Laboratory of Economics and Management

Sant'Anna School of Advanced Studies
Piazza Martiri della Libertà, 33 - 56127 PISA (Italy)
Tel. +39-050-883-343 Fax +39-050-883-344
Email: lem@sssup.it Web Page: http://www.lem.sssup.it/

LEM

Working Paper Series

Social Choice among Complex Objects:

Luigi Marengo°
Simona Settepanella°

° LEM-Scuola Superiore Sant'Anna

2010/02 January 2010

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Luigi Marengo[1] and Simona Settepanella[2]

[1] *LEM, Scuola Superiore Sant'Anna, Pisa, l.marengo@sssup.it*
[2] *LEM, Scuola Superiore Sant'Anna, and Department of Mathematics, University of Pisa, s.settepanella@sssup.it*

Abstract

We present a geometric model of social choice when the latter takes place among bundles of interdependent elements, that we will call objects. We show that the outcome of the social choice process is highly dependent on the way these bundles are formed. By bundling and unbundling the same set of constituent elements an authority has the power of determine the social outcome. We provide necessary and sufficient conditions under which a social outcome may be a local or global optimum for a set of objects, and we show that, by appropriately redefining the set of objects, intransitive cycles may be broken and the median voter may be turned into a loser.

Keywords: social choice; object construction power; agenda power; intransitive cycles; median voter.

JEL Classification: D03, D71, D72

Acknowledgments: We are greatly indebted to Corrado Pasquali and Mario Salvetti for sharing and discussing many of the ideas contained in this paper. We gratefully acknowledge the support given to this research by the European Commission, within the 6th FP Network of Excellence "DIME - Dynamics of Institutions and Markets in Europe" and the Specific Targeted Research Project "CO3 - Common Complex Collective Phenomena in Statistical Mechanics, Society, Economics and Biology". The usual caveat applies.

1 Introduction

Social choice theory usually assumes that agents are faced with a set of exogenously given and mutually exclusive alternatives. These alternatives are given in the sense that the pre-choice process through which they are constructed is not analyzed. Moreover, these alternatives are "simple", in the sense that are one-dimensional objects or, even when they are multidimensional, they are simply points in some portion of the homogeneous $\Re^n$ space and they lack an internal structure that limits the set of possible alternatives.

Many choices in real life situations depart substantially from this simple setting. Choices are often made among bundles of interdependent elements. Those bundles may be formed in a variety of ways, which in turn affect the selection process of a social outcome. Take, for instance, the typical textbook example of social choice: a group of friends deciding democratically what to do for the evening, for instance by pairwise majority voting. The textbook would start from such a choice set as $X = \{A, B, C, D, ldots\}$ where $A, B, C, D, \ldots$ could stand for *movie, concert, restaurant, dinner at home, ...*. At closer scrutiny, these alternatives are neither primitive nor exogenously given. Going to the movies or to a restaurant are labels for bundles of elements (e.g. with whom, where, when, movie genre, director, type of food, etc.) and everyone's preference is unlikely to be expressed before the labels get specified in their constituting elements. A skillful member of the group could easily obtain a social outcome close to the one he or she prefers by carefully crafting the objects $A, B, C, D, \ldots$ and possibly by designing a new set of objects.

Moving on, to more serious examples, candidates and parties in political elections stand for complex bundles of interdependent policies and personality traits. Committees and boards are called upon to decide upon packages of policies, e.g. a recruitment package that a university governing board has to approve. In principle, any combination of elements (subject to a budget or some other constraint) could be considered and compared (e.g. through majority voting) with any other, but in reality only a relatively small number of packages undergo examination. Typically, the bundling of elements into what we will call objects serve the purpose of reducing the number of alternatives to be examined, by decomposing the whole space of alternatives into smaller subspaces.

In this paper we present a model of social choice among bundles of elements, which we call *objects*, and model two non-standard features that objects are likely to have. First, generally objects are not simply aggregations of primitive components but have an internal structure that is likely to determine interdependencies and non-separabilities in individual preferences.

In the "what shall we do tonight" choice setting, my preferences on the with whom element is likely to be highly interdependent with the other elements, as I may well find a given person a perfect companion for an evening at the movies but relatively dislike her or his company if we finally decide to go to a restaurant.

Second, objects provide structure to the choice problem. Consider again the "what shall we do tonight" case. A possible reply to our point on bundles would be that the choice set X is underspecified and that we should start from a choice set formed by all possible combinations of the elements. However, for obvious combinatorial arguments this set, even in this simple example, would be so large that any exhaustive choice procedure, e.g. pairwise majority voting, could not be completed in a feasible time span. In our approach, objects decompose this computationally complex search space into quasi-separable subspaces (Simon 1982) by simplifying the computational task and making decisions possible.

There is also another way objects can contribute to making the determination of a social outcome easier. We will show that, by appropriate object construction, intransitive cycles that often characterize social decisions can always be eliminated. In general, coarse objects, i.e. those made of many elements, tend to produce many cycles, whereas fine objects do not. However finer objects do so by increasing the number of local optima and thereby making the social outcome more manipulable through the control of initial conditions and agendas.

Because of interdependencies that are likely to characterize individual preferences over objects, the way objects are constructed by bundling (or unbundling) elements can strongly impact on the outcome of social choice. We show that, in general, by appropriately constructing objects, the outcome of a social choice process, e.g. pairwise majority voting, may be heavily manipulated. An authority who has the power to construct objects may obtain a desired outcome even when the latter is chosen democratically. We will prove necessary and sufficient conditions for any social outcome to be a local or global optimum for a social decision rule under a specific set of objects. We will also show an algorithmic procedure to determine which set of objects, agendas and initial conditions can lead society, through a given social choice procedure, to select a given outcome.

The object construction power that we describe and analyze in this paper will be proven to be stronger and more general than the well-known agenda power (cf. theorem 1 below).

We will show that by appropriately designing objects it is possible to break any intransitive cycle that frequently characterizes social choice. As already mentioned, we will analyze how different sets of objects strike different balances in the trade-off between decidability and non-manipulability. Finally, we will discuss how another classic result, i.e. the so-called median voter theorem, may be reverted by appropriate modification of the set of objects, thus transforming the median voter into an outright loser.

