,3\}},\ \mathcal{A}_{\{3\}}\}$. The sequence of social outcomes

$$000 \to 110 \to 111 \to 011$$

is a domination path through A. Indeed (see Figure 7), the social outcome 110 is the best neighbor of 000 with respect to the object $\mathcal{A}_{\{1,2\}}$ (because 110 dominates 000, 010 and 100), the social outcome 111 is the best neighbor of 110 with respect to the object $\mathcal{A}_{\{3\}}$ (because 111 dominates 110), and the social outcome 011 is the best neighbor of 111 with respect to the object $\mathcal{A}_{\{1,2\}}$ (because 011 dominates 001, 101 and 111).

Note however that the social outcome 011 admits a best neighbor, 000, with respect to the object $\mathcal{A}_{\{2,3\}}$, so the domination path does not end in 011.

Let $\alpha = (h_1, \ldots, h_t)$ be an agenda of an objects scheme A. A domination

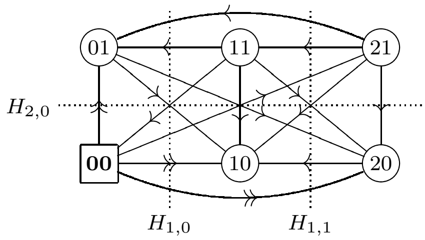

Figure 8: In the 2d social rule the social outcome 00 is a local optimum for the objects scheme $A = \{\mathcal{A}_{\{1\}},\ \mathcal{A}_{\{2\}}\}$.

path through A

$$x = x_0 \to x_1 \to \cdots \to x_s = y$$

is said to be *ordered along* α if the order of the objects $\mathcal{A}_{I_{h_*}}$ is given by α, i.e. if x_i is the best neighbor of x_{i-1} with respect to the object $\mathcal{A}_{I_{h_q+1}}$ where h_q is the remainder of the division of $i-1$ by t. Such a domination path will be denoted by $DP(x, y, A_\alpha)$.

Example 3.7. The domination path described in Example 3.5 is ordered along the agenda $(2, 1)$. The domination path described in Example 3.6 is ordered along the agenda $(1, 3, 1, 2)$.

Local optima A *local optimum* for an objects scheme A is a social outcome such that at least one domination path through A ends in it.

In general, more than one domination path ends in a local optimum and there may be more than one local optimum.

Example 3.8. It is easy to show that in the 2d social rule there are two different local optima for the objects scheme $A = \{\mathcal{A}_{\{1\}},\ \mathcal{A}_{\{2\}}\}$: 00 and 21. For instance, Figure 8 shows that 00 is a local optimum.

Example 3.9. In the 3d social rule the domination path

$$000 \to 110 \to 111 \to 011 \to 000$$

of the objects scheme $A = \{\mathcal{A}_{\{1,2\}},\ \mathcal{A}_{\{2,3\}},\ \mathcal{A}_{\{3\}}\}$ ordered along the agenda such that A_α is $(\mathcal{A}_{\{1,2\}},\ \mathcal{A}_{\{3\}}\ \mathcal{A}_{\{1,2\}},\ \mathcal{A}_{\{2,3\}})$ is a cycle (see Figure 9).

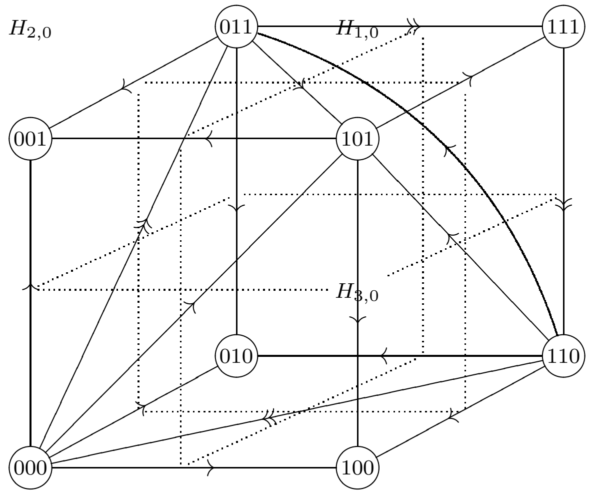

Figure 9: In the 3d social rule a domination path that is a cycle.

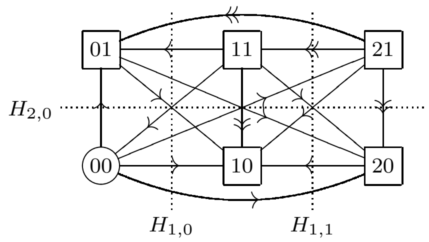

Figure 10: In the 2d social rule the basin of attraction of the social outcome 21 with respect to the objects scheme $A = \{\mathcal{A}_{\{1\}},\ \mathcal{A}_{\{2\}}\}$ is $\{10,\ 20,\ 01,\ 11,\ 21\}$.

Basin of attraction The *basin of attraction* $\Psi(x, A)$ of a local optimum x with respect to an objects scheme A is the set of the social outcomes y for which there exists a domination path $DP(y, x, A)$ that ends in x.

Example 3.10. In the 2d social rule the basin of attraction of the social outcome 21 with respect to the objects scheme $A = \{\mathcal{A}_{\{1\}},\ \mathcal{A}_{\{2\}}\}$ is $\{10,\ 20,\ 01,\ 11,\ 21\}$ (see Figure 10). Indeed, the following domination paths end in 21:

$$11 \to 21, \qquad 10 \to 11 \to 21, \qquad 01 \to 21, \qquad 20 \to 21.$$

Global optima A *global optimum* of an objects scheme A is a social outcome z whose basin of attraction is the whole set of social outcomes, i.e. $\Psi(z, A) = X$.

