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Scuola Superiore Sant'Anna

Piazza Martiri della Libertà, 33 - 56127 Pisa, Italy
ph. +39 050 88.33.43
institute.economics@sssup.it

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On the Configuration Spaces of Grassmannian Manifolds

Sandro Manfredini[¶]
Simona Settepanella[°]

[¶] Department of Mathematics, University of Pisa, Italy
[°] Institute of Economics and LEM, Scuola Superiore Sant'Anna, Pisa, Italy

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On the Configuration Spaces of Grassmannian Manifolds

Sandro Manfredini* Simona Settepanella†

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Abstract

Let $\mathcal{F}^i_h(k,n)$ be the ith ordered configuration space of all distinct points $H_1,\dots,H_h$ in the Grassmannian $Gr(k,n)$ of k-dimensional subspaces of $\mathbb{C}^n$, whose sum is a subspace of dimension i. We prove that $\mathcal{F}^i_h(k,n)$ is (when non empty) a complex submanifold of $Gr(k,n)^h$ of dimension $i(n-i)+hk(i-k)$ and its fundamental group is trivial if $i=min(n,hk)$, $hk\neq n$ and $n>2$ and equal to the braid group of the sphere $\mathbb{C}P^1$ if $n=2$. Eventually we compute the fundamental group in the special case of hyperplane arrangements, i.e. $k=n-1$.

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complex space, configuration spaces,
braid groups.

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*Department of Mathematics, University of Pisa, manfredi@dm.unipi.it

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1 Introduction

Let M be a manifold. The *ordered configuration space* of h distinct points in M, $\mathcal{F}_h(M) = \{(x_1, \ldots, x_h) \in M^h | x_i \neq x_j,\ i \neq j\}$, has been widely studied after it has been introduced by Fadell and Neuwirth [FaN] and Fadell [Fa] in the sixties. It is well known that for a simply connected manifold M of dimension ≥ 3, the *pure braid group* on h strings of M, $\pi_1(\mathcal{F}_h(M))$, is trivial, while in low dimension there are non trivial pure braids. For example, the pure braid group of the sphere $S^2 \approx \mathbb{C}P^1$ is not trivial with the following presentation:

$$\pi_1(\mathcal{F}_h(\mathbb{C}P^1)) \cong \left\langle \alpha_{ij}, 1 \leq i < j \leq h-1 \,\middle|\, (YB\,3)_{h-1}, (YB\,4)_{h-1}, D^2_{h-1} = 1 \right\rangle$$

where $D_k = \alpha_{12}(\alpha_{13}\alpha_{23})(\alpha_{14}\alpha_{24}\alpha_{34}) \cdots (\alpha_{1k}\alpha_{2k} \cdots \alpha_{k-1\ k})$ (see [Bi] and [FaH] for the Yang-Baxter relations $(YB, 3)_{h-1}$, $(YB, 4)_{h-1}$).
In a recent paper ([BaS]) Berceanu and Parveen introduced new configuration spaces. They stratify the classical configuration spaces $\mathcal{F}_h(\mathbb{CP}^n)$ with complex submanifolds $\mathcal{F}^i_h(\mathbb{CP}^n)$ defined as the ordered configuration spaces of all h points in $\mathbb{CP}^n$ generating a projective subspace of dimension i. Then they compute the fundamental groups $\pi_1(\mathcal{F}^i_h(\mathbb{CP}^n))$ proving that they are trivial except when $i = 1$ providing, in this last case, a presentation for $\pi_1(\mathcal{F}^1_h(\mathbb{CP}^n))$ similar to those of the pure braid group of the sphere.
In a subsequent paper ([MPS]), the authors apply similar techniques to the affine case, i.e. to $\mathcal{F}_h(\mathbb{C}^n)$, showing that the situation is similar except in one case. More precisely they prove that, if $\mathcal{F}^{i,n}_h = \mathcal{F}^i_h(\mathbb{C}^n)$ denotes the ordered configuration space of all h points in $\mathbb{C}^n$ generating an affine subspace of dimension i, then the spaces $\mathcal{F}^{i,n}_h$ are simply connected except for $i = 1$ or $i = n = h - 1$ and, in the last cases, they provide a presentation of the fundamental groups $\pi_1(\mathcal{F}^{i,n}_h)$.
In this paper we generalize the result in [BaS] to the Grassmannian manifold $Gr(k, n)$ parametrizing k-dimensional subspaces of $\mathbb{C}^n$. We define the ith ordered configuration space $\mathcal{F}^i_h(k, n)$ as the ordered configuration space of all distinct points $H_1, \ldots, H_h$ in the Grassmannian $Gr(k, n)$ such that the sum $(H_1 + \cdots + H_h)$ is an i-dimensional space.
We prove that the ith ordered configuration space $\mathcal{F}^i_h(k, n)$ is (when non empty) a complex submanifold of $Gr(k, n)^h$ and we compute its dimension.
As a corollary, we prove that if $n \neq hk$ and i =min(n, hk) then the ith ordered configuration space $\mathcal{F}^i_h(k, n)$ has trivial fundamental group except