A comment has to be made in order to clarify the meaning and limits of the present paper. Our model simply analyzes the properties of different bundles of the same finite set of elements (features in our terminology) into objects. All features are always decided on. We do not deal with another important (but different) phenomenon; that is, the power or capacity to select those issues society should decide on and those that society should neglect or consider secondary. In a world of potentially infinite issues, this focussing of attention effect might indeed be very important (Lakoff 2004, Riker 1982).

In order to formally analyze the properties of a social choice model with object construction power we will use some geometric properties of hyperplane arrangements and link them to graph theoretic representations. We believe that our paper also provides novel analytical tools for modeling choice problems that could be applied to a variety of different settings.

The paper is structured as follows: in section 2 we briefly discuss the similarities and differences between our approach and those already discussed in the literature. To our knowledge, the issue of object construction in social choice has never been addressed with our method. Indeed, our approach has close links with standard results on multidimensional voting and on agenda power, but there are fundamental differences that make our model new and somehow more general.

In section 3 we outline our geometric and algebraic model. A key ingredient of our analytic approach is the theory of hyperplane arrangements, whose basics are very succinctly summarized for the reader in appendix A. Then, in section 4 we draw the main results concerning how, through object construction, it is possible to manipulate social outcomes, create or eliminate agenda power, break or create intransitive cycles, and turn the median voter into a loser. We will also demonstrate that our object construction power is stronger and more general than the traditionally considered agenda power and that objects strike a balance in the trade-off between decidability and non-manipulability. Finally, in section 5 we draw some conclusions. In a companion paper (Marengo and Pasquali 2008) many analytical results obtained here are illustrated through examples and computational models. The interested reader can refer to it for algorithmic implementations of most of the arguments contained in the present paper.

2 Relation to Previous Literature

To our knowledge, the issue of object construction has not been dealt with by economic models. The literature on multidimensional voting models (Kramer 1972, Shepsle 1979, Denzau and Mackay 1981, Enelow and Hinich 1983)is Relatively close to our perspective .

In particular, Shepsle (1979) presents a model of majority voting in which institutions play a similar role to the one objects have in our own model, i.e. that of limiting the set of outcomes that undergo examination. Two institutional mechanisms are analyzed: jurisdictional restrictions – especially those induced by decentralization and division of labour among decision making units – and agenda limitations to the possible amendments to the status quo. Both limit the set of attainable outcomes and equilibria (called structure-induced equilibria) and can rule out cycles. There are at least two important differences between this perspective and ours. First, the problem tackled by all these papers is essentially the one arising from the sequential interdependency of voting: how we settle an issue today may change how we prefer to settle a related issue tomorrow. In our approach, we instead focus on interdependencies generated by how elements interact within the particular objects we are deliberating upon. Second, in Shepsle (1979), restrictions on attainable outcomes are placed by legal and organizational rules, that reduce the set of possible legal amendments to the status quo. Instead, in our approach restrictions are placed by the object construction process exerted by some agent or institution: once an object has been defined, all its instances are always generated and compared.

Enelow and Hinich (1983) consider a multi-issue case in which each issue is voted sequentially in time and when the agenda induces path-dependency, which might be mitigated by the agents' forecasting abilities.

Our work is closely linked to the literature on agenda power (McKelvey 1976, Plott and Levine 1978), and we will show that we generalize some of its results in the sense that even agenda power is subject to manipulation through object design. Moreover, our model presents some instances of a wide family of aggregation paradoxes in voting. Saari and Sieberg (2001) discuss the links between aggregation paradoxes in voting and similar aggregation paradoxes arising in statistics such as the so-called Simpson's paradox. Logrolling models (Buchanan and Tullock 1962) discuss some of these paradoxes which are similar to those in the present paper. Bernholz (1974) showed that logrolling implies cycles, therefore our result proving that cycles may be broken or created by appropriate object construction also extends to logrolling.

Our paper is also related to recent literature that has begun to analyze decision-making when agents group states of the world into coarse categories (Mullainathan 2000, Fryer and Jackson 2008). They show, among other things, that in these circumstances agents may be persuaded, meaning that uninformative messages may influence their decisions (Mullainathan, Schwartzstein, and Shleifer 2008). Our perspective is different and complementary: our objects are not categories based on similarities among the states of the world, but are bundles of different and separate elements with an internal structure of interdependencies and not sets of states of the world that agents cannot distinguish from each other.

Context-dependent voting has also been analyzed by some papers (Callander and Wilson 2006). In these papers context-dependency refers to the violation of the axiom of Independence of Irrelevant Alternatives (IIA), i.e. the assumption that the preference expressed by an agent between two outcomes x_i and x_j does not depend on the presence or absence of other outcomes in the choice set. Psychologists and marketing scholars have observed systematic violations of IIA (Kahneman and Tversky 2000). In our model we assume a different form of context dependency, meaning that preferences between two instantiations of an element (feature in our terminology) in general depend on the value taken by other traits. In the next section we argue why this form of non-separability is very likely to happen in our context of objects made up of interdependent features.

This paper is also meant to contribute to the development of rigorous analytical tools in social choice models. We provide here a geometric representation based on hyperplanes arrangement theory and algebraic topology. Indeed, geometric approaches have already been used in the literature on social choice. Donald Saari has greatly contributed to establishing general geometric representations of voting models and voting paradoxes (Saari 1994, Saari 2000a, Saari 2000b), and we will argue later that our representation is more general in many respects. Eckmann (1954), Eckmann, Ganea, and Hilton (1962), Chichilnisky (1980), Chichilnisky (1983) study the problem of the existence of a social decision function from a topological point of view and show that the paradoxes of social choice are partly a consequence of the topological structure of the spaces of ordinal preferences. On the other hand, Baryshnikov (1997) discusses the possibility of introducing topological methods in the combinatorial paradigm of social choice theory. Weinberger (2004) and Terao (2007) extend well-known results on social choice functions to, respectively, CW complexes and arrangements, thus obtaining new results for both mathematical objects.

In this respect, our model is a novel contribution to the analysis of the relation between discrete problems of social choice and their topological structure and it provides a bridge between a geometric and topological representation of a social choice problem to create a more general framework in which the topological space is manipulable through object construction.

3 Definitions and structure of the model

We assume that choices are made over bundles of elements or *features*. The number of features is finite and each feature may take one value out of a finite set of alternatives. We call $F = \{f_1, \ldots, f_n\}$ the set of such features, and in order to simplify the notation and without loss of generality, we assume that all features may take the same number $m+1$ of values: $f_i \in \{0, 1, 2, \ldots, m\}$ $\forall i = 1, \ldots, n$.