Remark 3.11. A generalized Condorcet winner is a global optimum with respect to the objects scheme

$$A = \left\{\mathcal{A}_{\{1,\dots,n\}}\right\}.$$

Example 3.12. In the 2d social rule there is no global optimum.

Local and global optima strictly depend on the choice of the objects scheme A. If an individual has the right to construct the objects, he or she will enjoy a power of influencing the outcome of the social choice. This is called "object construction power" (Marengo and Pasquali, 2011; Marengo and Settepanella, 2012).

To show this let us call *prominent distance* $d_p(x, y)$ between two social outcomes x and y the number of features for which x and y differ. The following result holds.

Theorem 3.13 (Marengo and Settepanella (2012))**.** *Let z be a social outcome. There exists an objects scheme A_z for which z is a local optimum if and only if the inequality $d_p(w, z) > 1$ holds for any social outcome w with $w \succ z$.*

Universal basin of attraction The *universal basin of attraction* of a social outcome z is the union of the basins of attraction of z with respect to each objects scheme, i.e. the set

$$\Psi(z) = \bigcup_{A \in \Pi(\mathcal{A}_{n,m})} \Psi(z, A),$$

where $\Pi(\mathcal{A}_{n,m})$ is the set of all possible objects schemes in $\mathcal{A}_{n,m}$.

Example 3.14. In the 2d social rule the universal basin of attraction of the social outcome 21 is $\{10,\ 20,\ 01,\ 11,\ 21\}$.

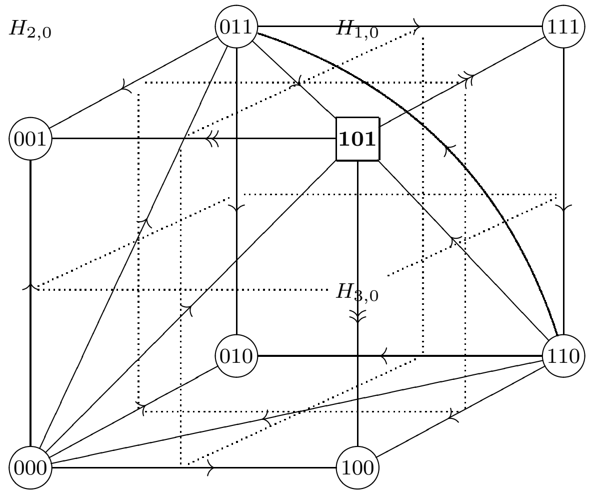

Figure 11: In the 3d social rule the social outcome 101 is a local optimum.

U-local optima A social outcome z is said to be a *u-local optimum* if its universal basin of attraction $\Psi(z)$ is the whole set of social outcomes X.

Remark 3.15. A generalized Condorcet winner is a global optimum that is necessarily a u-local optimum, and a u-local optimum is necessarily a local optimum for at least one objects scheme. In other words, the most demanding (and therefore the least likely to exist) notion of social optimum is the generalized Condorcet winner, followed by the global optimum, by the u-local optimum and, finally, by the local optimum, which on the contrary is the least demanding and the most likely to exist.

Example 3.16. In the 3d social rule there are no generalized Condorcet winners, but we can find global optima, u-local optima and local optima. Let us describe all the kinds of optima in details:

- The social outcome 101 is a local optimum for the objects scheme $A = \{\mathcal{A}_{\{1\}},\ \mathcal{A}_{\{2\}},\ \mathcal{A}_{\{3\}}\}$. Recall that $\mathcal{A}_{\{1\}} = \{H_{1,0}\}$, $\mathcal{A}_{\{2\}} = \{H_{2,0}\}$ and $\mathcal{A}_{\{3\}} = \{H_{3,0}\}$ and see Figure 11.

- The social outcome 011 is a global optimum for any agenda of the objects scheme $A = \{\mathcal{A}_{\{1,2\}},\ \mathcal{A}_{\{3\}}\}$. Recall that $\mathcal{A}_{\{1,2\}} = \{H_{1,0},\ H_{2,0}\}$ and $\mathcal{A}_{\{3\}} = \{H_{3,0}\}$ and see Figure 12.

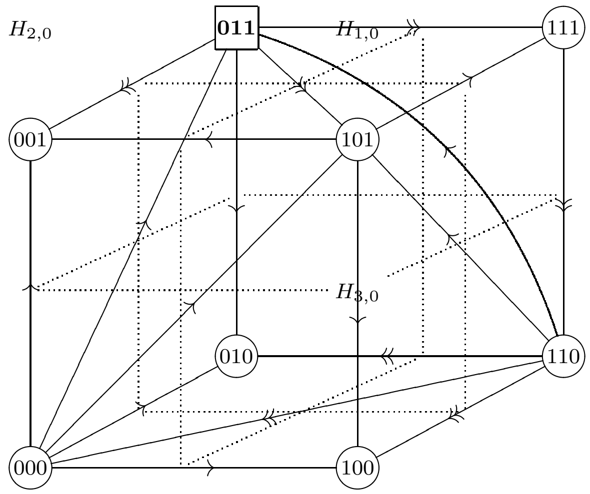

Figure 12: In the 3d social rule the social outcome 011 is a global optimum.