when $n = 2$ where we get the pure braid group of the sphere, that is

$$\begin{aligned} &\pi_1(\mathcal{F}_h^{min(n,hk)}(k,n)) = 0 \quad \text{if } (k,n) \neq (1,2) \\ &\pi_1(\mathcal{F}_1^1(1,2)) = \pi_1(\mathcal{F}_2(\mathbb{CP}^1)). \end{aligned} \tag{1}$$

In the particular case of hyperplane arrangements, i.e. $k = n-1$, we remark that the fundamental group of the ith ordered configuration space $\mathcal{F}_h^i(n-1,n)$ vanishes except when $n = 2$. Moreover using a dual argument we get similar results for the fundamental group of the ordered configuration space of all distinct k-dimensional subspaces $H_1, \dots, H_h$ in $\mathbb{C}^n$ such that the intersection $(H_1 \cap \dots \cap H_h)$ is an i-dimensional subspace.
We conjecture that a similar result to the one obtained in [BaS] for projective spaces holds also for Grassmannian manifolds and the fundamental group of the ith ordered configuration space $\mathcal{F}_h^i(k,n)$ vanishes except for low values of i. This will be the object of a forthcoming paper together with a generalization of the Pappus's construction in [BaS].

2 Main Section

For $0 < k < n$, let us consider the Grassmannian manifold $Gr(k,n)$ parametrizing k-dimensional subspaces of the n-dimensional complex space $\mathbb{C}^n$, and its ordered configuration spaces $\mathcal{F}_h(Gr(k,n))$.

The spaces $\mathcal{F}_h^i(k,n)$. Let's define the ith ordered configuration space $\mathcal{F}_h^i(k,n)$ as the space of all distinct points $H_1, \dots, H_h$ in the Grassmannian $Gr(k,n)$ whose sum is an i-dimensional subspace of $\mathbb{C}^n$, i.e.

$$\mathcal{F}_h^i(k,n) = \{(H_1, \dots, H_h) \in \mathcal{F}_h(Gr(k,n)) \mid \dim(H_1 + \dots + H_h) = i\}.$$

It is an easy remark that the following results hold:

1. in order to get a non empty set, $h = 1$ forces $i = k$ and we have $\mathcal{F}_1^k(k,n) = Gr(k,n)$;

2. in order to get a non empty set, $i = 1$ forces $k = h = 1$, and we have $\mathcal{F}_1^1(1,n) = Gr(1,n) = \mathbb{CP}^{n-1}$;

3. for $h \geq 2$, $\mathcal{F}_h^i(k,n) \neq \emptyset$ if and only if $i \geq k+1$ and $i \leq \min(hk,n)$;

4. for $i = hk \leq n$, then the h subspaces giving a point of $\mathcal{F}_h^{hk}(k,n)$ are in direct sum;

5. for $h \geq 2$, $\mathcal{F}_h(Gr(k,n)) = \coprod_{i=2}^{n} \mathcal{F}_h^i(k,n)$;

6. for $h \geq 2$, the adjacency of the strata is given by

$$\overline{\mathcal{F}_h^i(k,n)} = \mathcal{F}_h^i(k,n) \coprod \mathcal{F}_h^{i-1}(k,n) \coprod \dots \coprod \mathcal{F}_h^2(k,n).$$

By the above remarks, it follows that the case $h = 1$ is trivial, so from now on we will consider $h > 1$ (and hence $i > k$).
We want to show that $\mathcal{F}_h^i(k,n)$ is (when non empty) a complex submanifold of $Gr(k,n)^h$ and compute its dimension. In order to do it we need to briefly recall a few easy facts and introduce some notation.

The determinantal variety. Let's recall that the determinantal variety $D_r(m,m')$ is the variety of $m \times m'$ matrices with complex entries of rank less than or equal to $r \leq \min(m,m')$. It is an analytic (algebraic, in fact) variety of dimension $r(m+m'-r)$ whose set of singular points is given by those matrices of rank less than r. From now on, $D_r(m,m')^*$ will denote the set of non-singular points of the determinantal variety $D_r(m,m')$, that is the set of $m \times m'$ matrices of rank equal to r.