The space of social outcomes is given by the $(m+1)^n$ $n-$tuples specifying one value for every feature. We call $X = \{x_1, \ldots, x_{(m+1)^n}\}$ the set of all possible social outcomes and x_i a generic element thereof.

Example 1 *Let us consider an example of* 3 *features taking a value out of the binary set* $\{0, 1\}$. *This is equivalent to, e.g., the case presence or absence of three possible traits. Thus the space* X *of possible social outcomes is a set of* 8 *ordered* 3*-tuples of the form* (f_1, f_2, f_3) *for* $f_i \in \{0, 1\}$, $i = 1, 2, 3$.

Let us choose an hyperplane arrangement[1] in $\mathbb{R}^n$

$$\mathcal{A}_{n,m} = \{H_{i,j}\}_{\substack{1 \leq i \leq n \\ 0 \leq j \leq m-1}},$$

where $H_{i,j}$ is the hyperplane of equation $y_i = j$.

Let $\mathcal{C}$ be the set of chambers of the arrangement $\mathcal{A}_{n,m}$; then each social outcome $x_i = (i_1, \ldots, i_n)$ corresponds to the chamber C_i which contains the open set:

$$\{(f_1, \ldots, f_n) \in \mathbb{R}^n \mid i_j - 1 < f_j < i_j,\ j = 1, \ldots, n\}.$$

Example 2 *In example 1 the associated arrangement* $\mathcal{A}_{3,2}$ *is simply the arrangement given by the coordinate hyperplanes of an orthogonal Cartesian system in* $\mathbb{R}^3$.

[1] For the reader who is unfamiliar with hyperplane arrangements, we provide some basic notions in Appendix A. Moreover the interested reader can find some basic definitions and notations in order to better understand the model below in (Settepanella 2010)

Consider a population of ν agents. Each agent i is characterized by a system of preferences $\succeq_i$ over the set of social outcomes defined in the space $\mathcal{P}$. A social decision rule $\mathcal{R}$ is a function:

$$\begin{aligned} \mathcal{R} : \mathcal{P}^{\nu} & \longrightarrow \overline{\mathcal{P}} \\ (\succeq_1, \ldots, \succeq_{\nu}) & \longmapsto \succeq_{\mathcal{R}(\succeq_1, \ldots, \succeq_{\nu})} \end{aligned}$$

which determines a system of social preferences $\succeq_{\mathcal{R}(\succeq_1,\ldots,\succeq_{\nu})}$ from the preferences of ν individual agents. Any type of social decision rule may fit into our framework, provided individual preferences are expressed sincerely. For the time being we rule out the possibility of strategic misrepresentation of a person's preferences.

We assume that individual preferences are transitive over social outcomes. However, because of interdependencies and non-separabilities, transitivity might not hold for all subsets of features. Indeed, a generic agent may prefer a certain value for a given feature and for a given $n-1$-tuple of values of the other features, but a different value of the same feature when the other features are set to different values. It is well known that individual transitivity does not guarantee transitivity for a social rule (Condorcet de Caritat marquis de 1785).

In order to completely describe $\overline{\mathcal{P}}$, let us mention that if $\Delta = \{(x_i, x_i) \in X \times X\}$ is the *diagonal* of the Cartesian product $X \times X$, then an element $\succeq_{\mathcal{R}} \in \overline{\mathcal{P}}$ defines a subset:

$$Y_{1,\succeq_{\mathcal{R}}} \subset X \times X \setminus \Delta$$

as follows: a pair (x_i, x_j) is in $Y_{1,\succeq_{\mathcal{R}}}$ if and only if $x_i \succeq_{\mathcal{R}} x_j$.

Without loss of generality we will indicate by $\succeq_{\mathcal{R}} \in \overline{\mathcal{P}}$ a general element in the image of $\mathcal{R}$.

We call $Y_{0,\succeq_{\mathcal{R}}}$ the set of *relevant* social outcomes, i.e. the set of all social outcomes on which a social preference is expressed:

$$Y_{0,\succeq_{\mathcal{R}}} = \{x_i \in X \mid \exists x_j : (x_i, x_j) \in Y_{1,\succeq_{\mathcal{R}}} \text{ or } (x_j, x_i) \in Y_{1,\succeq_{\mathcal{R}}}\}.$$

The subset $Y_{0,\succeq_{\mathcal{R}}} \subseteq X$ is in one to one correspondence with a subset $\mathcal{C}_{\succeq_{\mathcal{R}}} \subset \mathcal{C}$ of the set of chambers in $\mathcal{A}_{n,m}$:

$$C_i \in \mathcal{C} \Leftrightarrow x_i \in Y_{0,\succeq_{\mathcal{R}}}.$$

Example 3 *Let us consider the space of social outcomes X as in example 1 with the following rule $\succ_{\mathcal{R}}$:*

$(0,0,0)$ *preferred to all, except* $(1,1,0) \succ_{\mathcal{R}} (0,0,0), (0,0,1) \succ_{\mathcal{R}} (0,0,0)$;
$(0,1,0) \prec_{\mathcal{R}} (0,1,1), (0,1,0) \prec_{\mathcal{R}} (1,1,1), (0,1,0) \prec_{\mathcal{R}} (1,0,0)$,
$(0,1,0) \succ_{\mathcal{R}} (1,0,1), (0,1,0) \succ_{\mathcal{R}} (1,1,0), (0,1,0) \prec_{\mathcal{R}} (0,0,1)$;
$(0,1,1) \succ_{\mathcal{R}} (1,1,1), (0,1,1) \succ_{\mathcal{R}} (1,0,0), (0,1,1) \succ_{\mathcal{R}} (1,0,1)$,
$(0,1,1) \succ_{\mathcal{R}} (1,1,0), (0,1,1) \prec_{\mathcal{R}} (0,0,1)$;
$(1,1,1) \succ_{\mathcal{R}} (1,0,0), (1,1,1) \succ_{\mathcal{R}} (1,0,1), (1,1,1) \succ_{\mathcal{R}} (1,1,0), (1,1,1) \succ_{\mathcal{R}} (0,0,1)$;
$(1,0,0) \succ_{\mathcal{R}} (1,0,1), (1,0,0) \succ_{\mathcal{R}} (1,1,0), (1,0,0) \prec_{\mathcal{R}} (0,0,1)$;
$(1,0,1) \succ_{\mathcal{R}} (1,1,0), (1,0,1) \prec_{\mathcal{R}} (0,0,1)$;
$(1,1,0) \prec_{\mathcal{R}} (0,0,1)$.