- The social outcome 000 is a u-local optimum. Recall that we can consider different objects schemes and see Figure 13. For the two-arrow arcs that have 000 as an endpoint we have used the objects (with two hyperplanes) $\mathcal{A}_{\{1,3\}}$ and $\mathcal{A}_{\{2,3\}}$. For the other two-arrow arcs we have used the objects (with one hyperplane) $\mathcal{A}_{\{2\}}$ and $\mathcal{A}_{\{3\}}$. Note that in the latter case we have to add another object, say $\mathcal{A}_{\{1,3\}}$, in order to have an objects scheme.
- Finally, in this social decision rule there is no generalized Condorcet winner. Indeed there is a cycle (the two-arrow arcs in Figure 14), and the three remaining social outcomes, which do not belong to this cycle, are dominated (as also shown in Figure 14).

4 Decidability

In the classical framework of the search for a Condorcet winner, a given social outcome z is an optimum if and only if it dominates all the other social outcomes. Therefore, the probability $P(z)$ that a given social outcome z is a generalized Condorcet winner for a social rule on M social outcomes is given by the ratio between the number of graphs with $M - 1$ nodes and

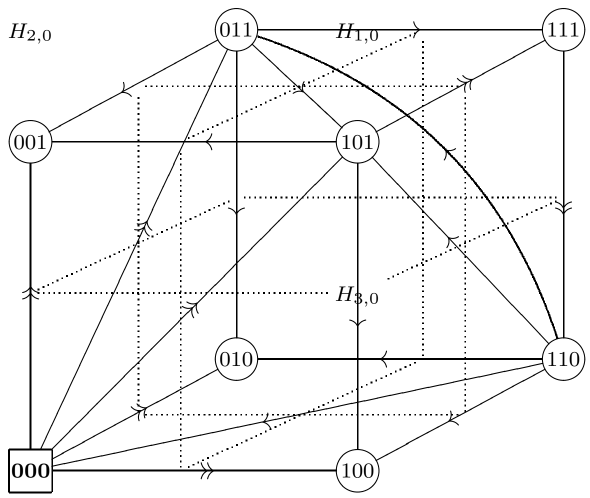

Figure 13: In the 3d social rule the social outcome 000 is a u-local optimum.

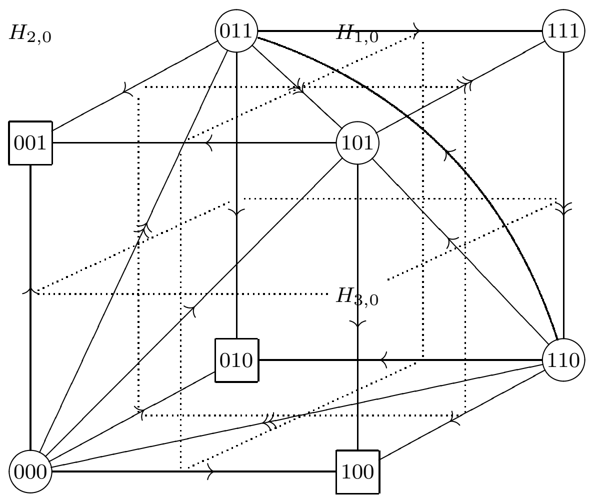

Figure 14: In the 3d social rule there is no generalized Condorcet winner.

the number of graphs with M nodes, i.e.

$$P(z) = \frac{2^{\binom{M-1}{2}}}{2^{\binom{M}{2}}} = \frac{1}{2^{M-1}}. \tag{1}$$

In our model, global optima play the role of generalized Condorcet winners, but also a local optimum can be an optimum if the agents vote starting from a particular social outcome. The probability $\text{Prob}_n(z)$ that a randomly chosen z is a local optimum, when the number of features is n, is given by the ratio between the number of the graphs with M nodes and with $\sum_{i=1}^n m_i - n$ fixed arcs, and the number of all graphs with M nodes, i.e.

$$\text{Prob}_n(z) = \frac{2^{\binom{M}{2} - (\sum_{i=1}^n m_i - n)}}{2^{\binom{M}{2}}} = \frac{1}{2^{\sum_{i=1}^n m_i - n}} = \frac{2^n}{2^{\sum_{i=1}^n m_i}}. \tag{2}$$

It is clear that, if n is greater than 1, the probability that z be a local optimum is far greater than the probability that z be a generalized Condorcet winner. Let us define a function $F : \mathbb{N}^3 \longrightarrow \mathbb{Q}$, depending on n, $M = \prod_{i=1}^n m_i$ and $\sigma = \sum_{i=1}^n m_i$, as the ratio between the probability of a social outcome to be a generalized Condorcet winner and that to be a local optimum in the new model, i.e.

$$F(n, M, \sigma) = \frac{2^n}{2^{\sum_{i=1}^n m_i}} 2^{M-1} = 2^{n+M-(\sigma-1)}.$$

Clearly $F(n, M, \sigma) > 1$ if and only if $n > 1$.

The function F provides a precise characterization of the improvement on decidability from the generalized-Condorcet framework to the new model. Obviously the non uniqueness of local optima and their dependence from the chosen objects scheme introduce a manipulability issue that we are going to analyze in greater detail below.

The algorithm ComputeUniversalBasin It is worth noting that finding optima (or, equivalently, basins of attraction) is not straightforward. Given the combinatorial nature of our problem, the number of possible objects and agendas is in general very high and a simple brute-force algorithm would take far more than exponential time.