A system of local coordinates for $Gr(k,n)^h$. Let $V_0 \subset \mathbb{C}^n$ be a subspace of dimension $\dim V_0 = n-k$, then the set

$$U_{V_0} = \{H \in Gr(k,n) \mid H \oplus V_0 = \mathbb{C}^n\}$$

is an open dense subset of $Gr(k,n)$.
Fix a basis $B = \{w_1, \dots, w_k, v_1, \dots, v_{n-k}\}$ of $\mathbb{C}^n$ such that $\{v_1, \dots, v_{n-k}\}$ is a basis of V_0. Then we get a (complex) coordinate system on U_{V_0} as follows. Let H be an element in U_{V_0}, then the affine subspaces $V_0 + w_j$ intersect H in one point u_j for any $j = 1, \cdots, k$ and $\{u_1, \cdots, u_k\}$ will be a basis of H. Hence H is uniquely determined by a $n \times k$ matrix of the form $\begin{pmatrix} I \\ A \end{pmatrix}$, where I is the $k \times k$ identity matrix and A is the $(n-k) \times k$ matrix of the coordinates of $u_1 - w_1, \dots, u_k - w_k$ with respect to the vectors $\{v_1, \dots, v_{n-k}\}$. The coefficients of A give complex coordinates in $U_{V_0} \cong \mathbb{C}^{k(n-k)}$.

Let $(H_1,\dots,H_h)$ be a point in $Gr(k,n)^h$, the open sets $U_{H_1},\dots,U_{H_h}$ in the Grassmannian manifold $Gr(n-k,n)$ have non empty intersection, that is there exists an element $V_0\in Gr(n-k,n)$ such that $V_0\oplus H_j=\mathbb{C}^n$ for all $j=1,\dots,h$. Thus, $Gr(k,n)^h$ is covered by the open sets $U^h_{V_0}$ as V_0 varies in $Gr(n-k,n)$. Taking a basis as defined above, each element in $U^h_{V_0}$ is uniquely determined by a $n\times hk$ matrix of the form $\begin{pmatrix} I & I & \cdots & I\\ A_1 & A_2 & \cdots & A_h\end{pmatrix}$ and the coefficients of $\begin{pmatrix} A_1 & A_2 & \cdots & A_h\end{pmatrix}$ give complex coordinates in $U^h_{V_0}\cong\mathbb{C}^{hk(n-k)}$.

A system of local coordinates for $\mathcal{F}^i_h(k,n)$. In terms of the above coordinates, $(H_1,\dots,H_h)\in U^h_{V_0}$ belongs to $\mathcal{F}^i_h(k,n)$ if and only if $A_j\neq A_l$ if $j\neq l$ and $\operatorname{rank}\begin{pmatrix} I & I & \cdots & I\\ A_1 & A_2 & \cdots & A_h\end{pmatrix}=i$. Let us remark that

$$\begin{aligned}\operatorname{rank}\begin{pmatrix} I & I & \cdots & I\\ A_1 & A_2 & \cdots & A_h\end{pmatrix} &=\operatorname{rank}\begin{pmatrix} I & I & \cdots & I\\ 0 & A_2-A_1 & \cdots & A_h-A_1\end{pmatrix}\\ &=k+\operatorname{rank}\begin{pmatrix} A_2-A_1 & \cdots & A_h-A_1\end{pmatrix}.\end{aligned}$$

We can then change coordinates taking the coefficients of $B_j=A_j-A_1$ instead of those of A_j for $j=2,\dots,h$. In these new coordinates $U_{V_0}\cap\mathcal{F}^i_h(k,n)$ corresponds in $\mathbb{C}^{hk(n-k)}$ to $\mathbb{C}^{k(n-k)}\times D_{i-k}(n-k,hk-k)^*$ minus the closed sets given by $B_j=0$ for $2\leq j\leq h$ and by $B_j=B_l$ for $2\leq j,l\leq h$, $j\neq l$. So far we have proved the following theorem.

Theorem 2.1. *The ith ordered configuration space $\mathcal{F}^i_h(k,n)$ is a complex submanifold of the Grassmannian manifold $Gr(k,n)$ of dimension*

$$d^i_h(k,n)=i(n-i)+hk(i-k).$$

The computation on dimension comes from the easy equality

$$k(n-k)+(i-k)(n-k+hk-k-(i-k))=i(n-i)+hk(i-k).$$

Moreover, $d^i_h(k,n)=hk(n-k)$ if and only if $i=n$ or $i=hk$ and so, as a function of i, $d^i_h(k,n)$ is strictly increasing for $i\leq\min(n,hk)$.