For simplicity we will denote by $f_1f_2f_3$ the 3-tuple (f_1, f_2, f_3). Thus $Y_{0,\succeq_{\mathcal{R}}} = \{f_1f_2f_3 \mid f_j = 0$ or $1\} = X$.
$Y_{1,\succeq_{\mathcal{R}}}$ is given by all pairs $(f_1f_2f_3, g_1g_2g_3)$ such that $f_1f_2f_3 \succ_{\mathcal{R}} g_1g_2g_3$; i.e., for example, $(000,101),(011,000) \in Y_{1,\succeq_{\mathcal{R}}}$ *while* $(000,011) \notin Y_{1,\succeq_{\mathcal{R}}}$.

If social preferences are complete over X, then $Y_{0,\succeq_{\mathcal{R}}} = X$. In the sequel we will consider only complete preferences, although our framework may easily accommodate the more general case where $Y_{0,\succeq_{\mathcal{R}}} \subseteq X$.

We can represent the sets $Y_{0,\succeq_{\mathcal{R}}}$ and $Y_{1,\succeq_{\mathcal{R}}}$ respectively as the set of vertices and edges of an **oriented graph** $\mathcal{Y}_{\succeq_{\mathcal{R}}}$. Two vertices x_i and x_j in $Y_{0,\succeq_{\mathcal{R}}}$ are connected by an edge if and only if $(x_i, x_j) \in Y_{1,\succeq_{\mathcal{R}}}$ or $(x_j, x_i) \in Y_{1,\succeq_{\mathcal{R}}}$, while the orientation is from x_i to x_j in the former case and from x_j to x_i in the latter. In a natural way this construction applies to all subsets $Y \subset X \times X \setminus \Delta$.

Without loss of generality we will denote by x_i the vertices of $\mathcal{Y}_{\succeq_{\mathcal{R}}}$ and by (x_i, x_j) its edges.

Example 4 *The social preferences of example 3 are fully described by the oriented graph represented in figure 1.*

Notice that the assumption of complete preferences guarantees that we will deal only with *connected* graphs.

A *cycle of length h* in the oriented graph $\mathcal{Y}_{\succeq_{\mathcal{R}}}$ is a subgraph γ_I with vertices $\{x_{i_1}, \ldots, x_{i_h}\}$ and edges $\{(x_{i_1}, x_{i_2}), (x_{i_2}, x_{i_3}), \ldots, (x_{i_h}, x_{i_1})\}$. It corresponds to a Condorcet-Arrow cycle, i.e. to the sequence $x_{i_1} \succeq_{\mathcal{R}} x_{i_2} \succeq_{\mathcal{R}} \ldots \succeq_{\mathcal{R}} x_{i_h} \succeq_{\mathcal{R}} x_{i_1}$. In example 3 and in the corresponding graph of figure 1 we find a three outcomes cycle $000 \succ_{\mathcal{R}} 101 \succ_{\mathcal{R}} 011 \succ_{\mathcal{R}} 000$.

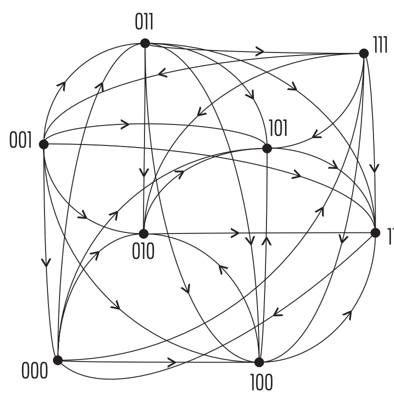

Figure 1: The oriented graph derived from example 3

As shown by Salvetti (1987), we can construct a CW-complex $\mathcal{S}(\mathcal{A})$ having the homotopy type of the complement $M(\mathcal{A})$ when the arrangement $\mathcal{A}$ is the complexification of a real one.

We briefly recall here the construction of this complex. In the next section we will give a more detailed account of how it can be applied to our social choice problem.

Let $\mathcal{A} = \{H\}$ be a finite affine hyperplane arrangement in $\mathbb{R}^n$. Assume $\mathcal{A}$ is essential, so that the minimal dimensional non-empty intersections of hyperplanes are points (that we call *vertices* of the arrangement). Equivalently, the maximal elements of the associated *intersection lattice* $L(\mathcal{A})$ have rank n (Orlik and Terao 1992).

Let $M(\mathcal{A}) = \mathbb{C}^n \setminus \bigcup_{H \in A} H_{\mathbb{C}}$ be the complement to the complexified arrangement. The regular CW-complex $\mathcal{S}(\mathcal{A})$ can be characterized as follows: let $\mathbf{S} := \{F^k\}$ be the stratification of $\mathbb{R}^n$ into facets F^k which is induced by the arrangement (Bourbaki 1968), where the exponent k stands for codimension. Then $\mathbf{S}$ has standard partial ordering

$$F^i \prec F^j \quad \text{iff} \quad clos(F^i) \supset F^j$$

where $clos(F^i)$ is the closure of F^i.

The k-cells of $\mathcal{S}(\mathcal{A})$ bijectively correspond to pairs $[C \prec F^k]$ where $C = F^0$ is a chamber of $\mathbf{S}$.

Let $|F|$ be the affine subspace spanned by F, and let us consider the sub-arrangement $\mathcal{A}_F = \{H \in \mathcal{A} : F \subset H\}$. A cell $[C \prec F^k]$ is in the boundary of $[D \prec G^j]$ $(k < j)$ iff:

i) $F^k \prec G^j$

ii) the chambers C and D are contained in the same chamber of $\mathcal{A}_{F^k}$.

The previous conditions are equivalent to saying that C is the chamber of $\mathcal{A}$ which is"closest" to D among those containing F^k in their closure.

It is possible to realize $\mathcal{S}(\mathcal{A})$ inside $\mathbb{C}^n$ with explicitly given attaching maps of the cells (Salvetti 1987). Salvetti complex provides a very useful model for our construction. Indeed, each element $\succeq_{\mathcal{R}} \in \overline{\mathcal{P}}$ is equivalent to a subcomplex of the 1-skeleton of $\mathcal{S}(\mathcal{A}_{n,m})$.