The algorithm ComputeUniversalBasin (Amendola and Settepanella, 2012) computes the universal basin of attraction of a social outcome z for a social rule $\succ$. If the social rule $\succ$ is defined on M social outcomes, the algorithm ComputeUniversalBasin computes the universal basin of attraction of z in $O(M^3 \log M)$ time. The algorithm has been implemented in the computer program `FOSoR` (Amendola, 2011a). `FOSoR` reads a social rule and

- computes the universal basin of attractions,
- checks whether a social outcome is a local (or an u-local) optimum,
- checks whether a social outcome is in the universal basin of attraction of another one,
- checks whether there is a local (or an u-local) optimum,
- finds the number of local (or u-local) optima,
- given two social outcomes, finds an objects scheme (if any) for which there is a domination path from one to the other.

4.1 Numerical results

In this section we will give numerical results obtained by means of the computer program `FOSoRStat` (Amendola, 2011b), which repeatedly (in this case 1,000,000 times) generates a random social rule and applies the algorithm ComputeUniversalBasin to find all optima.

The generalized Condorcet winner We start by showing in Figure 15 the likelihood that a generalized Condorcet winner exists. This can be computed with the formula

$$\frac{M}{2^{M-1}}.$$

The figure shows that the probability of finding a generalized Condorcet winner quickly vanishes as the number of social outcomes increases and that for as little as 10 social outcomes it is already practically zero.

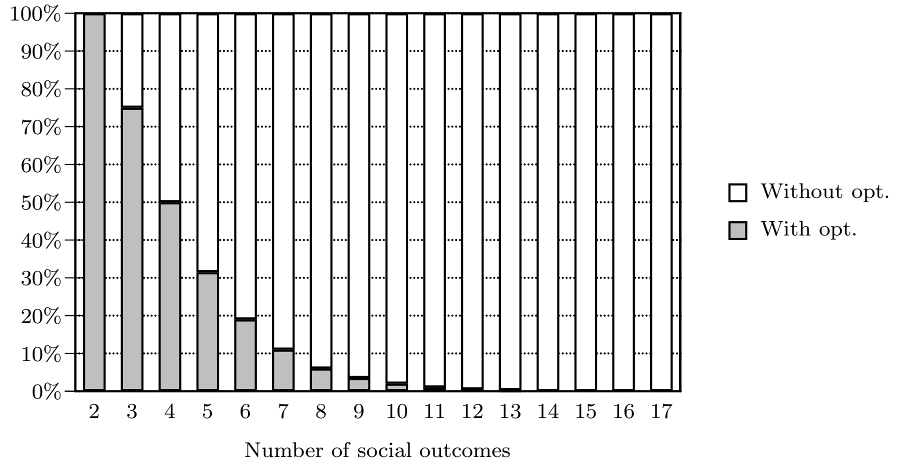

Figure 15: The probability that a social rule has a generalized Condorcet winner.

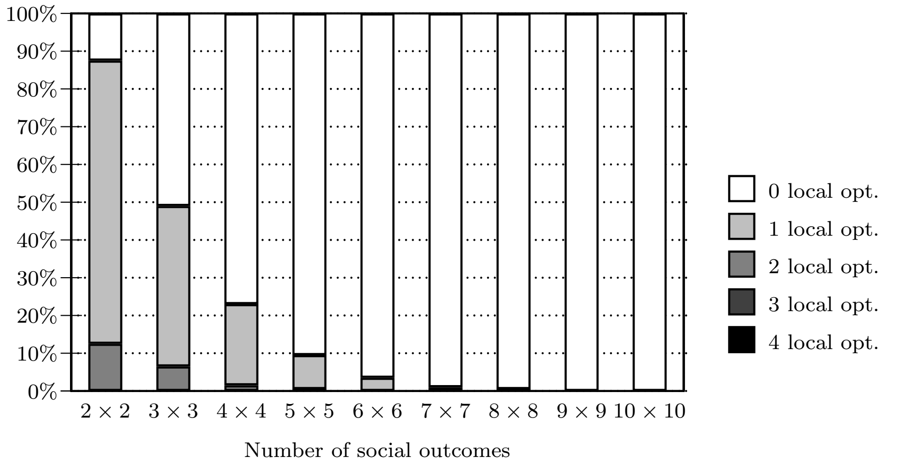

Figure 16: The probability that a social rule with two features has a given number of local optima.

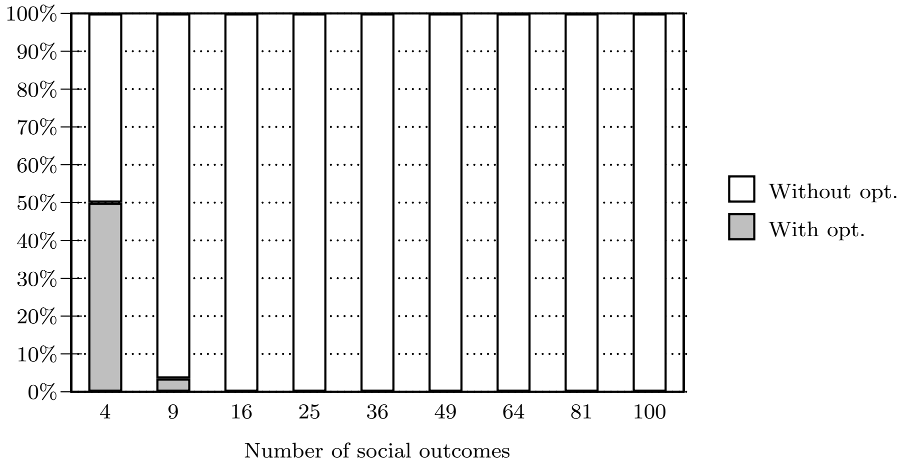

Figure 17: The probability that a social rule with two features has a generalized Condorcet winner.