The fundamental group of $\mathcal{F}^{\min(n,hk)}_h(k,n)$. The space $\mathcal{F}^{min(n,hk)}_h(k,n)$ is an open subset of the ordered configuration space $\mathcal{F}_h(Gr(k,n))$ and all other (non void) $\mathcal{F}^j_h(k,n)$ have strictly lower dimension. Moreover, if $i=n$ the

difference of dimensions $d_h^i(k,n) - d_h^{i-1}(k,n)$ equals $1+hk-n$ and if $i = hk$ it equals $1+n-hk$. Then if $n \neq hk$, all (non void) $\mathcal{F}_h^j(k,n)$ with $j <\min(n,hk)$ have real codimension at least 4 in $\mathcal{F}_h(Gr(k,n))$. Then, if $n \neq hk$ and $i = \min(n,hk)$, the fundamental group of $\mathcal{F}_h^i(k,n) = \mathcal{F}_h(Gr(k,n)) \backslash \overline{\mathcal{F}_h^{i-1}(k,n)}$ is the same as the fundamental group of $\mathcal{F}_h(Gr(k,n))$ (since, by the adjacency of the strata, the closure $\overline{\mathcal{F}_h^{i-1}(k,n)}$ is the finite union of complex subvarieties of $\mathcal{F}_h(Gr(k,n))$ of real codimension at least 4).
Let us recall that the complex Grassmannian manifolds $Gr(k,n)$ are simply connected and have real dimension at least 4 except $Gr(1,2) = \mathbb{CP}^1$ and that for a simply connected manifold of real dimension at least 3 the pure braid groups vanish, i.e. $\pi_1(\mathcal{F}_h(Gr(k,n))) = 0$ if $(k,n) \neq (1,2)$.
Then we get the following corollary:

Corollary 2.2. *The fundamental group of the ith ordered configuration space $\mathcal{F}_h^i(k,n)$ vanishes if $n \neq hk$ and $i = \min(n,hk)$ except when $n = 2$ for which we get the pure braid group of the sphere.*

The dual case. Let $Gr(k,n)^*$ be the Grassmannian manifold parametrizing k-dimensional subspaces in the dual space $(\mathbb{C}^n)^*$. Then we can define the ith dual ordered configuration space $\mathcal{F}_h^i(k,n)^*$ as

$$\mathcal{F}_h^i(k,n)^* = \{(H_1,\ldots,H_h) \in \mathcal{F}_h(Gr(k,n)^*) \mid \dim(H_1 \cap \cdots \cap H_h) = i\}.$$

The spaces $\mathcal{F}_h^i(k,n)^*$ stratify the ordered configuration space $\mathcal{F}_h(Gr(k,n)^*)$ of the Grassmannian manifold $Gr(k,n)^*$.
Taking annihilators, we get homeomorphisms Ann: $Gr(n-k,n) \to Gr(k,n)^*$ which induce homeomorphisms between the $(n-i)$th ordered configuration space $F_h^{n-i}(n-k,n)$ and the ith dual ordered configuration space $F_h^i(k,n)^*$. As a consequence we get that the spaces $F_h^{\max(0,n-hk)}(n-k,n)^*$ are simply connected manifolds except when $n = 2$. In this case the fundamental group is the pure braid group of the sphere.

***i*th ordered configuration spaces of hyperplane arrangements.** Let us remark that when $k = n-1$ we get an h-uple of hyperplanes in $\mathbb{C}^n$, i.e. an ordered arrangement of hyperplanes. In this case, if $h = 1$ then $i = n-1$ and we get that the ith ordered configuration space is simply the Grassmannian manifold, i.e. $\mathcal{F}_1^{n-1}(n-1,n) = Gr(n-1,n)$. If $h > 1$, since the sum of two (different) hyperplanes is the whole space $\mathbb{C}^n$, we get that $h \geq 2$ forces $i = n$ and the following equalities hold

$$\mathcal{F}_h^n(n-1,n) = \mathcal{F}_h(Gr(n-1,n)) = \mathcal{F}_h(\mathbb{CP}^{n-1}).$$

Hence, the fundamental group of the ith ordered configuration space of hyperplane arrangements $\mathcal{F}_h^i(n-1,n)$ vanishes except when $n = 2$. In this case we get the fundamental group of the sphere $\mathbb{CP}^1$.
Taking duals, we have $\mathcal{F}_h^i(n-1,n)^* \cong \mathcal{F}_h^{n-i}(1,n)$ and the fundamental groups of the latter spaces have been computed in [BaS]. We get that the space of h-uples of distinct hyperplanes in $\mathbb{C}^n$ whose intersection has dimension equal to i is simply connected except for $i = n-1$.

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