Thus, let us characterize the 0 and 1-dimensional cells (also called 0 and 1-skeletons) of Salvetti complex and show how they are related to the graph $\mathcal{Y}_{\succeq_{\mathcal{R}}}$ for a given social rule $\succeq_{\mathcal{R}}$.

Let $\mathcal{A}_{n,m}$ be the arrangement associated to the space of social outcomes and $\mathcal{S}(\mathcal{A}_{n,m})$ the correspondent Salvetti complex. The set of generators $\mathcal{S}_0(\mathcal{A}_{n,m})$ of the 0-skeleton of the complex $\mathcal{S}(\mathcal{A}_{n,m})$ is in a one-to-one correspondence with the set of chambers in $\mathcal{A}_{n,m}$, i.e. with the set of social outcomes X. Thus:

$$\mathcal{S}_0(\mathcal{A}_{n,m}) = Y_{0,\succeq_{\mathcal{R}}} = X$$

for any given social rule $\succeq_{\mathcal{R}}$.

Example 5 *The* 0*-skeleton* $\mathcal{S}_0(\mathcal{A}_{3,2})$ *associated to example 1 is given by* 8 *points, one in each chamber of the coordinate arrangement* $\mathcal{A}_{3,2}$*. Thus they can be represented as the vertices of a cube, as shown in figure 1.*

Let us recall that two chambers C_i and C_j are said to be *adjacent* if and only if they are separated by only one hyperplane H. If x_i and x_j are the 0-cells corresponding to C_i and C_j, then we can consider the edge (x_i, x_j) between them.

The generators of the 1-skeleton can be described as the elements in the set:

$$\mathcal{S}_1(\mathcal{A}_{n,m}) = \{(x_i, x_j) \in X \times X \setminus \Delta \mid x_i \text{ and } x_j \text{ are adjacent}\}.$$

Two elements in $\mathcal{S}_1(\mathcal{A}_{n,m})$ are *consecutive* if and only if the second entry of the first pair is equal to the first entry of the second pair: for example, (x_i, x_j) and (x_j, x_k). Given a subset of consecutive elements in $\mathcal{S}_1(\mathcal{A}_{n,m})$

$$\{(x_{i_1}, x_{i_2}), (x_{i_2}, x_{i_3}), \ldots, (x_{i_{k-2}}, x_{i_{k-1}}), (x_{i_{k-1}}, x_{i_k})\} \in \mathcal{S}_1(\mathcal{A}_{n,m})$$

we define their *formal sum* as:

$$(x_{i_1}, x_{i_k}) = \sum_{j=1}^{k-1} (x_{i_j}, x_{i_{j+1}}). \tag{1}$$

Let C_i and C_j be two adjacent chambers separated by the hyperplane H. We say that we move from C_i to C_j across H if we move along the edge (x_i, x_j). Thus, by moving across hyperplanes, we can reach any chamber of the arrangement starting from a given one. If we only cross each hyperplane once, we call this path *minimal*. Obviously, there are many minimal paths, depending on which order we cross hyperplanes, however, they are all homotopically equivalent (Salvetti 1987).

It follows that given a rule $\succeq_{\mathcal{R}}$, any edge $(x_i, x_j) \in Y_{1,\succeq_{\mathcal{R}}}$ can be written as a formal sum of a minimal number of consecutive elements in $\mathcal{S}_1(\mathcal{A}_{n,m})$. The number of elements is exactly the number of hyperplanes that separate the two social outcomes $x_i, x_j \in X$.

Moreover, let $(x_i, x_j) \in Y_{1,\succeq_{\mathcal{R}}}$ be an edge given by a formal sum with coefficient 1 of edges which are in $Y_{1,\succeq_{\mathcal{R}}}$. Then, under the assumption of transitive preferences, it can be deleted.

Example 6 *Given the rule $\succ_{\mathcal{R}}$ in the example 3, we may notice, for instance, that the element $(000, 101) \in Y_{1,\succeq_{\mathcal{R}}}$ can be represented in two ways as a formal sum of a minimal number of consecutive elements in $\mathcal{S}_1(\mathcal{A}_{3,2})$ as follows:*

$$(000, 101) = (000, 100) + (100, 101) \text{ or } (000, 101) = (000, 001) + (001, 101)$$

By proceeding this way the graph in figure 1 can be reduced into the one in figure 2. To be precise, the graph in figure 2 does not exactly represent Salvetti's complex, as the latter is in the complexification of $\mathcal{A}_{3,2}$, i.e. in $\mathbb{R}^6$, which allows paths to "go around" hyperplanes. However figure 2 gives a useful, though not totally precise, visual representation.

Clearly, there are many ways we can reduce the set $Y_{1,\succeq_{\mathcal{R}}}$ to a basic number of elements, and this number is not unique since we are not in a vector field. However, all the reduced graphs represent exactly the same pairwise social preferences.

The graph of fig. 2 corresponds to reducing $Y_{1,\succeq_{\mathcal{R}}}$ to the following base:

$$\begin{aligned}&\{(000, 011), (000, 010), (000, 100), (110, 000), (001, 011), (011, 111),\\&(111, 110), (111, 001), (010, 110), (100, 101), (101, 110)\}\end{aligned} \tag{2}$$

but there is also another base given by the following 10 *elements:*

$$\begin{aligned}&\{(000, 010), (001, 000), (000, 011), (010, 101), (100, 010),\\&(011, 111), (111, 100), (101, 110), (111, 001), (110, 000)\}.\end{aligned}$$

which is represented in graph 3.

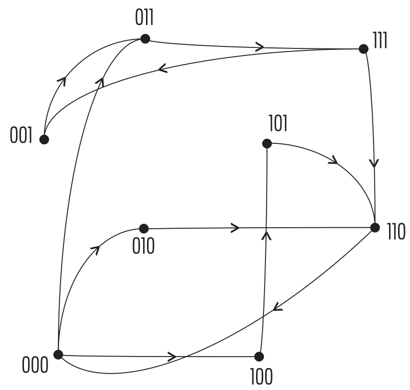

Figure 2: Reduced graph with a base of 11 elements

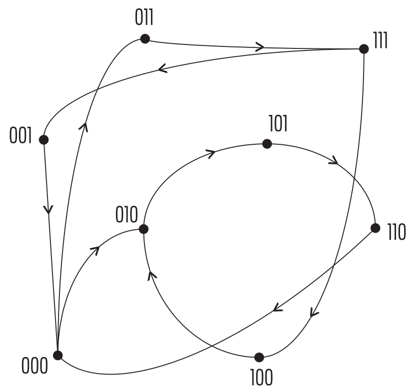

Figure 3: Reduced graph with a base of 10 elements

However if we move along these two graphs, pairwise preferences are exactly the same and correspond to those of example 3.