Social rules with two features In Figure 16 we show the probability that a social rule with two features has a given number of local optima. Compare it with Figure 17, where the probability that a social rule with the same number of social outcomes has a generalized Condorcet winner. We can deduce that local optima are much more likely to exist but in general there may exist more than one of them (we counted up to four of them).

In Figure 18 we show the probability that a social rule with two features has a given number of u-local optima.

Binary features An important case is when we consider only *binary features*, that could for instance model cases in which agents must take a set of interrelated yes/no decisions. In Figure 19 we plot the probability that a social rule with binary features has a given number of local optima, depending on the number of features. We point out that the number of social outcomes, that is 2^n, grows very fast as the number n of features increases. The probability of finding local optima is of course one if there is only one feature (and therefore 2 social outcomes) and decreases slowly as the number of features increases, and seemingly stabilizes just above 60%. Up to 9 different local optima may be found with 9 binary features.

An analogous behaviour occurs in the case of u-local optima, as shown in Figure 20.

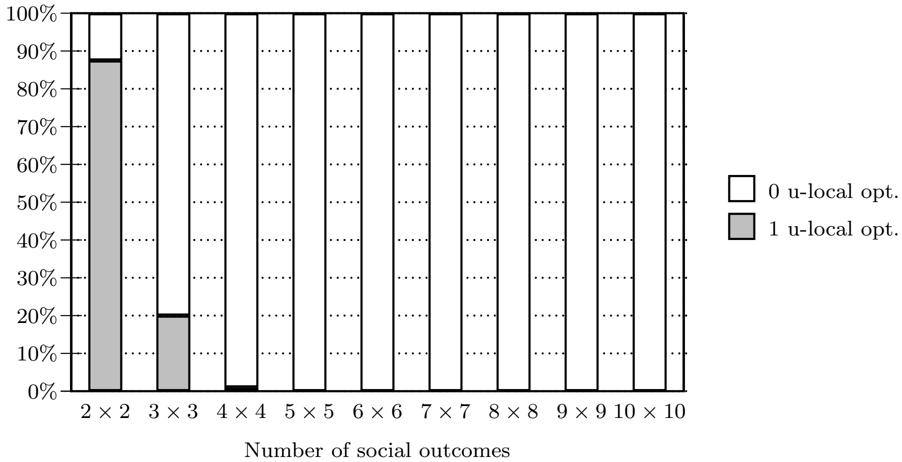

Figure 18: The probability that a social rule with two features has a given number of u-local optima.

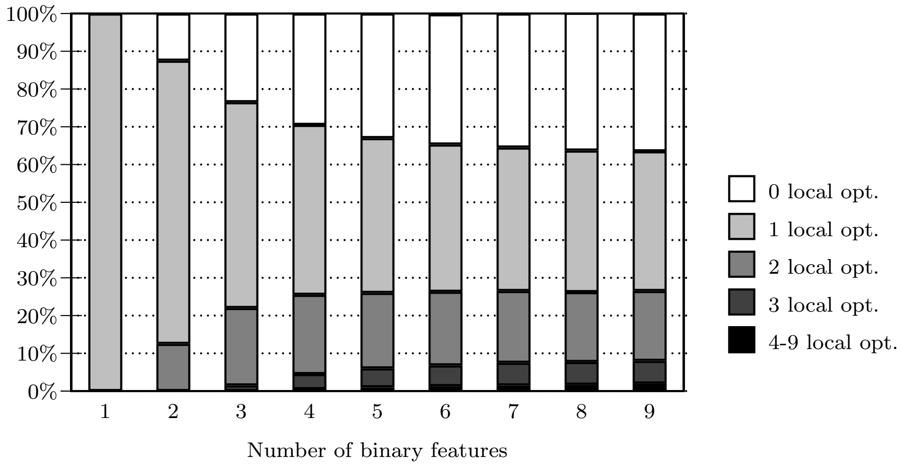

Figure 19: The probability that a social rule with binary features has a given number of local optima, depending on the number of features.

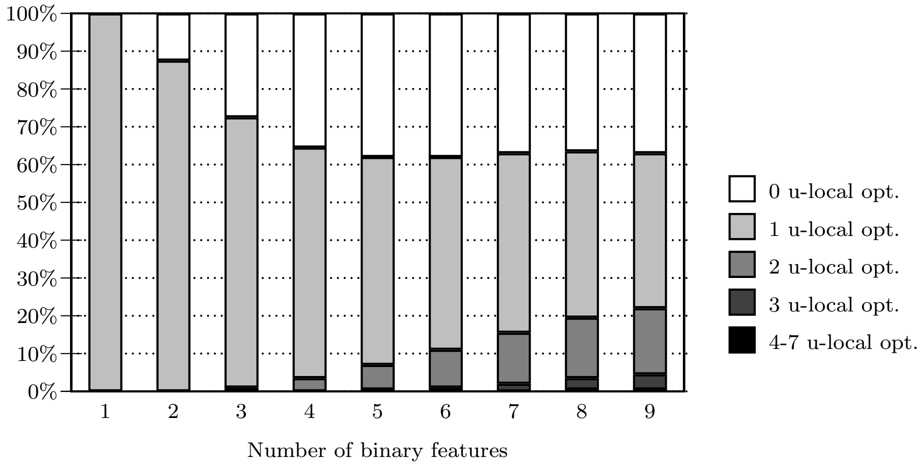

Figure 20: The probability that a social rule with binary features has a given number of u-local optima, depending on the number of features.

We have not shown what happens for generalized Condorcet winners because, as shown in Figure 15, if the number of features is greater than 4 (and hence the number of social outcomes is greater that $2^4 = 16$) the probability that a social rule has a generalized Condorcet winner is almost zero.

Objects schemes Figure 21 plots the probability that a social rule with 6 binary features (hence, with $2^6 = 64$ social outcomes) has a given number of local optima, depending on how such features are bundled together into different objects schemes. In particular, we consider the case in which, respectively, there is only one object of 6 features (i.e. the generalized-Condorcet case), two objects of 3 features each, three objects of 2 features each, or, finally, six objects of 1 feature each. Note that if the number of objects increases (and hence the number of features in each object decreases) the number of local optima increases up to 7.

Cardinality of features Figure 21 can be regarded as the plot of the probability that a social rule defined on 64 social outcomes has a given number of local optima, depending on how they are represented by more or less features taking a smaller or larger number of values. In particular, we consider the case in which, respectively, there is only one feature taking 64 different values (i.e. the generalized-Condorcet case), two features taking 8 values each, three

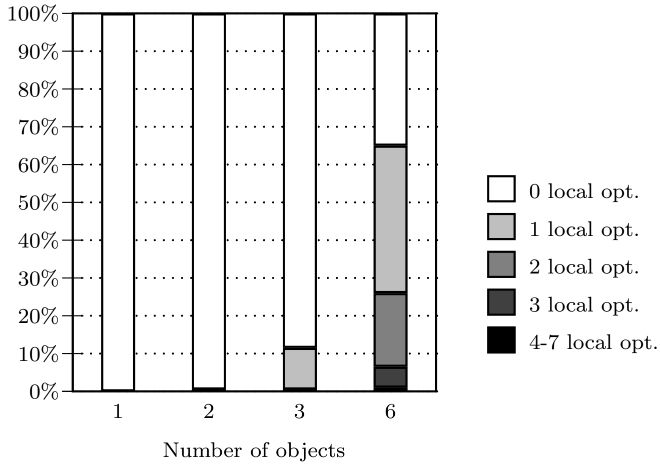

Figure 21: The probability that a social rule with 6 binary features has a given number of local optima, depending on the objects scheme.

features taking 4 values each, or, finally, six binary features.

In Figure 22 we do the same for the case of u-local optima. Although the notion of u-local optimum is more restrictive than that of local optimum, also in this case if the number of features increases (and hence the number of values taken by each feature decreases) the probability that a social rule has a fixed number of u-local optima increases. Moreover, we have found cases with up to 6 different u-local optima.

5 Probability of Local Optima

In this section we compute the probability to finds local optima in the binary case. We will prove that a local optimum is actually reached in more than 60% of the cases, even for a large number of alternatives, as already shown numerically in the previous section. Thus in our model the social choice process converges to some "acceptable", though manipulable, choice in the majority of cases even for decisions involving a considerable number of social outcomes, whereas – we have already shown – the probability of finding a Condorcet winner quckly vanishes in such cases.

Recall that $\mathrm{Prob}_n(z)$ is the probability that a randomly chosen z is a local optimum, when the number of features is n. Similarly, we indicate with $\mathrm{Prob}_n(z \wedge w)$ the probability that two randomly chosen social outcomes z

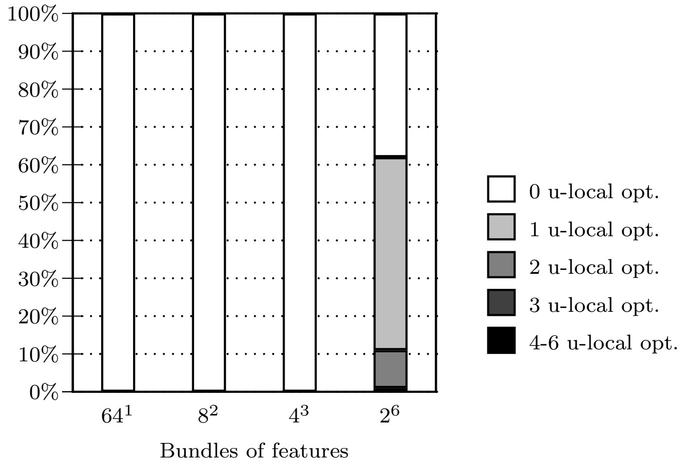

Figure 22: The probability that a social rule defined on 64 social outcomes has a fixed number of u-local optima, depending on the subdivision in features.

and w, with $z \neq w$, are simultaneously local optima, and with $\text{Prob}_n(z \mid w)$ the probability that z is a local optimum if w is.