The reduced graph carries very useful information. For example it is relatively simple to find the fundamental cycles that generate all the others. Looking at the graph in fig. 2 it is clear that, for this base, the foundamental cycles are:

$$\begin{aligned}
&001 \succ_{\mathcal{R}} 011 \succ_{\mathcal{R}} 111 \succ_{\mathcal{R}} 001,\\
&000 \succ_{\mathcal{R}} 010 \succ_{\mathcal{R}} 110 \succ_{\mathcal{R}} 000,\\
&000 \succ_{\mathcal{R}} 011 \succ_{\mathcal{R}} 111 \succ_{\mathcal{R}} 110 \succ_{\mathcal{R}} 000,\\
&000 \succ_{\mathcal{R}} 100 \succ_{\mathcal{R}} 101 \succ_{\mathcal{R}} 110 \succ_{\mathcal{R}} 000.
\end{aligned} \tag{3}$$

By construction, the first one depends on the existence of the edge $(111, 001)$, *while the other three depend on* $(110, 000)$. *Later in this paper we will use this information in order to determine which set of objects can break these cycles.*

So far we have described the cells of a complex. Obviously there is a boundary map on Salvetti's complex which is also very informative. We will introduce the boundary only for the 1-skeleton of the complex $\mathcal{S}(\mathcal{A}_{n,m})$. Let us consider the free abelian groups $S_1(\mathcal{A}_{n,m})$ and $S_0(\mathcal{A}_{n,m})$ generated by $\mathcal{S}_1(\mathcal{A}_{n,m})$ and $\mathcal{S}_0(\mathcal{A}_{n,m})$ considering all the formal sums with integer coefficients.

The boundary map $\partial : S_1(\mathcal{A}_{n,m}) \longrightarrow S_0(\mathcal{A}_{n,m})$ is defined as follows. Given a generator $(x_i, x_j) \in \mathcal{S}_1(\mathcal{A}_{n,m})$, then:

$$\partial(x_i, x_j) = x_j - x_i.$$

The map extends to $S_1(\mathcal{A}_{n,m})$ by linearity.

Given a social rule $\succeq_{\mathcal{R}}$ we obtain a graph $\mathcal{Y}_{\succeq_{\mathcal{R}}}$ which is, as we have seen above, a subcomplex of Salvetti's complex in 0 and 1-dimensions. We can compute its first homology group $H_1(\mathcal{Y}_{\succeq_{\mathcal{R}}})$ that is the free abelian group generated by all cycles in $\mathcal{Y}_{\succeq_{\mathcal{R}}}$.

Studying the first homology group of our graph is equivalent to studying the Condorcet-Arrow's cycles of the corresponding social rule. For example, if $H_1(\mathcal{Y}_{\succeq_{\mathcal{R}}}) = 0$ one can infer that the social rule $\mathcal{R}$ has generated transitive social preferences.

Remark 3.1 *The boundary operator transforms a geometric problem into an algebraic one and makes it possible to develop an algorithm that computationally performs the reduction of a given directed graph. We may begin by noticing that if two formal sums with coefficients* 1 *in* $Y_{1,\succ_{\mathcal{R}}}$ *are equivalent from a social point of view, i.e. they correspond to the same pairwise preferences, then they have the same boundary. Indeed, by linearity, the boundary of the sum in (1) is given by:*

$$\partial \sum_{j=1}^{k-1} (x_{i_j}, x_{i_{j+1}}) = \sum_{j=1}^{k-1} \partial(x_{i_j}, x_{i_{j+1}}) =$$
$$\sum_{j=1}^{k-1} x_{i_{j+1}} - \sum_{j=1}^{k-1} x_{i_j} = x_{i_k} - x_{i_1} = \partial(x_{i_1}, x_{i_k}).$$

Thus the boundary of such a formal sum depends only by its initial and final points.

Example 7 *In example 3 the boundary of each edge in $Y_{1,\succ_{\mathcal{R}}}$ is*

$$\partial(f_1 f_2 f_3, g_1 g_2 g_3) = g_1 g_2 g_3 - f_1 f_2 f_3.$$

We get a boundary matrix of 28 *columns, one for each edge, and* 8 *rows, one for each vertex, with two entries equal* 1 *and* −1 *(those corresponding to the vertices of the edge in question) in each column and all the other entries equal to 0. Now our algorithmic implementation must simply delete any column which is the sum of two or more other columns. In our example, we can get the following reduced matrix:*

$$\begin{bmatrix} -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

where the columns are ordered following the order of set (2) in the example 6 above and the rows are ordered lexicographically. The above matrix represents the boundary map of the graph in figure 2. Moreover, if the sum of three or more columns is the zero column, then we have a cycle. Indeed, for example, the sum of the 5*-th* 6*-th and* 8*-th columns is* 0 *and it corresponds to the first cycle described in (3).*

Our graph representation enables us to determine the number of cycles in a very straightforward way. Indeed, if $\mathcal{G}$ is a graph, the number of cycles is the rank of the first homology group $H_1(\mathcal{G})$ and in graph theory there is a theorem (see for instance Massey (1981)) which states that:

$$\text{rank}(H_0(\mathcal{G})) - \text{rank}(H_1(\mathcal{G})) = \sharp(G_0) - \sharp(G_1).$$

where $\sharp(G_0) - \sharp(G_1)$ is the *Euler Characteristic* of the graph $\mathcal{G}$, i.e. the difference between the number of vertices and the number of edges. Moreover, we are typically dealing with connected graphs and therefore, recalling that the rank of the 0-homology group of a connected graph is 1, we have $\text{rank}(H_1(\mathcal{G})) = 1 - \sharp(G_0) + \sharp(G_1)$.