Lemma 5.1. *Suppose that each feature can assume only two values, i.e. $m_i = 2$ for each $i = 1, \ldots, n$. Let z and w, with $z \neq w$, be two randomly chosen social outcomes. The following holds:*

$$\lim_{n \to \infty} \frac{\text{Prob}_n(z \wedge w)}{\text{Prob}_n(z) \cdot \text{Prob}_n(w)} = 1.$$

Proof. First of all, note that, by means of Equation (2), the unconditional probability $\text{Prob}_n(z)$ in the binary case is

$$\text{Prob}_n(z) = \frac{1}{2^n}. \tag{3}$$

Write $\text{Prob}_n(z \wedge w)$ as

$$\text{Prob}_n(z \wedge w) = \text{Prob}_n(z \mid w) \cdot \text{Prob}_n(w).$$

The probability that z is a local optimum, given that w is a local optimum, is easily decomposed as

$$\text{Prob}_n(z \mid w) = 0 \cdot \text{Prob}_n(d_p(z, w) = 1) + \frac{1}{2^n} \cdot \text{Prob}_n(d_p(z, w) > 1),$$

where the coefficients, 0 and $\frac{1}{2^n}$, follow from Theorem 3.13. Namely, either the prominent distance between two social outcomes is 1 so z cannot be a local optimum (since w is), or the prominent distance between two social outcomes is greater than 1 so the fact that z is a local optimum is independent from the fact that w is.

Note that the number of pairs of social outcomes with prominent distance equal to 1 is obtained by summing n (the total number of prominent neighbors for each social outcome) for 2^n times (the total number of social outcomes) and dividing the final result by two (every arc is counted twice). Considering that the total number of pairs of social outcomes (or the total number of edges of the complete graph) is $\frac{2^n\,(2^n-1)}{2}$ we obtain:

$$\begin{aligned}\mathrm{Prob}_n\left(d_p\left(z,w\right)>1\right) &= 1-\mathrm{Prob}_n\left(d_p\left(z,w\right)=1\right)= \\ &= 1-\frac{\frac{n\,2^n}{2}}{\frac{2^n\,(2^n-1)}{2}}= \\ &= 1-\frac{n}{2^n-1}\to 1.\end{aligned}$$

Hence we have

$$\mathrm{Prob}_n\left(z\mid w\right)\sim\mathrm{Prob}_n\left(z\right),$$

so

$$\mathrm{Prob}_n\left(z\wedge w\right)\sim\mathrm{Prob}_n\left(z\right)\cdot\mathrm{Prob}_n\left(w\right),$$

which is our thesis. □

The above lemma essentially proves that the events "z is a local optimum" and "w is a local optimum" are asymptotically independent.

We can now state and prove the theorem that formalizes our main result:

Theorem 5.2. *Suppose that each feature can assume only two values, i.e.* $m_i = 2$ *for each* $i=1,\ldots,n$. *Let* $K(M)$ *denote the number of local-optima. Then for each* $k=0,1,\ldots$ *we have*

$$\lim_{n\to\infty}\mathrm{Prob}\left(K\left(M\right)=k\right)=e^{-1}\,\frac{1}{k!}. \tag{4}$$

In particular, the probability to have at least one local optimum converges to

$$\lim_{n\to\infty}\mathrm{Prob}\left(K\left(M\right)\geq 1\right)=1-\frac{1}{e}\approx 63.2\%. \tag{5}$$

Proof. Let R be a random binary variable such that $R = 1$ represents the success to be a local optimum and $R = 0$ the corresponding failure, with the success probability

$$p = \text{Prob}\,(R = 1) = \frac{1}{2^n} = \frac{1}{M}.$$

For Lemma 5.1 we have that if M is large enough or, more precisely, when $k \ll \#X = M$, the events $R = 1$ are close to being independent and therefore the total number of local optima is binomially distributed:

$$\text{Prob}\,(K(M) = k) = \binom{M}{k} p^k (1-p)^{M-k} =: B_{(M,p)}\,(k)$$

As M goes to infinity the probability p goes accordingly to zero and the product $\lambda \equiv p\,M$ remains constant and equal to 1. Hence by the law of small numbers (Theorem 2 of Arratia, Goldstein, and Gordon, 1989; Falk, Hü, and Reiss, 2004) the binomial distribution of $K(M)$ converges to a Poisson distribution with mean $\lambda = 1$ in (4), that is

$$\lim_{\substack{M\to\infty \\ p\cdot M=1}} B_{(M,p)}\,(k) = e^{-1}\frac{1}{k!}, \qquad k = 0, 1, \ldots.$$

It follows that

$$\lim_{n\to\infty} \text{Prob}\,(K\,(M) = k) = 1 - \lim_{n\to\infty} \text{Prob}\,(K\,(M) = 0) = 1 - \frac{1}{e} \tag{6}$$

which is our thesis. □

The result stated above confirms the asymptotical behavior of the numerical results described in Figure 19.

Remark 5.3. By Theorem 2 of Arratia, Goldstein, and Gordon (1989) it can be shown that

$$\left|\text{Prob}\,(K\,(M) = 0) - \frac{1}{e}\right| \leq \frac{2n}{2^n - 1}$$

for all $n \geq 1$. Hence for $n \geq 10$, a locally optimal outcome is actually reached in more than 60% of the cases.

Moreover, by the numerical results shown in Figure 20, the above result seems to apply also to u-local optima. A remarkable consequence of this would be that the decision in a group is independent from the status quo. Indeed an agent who has the power to choose the objects scheme has the power to achieve his or her preferred local optimum independently of the status quo, just by changing the objects scheme according to the status quo.

6 Conclusions

In this paper we have shown that in the more general framework of choice among bundles of interdependent elements less stringent kinds of optima (u-local and, even more, local optima) than the generalized Condorcet winners are much more likely to exist. But since, in general, there exists a multiplicity of such optima, it will be possible for an authority to select one of them by changing the way features are bundled together in what we call the objects of choice, or by controlling the initial condition and/or the agenda.