Remark 3.2 *Notice that Salvetti's complex is a CW-complex in $\mathbb{C}^n$, but it has a underlying real structure which is a purely simplicial complex. This structure can be used in order to recast and generalize some existing geometric models of voting such as those provided by Saari (1994).*

4 Object construction and social outcomes

In this section we develop our main results. We define objects as bundles of features, and show that, in general, by appropriate manipulations of the set of objects, almost every social outcome may be obtained from a given social rule (e.g. majority voting). Such "object construction power", that is, the power of determining social outcomes by appropriately bundling or unbundling features, is stronger than the agenda power traditionally studied in the literature (McKelvey 1976). We then show that this power of manipulation also includes the possibility of breaking or creating intransitive cycles *à* la Condorcet-Arrow and of overturning the median voter effect. Finally we discuss the emerging trade-off between non manipulability and decidability. Coarser objects make social decisions less manipulable but they also increase the likelihood of intransitive cycles and the time required to reach a socially optimal outcome (if any). On the other hand finer objects make cycles less likely and reach a social outcome faster, but the number of locally optimal social outcomes greatly increases and, therefore, the scope for manipulability becomes broader.

4.1 Objects in social choice

With the notation introduced above, let $\mathcal{A}_{n,m}$ be the hyperplane arrangement defined by n features and m possibilities for each of them, and let $\succeq_{\mathcal{R}}$ be the social preferences over the set of social outcomes X. Given a subset $I \subset \{1, \ldots, n\}$, an *object* $\mathcal{A}_I$ is a non empty subset of the arrangement $\mathcal{A}_{n,m}$ of the form

$$\mathcal{A}_I = \{H_{i,j}\}_{\substack{i \in I \\ 0 \leq j \leq m-1}} .$$

The cardinality of $\mathcal{A}_I$ is called *size* of the object $\mathcal{A}_I$. We will also denote by $\mathcal{A}_{I^c} = \mathcal{A}_{n,m} \setminus \mathcal{A}_I$ the complement of the arrangement $\mathcal{A}_I$ in $\mathcal{A}_{n,m}$.

A set of objects $A = \{\mathcal{A}_{I_1}, \ldots, \mathcal{A}_{I_k}\}$ such that $\cup_{j=1}^k I_j = \{1, \ldots, n\}$ is an *objects scheme*. Notice that an objects scheme does not have to partition the set of features, as some of them may belong to more than one object. However, we require that the union of all objects covers all the features, otherwise the remaining features would not be decided on and could be simply omitted from the model.

Let x_j be a social outcome in X, i.e. a chamber of $\mathcal{A}_{n,m}$, then the *object instantiation* $x_j(\mathcal{A}_I)$ is the chamber of the subarrangement $\mathcal{A}_I$ which contains the chamber corresponding to x_j, as shown, for instance, in figure 4.

We can also define an operator between instantiations of different objects

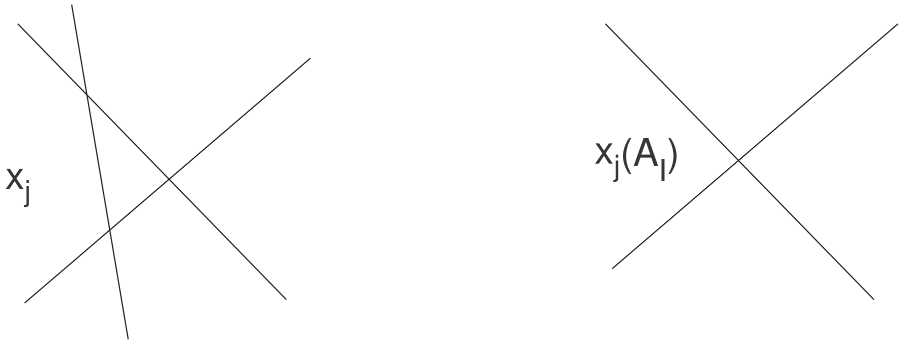

Figure 4: A graphical characterization of object instantiation

as follows:

$$x_i(\mathcal{A}_I) \vee x_j(\mathcal{A}_J) = z(\mathcal{A}_I \cup \mathcal{A}_J)$$

where z is the chamber of the arrangement $\mathcal{A}_I \cup \mathcal{A}_J$ obtained as intersection between $x_i(\mathcal{A}_I) \cap x_j(\mathcal{A}_J \setminus \mathcal{A}_I)$.

Moreover, the *size* of an objects scheme is the size of its largest object:

$$\mid A \mid = \max\{\mid \mathcal{A}_{I_1} \mid, \ldots, \mid \mathcal{A}_{I_k} \mid\}.$$

Given an objects scheme $A = \{\mathcal{A}_{I_1}, \ldots, \mathcal{A}_{I_k}\}$, we say that a social outcome x_j is a *preferred neighbor* of a social outcome x_i with respect to an object $\mathcal{A}_{I_h} \in A$ if the following conditions hold:

1. $x_j \succeq_{\mathcal{R}} x_i$,
2. $x_j(\mathcal{A}_{I_h^c}) = x_i(\mathcal{A}_{I_h^c})$, i.e. x_i and x_j belong to the same chamber of the arrangement $\mathcal{A}_{I_h^c}$,
3. $x_j(\mathcal{A}_{I_h}) \neq x_i(\mathcal{A}_{I_h})$, i.e. x_i and x_j belong to different chambers of the arrangement $\mathcal{A}_{I_h}$.

Let us define the set:

$$\Phi(x_i, \mathcal{A}_{I_h}) = \{x_j \mid x_j \text{ is a preferred neighbor of } x_i \text{ with respect to } \mathcal{A}_{I_h}\}$$

and $\Phi(x_i, A) = \cup_{j=1}^{k} \Phi(x_i, \mathcal{A}_{I_j})$.

The set of *best neighbors* $B(x_i, \mathcal{A}_{I_h}) \subset \Phi(x_i, \mathcal{A}_{I_h})$ with respect to an object $\mathcal{A}_{I_h} \in A$ is defined as:

$$B(x_i, \mathcal{A}_{I_h}) = \{x_j \in \Phi(x_i, \mathcal{A}_{I_h}) \mid x_j \succeq_{\mathcal{R}} x_k \ \forall x_k \in \Phi(x_i, \mathcal{A}_{I_h})\};$$

and $B(x_i, A) = \cup_{j=1}^{k} B(x_i, \mathcal{A}_{I_j})$.