This model is meant to be a first step in the direction of modeling social choice when the alternatives are not given a priori, but the model explicitly addresses the fundamental pre-choice problem of the construction of alternatives. Our paper shows that such a construction process faces a fundamental trade off between decidability and manipulability: if many features are bundled together social optima are unlikely to exist, and even when they do exist a concrete choice procedure (e.g. pairwise majority voting) will indeed locate them but with an unrealistically large number of operations. On the contrary, if features are considered separately or in small bundles containing only few of them, the likelihood of finding a social optimum (local or u-local) sharply increases and the time required to find it sharply decreases, but a problem of multiplicity of optima, and therefore of manipulability of the entire process, arises. In particular, an authority who has the power to determine the bundles, the order with which they are examined and the initial status quo, will enjoy a considerable power of making the social choice process converge to a preferred outcome. Somehow we can say that in our framework decidability can only be assured by introducing manipulability and power.

Although quite general, an important limitation of our model is that it does not allow for strategic misrepresentation of preferences. A further generalization of our model which accounts for the latter is left to future investigation.

Finally, as already pointed out in the introduction, it could be interesting to apply a similar argument to aggregation paradoxes such as the paradox of multiple elections introduced by Brams, Kilgour, and Zwicker (1998), and to compare in the general case (with $m_i > 2$) the set of local optima with existing best sets, studying also how rapidly it goes to the empty set.

A Appendix

In this appendix we reproduce the model described in Section 3 in a purely algebraic way, as preferences on combinatorial domains.

Preferences on combinatorial domains A *feature* f_i can be defined as an element $\{0, \ldots, m_i - 1\}$ of the non-negative integers. The set $\{f_1, \ldots, f_n\}$ of features is denoted by F. The combinatorial domain $X = f_1 \times \ldots \times f_n$ is the set of all social outcomes and $x = (x_1, \ldots, x_n)$ denotes an element in X. Let us remark that, with this notation, we totally loose the spacial structure of R^n and the utility of thinking of $x \in X$ as a point in the real n-dimensional space.

A tournament $T = (X, \succ)$ is an orientation of a complete graph on X, in which case $\succ$ can equivalently be seen as a complete and asymmetric relation on X.

A social outcome $x \in X$ is said to be a *generalized Condorcet winner* of a tournament $T = (X, \succ)$ if $x \succ y$ for all y distinct from x. The probability that a randomly chosen social outcome in X is a generalized Condorcet winner is given in equation (1).

Every subset of features $\{f_i\}_{i \in I}$, with $I \subset \{1, \ldots, n\}$, induces an equivalence relation $\sim_I$ over X such that for all elements $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in X,

$$(x_1, \ldots, x_n) \sim_I (y_1, \ldots, y_n) \qquad \text{if and only if} \qquad x_j = y_j \text{ for all } j \notin I.$$

For each $x \in X$ and each subset of features $\{f_i\}_{i \in I}$, the equivalence relation $\sim_I$ induces an equivalence class

$$[x]_{\sim_I} = \{y \in X : x \sim_I y\}.$$

Given a tournament $T = (X, \succ)$, for each subset of features $\{f_i\}_{i \in I}$ and each $x \in X$ one can associate, if it exists, the maximum element in $[x]_{\sim_I}$, i.e.

$$\max_{\succ}([x]_{\sim_I}) = \big\{y \in [x_I]_{\sim_I} : y \succ z \text{ for all } z \in [x]_{\sim_I} \setminus \{y\}\big\}.$$

Observe that, since $\succ$ is asymmetric, the cardinality of $\max_{\succ}([x_i]_{\sim_i})$ is either 0 or 1 and that, if $I = \{1, \dots, n\}$, this maximum coincides, if it exists, with the generalized Condorcet winner.

With the above notations, we have the following:

- an *objects scheme* is a set $A = \big\{\{f_i\}_{i \in I_j}\big\}_{1 \le j \le k}$ of subsets of features such that $\cup_{1 \le j \le k} I_j = \{1, \dots, n\}$, i.e. all features are considered at least once;
- an *agenda* α is an order, with repetitions, of the indices $j \in \{1, \dots, k\}$;
- the *process* starting from an initial element $x_0 \in X$ determines a subgraph $T_{x_0,A,\alpha}$ of $T = (X, \succ)$ that depends from $x_0 \in X$, the objects scheme and the fixed agenda.

An element $x \in X$ is a *local optimum* for the objects scheme A if it exists an $x_0 \in X$ and an agenda α such that x is the generalized Condorcet winner in the subgraph $T_{x_0,A,\alpha}$. Marengo and Settepanella (2012) show that the fact that x is the generalized Condorcet winner in a subgraph $T_{x_0,A,\alpha}$ is independent of x_0 and α, that is, if x is a local optimum then it is a local optimum for $T_{x,A,\alpha}$ for any agenda α and that, given an agenda α there is always an element $y \in X$ such that x is the generalized Condorcet winner in the subgraph $T_{y,A,\alpha}$. Moreover they noticed that a necessary and sufficient condition for $x \in X$ to be local optimum for at least an objects scheme A is that

$$x = \max_{\succ}([x]_{\sim_{\{i\}}}) \quad \text{for all } i \in \{1, \dots, n\},$$

i.e. x is the generalized Condorcet winner in each subtournament $([x]_{\sim_{\{i\}}}, \succ)$.

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