We call $P(x_i, x_j, A)$ *a path through A, starting from x_i and ending in x_j* a succession of best neighbors with respect to the objects in A, i.e. a succession:

$$x_i = x_{i_0} \preceq_{\mathcal{R}} x_{i_1} \preceq_{\mathcal{R}} \ldots \preceq_{\mathcal{R}} x_{i_{s+1}} = x_j \tag{4}$$

such that there exist objects $\mathcal{A}_{I_{h_0}}, \ldots \mathcal{A}_{I_{h_s}} \in A$ with $x_{i_{t+1}} \in B(x_{i_t}, \mathcal{A}_{I_{h_t}})$ for all $0 \leq t \leq s$.

A social outcome x_j is *reachable* from x_i with respect to an objects scheme A if and only if there exists a path $P(x_i, x_j, A)$. A social outcome x_i is a *local optimum* for A if and only if for all $x_j \in \Phi(x_i, A)$, $(x_j, x_i) \notin Y_{1,\succeq_{\mathcal{R}}}$. In the case of strict preferences, this amounts to the condition $\Phi(x_i, A) = \emptyset$.

Let $A = \{\mathcal{A}_{I_1}, \ldots, \mathcal{A}_{I_k}\}$ be a scheme; we call *agenda* α an ordered tuple of indices $(h_0, \ldots, h_t)$ such that the set $\{h_0, \ldots, h_t\} = \{1, \ldots, k\}$. Then an agenda α states the order in which objects $\mathcal{A}_{I_i}$ are decided on. In general all objects of the given scheme appear at least once in an agenda, but they may appear more than once. In fact, because of non-separabilities, it is possible that after modifications of other objects, an object previously set into what appeared to be a locally optimal social outcome, may undergo further improvements. In order to study only the stable limit properties of agendas, we suppose that the agenda is repeated over and over again until either a stable optimum or a cycle is reached.

Let $\alpha = (h_0, \ldots, h_t)$ be an agenda and A_α be the ordered tuple of objects in A, we say that a path

$$x_{i_0} \preceq_{\mathcal{R}} x_{i_1} \preceq_{\mathcal{R}} \ldots \preceq_{\mathcal{R}} x_{i_s} \tag{5}$$

is ordered along α iff $x_{i_1} \in B(x_{i_0}, \mathcal{A}_{I_{h_0}})$ and, if $x_{i_j} \in B(x_{i_{j-1}}, \mathcal{A}_{I_{h_q}})$ then $x_{i_{j+1}} \in B(x_{i_j}, \mathcal{A}_{I_{h_{q+1}}})$, with the condition that $h_{t+1} = h_0$. Such a path is denoted by $P(x_{i_0}, x_{i_s}, A_\alpha)$.

We say that a path $P(x, A_\alpha)$ is *maximal* if and only if it ends up in a local optimum or a limit cycle.

For a given social outcome x_i and an objects scheme A, we define the *basin of attraction* of x_i as the set:

$$\Psi(x_i, A) = \{x_j \mid \exists \text{ a maximal path } P(x_j, x_i, A)\}.$$

Clearly $\Psi(x_i, A) \neq \emptyset$ iff x_i is a local optimum for some agenda. While the *ordered basin of attraction* of x_i is:

$$\Psi(x_i, A_\alpha) = \{x_j \mid P(x_j, A_\alpha) \text{ ends up in } x_i\}$$

and $\Psi(x_i, A_\alpha) \neq \emptyset$ iff x_i is a local optimum for the agenda α.

Let $\Lambda(A)$ be the set of all possible agendas for an objects scheme A, then:

$$\Psi(x_i, A) = \cup_{\alpha \in \Lambda(A)} \Psi(x_i, A_\alpha).$$

A social outcome $z \in X$ is a *global optimum for an agenda* α if and only if $\Psi(x_i, A_\alpha) = X$. It is a *global optimum for the objects scheme* A if and only if $\Psi(x_i, A_\alpha) = X$ for all agendas $\alpha \in \Lambda(A)$, i.e. it is a global optimum for all agendas.

It is easy to verify that local and global optima strictly depend on the choice of the objects scheme A.

Example 8 *Let us go back to example 3 and denote by H_i the hyperplane with equation $x_i = 0$ for $i = 1, 2, 3$. If the objects scheme is given by $A = \{\{H_1, H_2, H_3\}\}$, or by $A = \{\{H_1, H_2\}, \{H_3\}\}$ then the social rule always produces a cycle, that may be either* $001 \succ_{\mathcal{R}} 011 \succ_{\mathcal{R}} 111 \succ_{\mathcal{R}} 001$ *or* $000 \succ_{\mathcal{R}} 010 \succ_{\mathcal{R}} 110 \succ_{\mathcal{R}} 000$ *depending on the chosen agenda and initial condition. While if $A = \{\{H_1, H_3\}, \{H_2\}\}$ or $A = \{\{H_2, H_3\}, \{H_1\}\}$ then there exists a unique global optimum* 001 *for all agendas.*

The dependence of the optimum on the objects scheme is very strong. Indeed, there are many examples in which two different social outcomes $z_1, z_2 \in X$ are global optima for two different choice of objects schemes. More precisely, there exist two objects schemes A_1 and A_2 such that $\Psi(z_i, A_i) = X$ and $\Psi(x, A_i) = \emptyset$ for all social outcomes $x \neq z_i$ and $i = 1, 2$.

Example 9 *Let us consider the space of social outcomes X as in example 1 with the following preferences $\succ_{\mathcal{R}}$:*

$(0,0,0)$ *preferred to all except* $(0,1,1) \succ_{\mathcal{R}} (0,0,0)$
$(0,1,1)$ *preferred to all except* $(1,0,1) \succ_{\mathcal{R}} (0,1,1)$
$(1,0,1) \prec_{\mathcal{R}} (1,0,0), (1,0,1) \succ_{\mathcal{R}} (1,1,1), (1,0,1) \succ_{\mathcal{R}} (0,0,1),$
$(1,0,1) \prec_{\mathcal{R}} (1,1,0), (1,0,1) \prec_{\mathcal{R}} (0,1,0)$
$(0,0,1) \succ_{\mathcal{R}} (1,1,1), (0,0,1) \prec_{\mathcal{R}} (0,1,0), (0,0,1) \prec_{\mathcal{R}} (1,1,0), (0,0,1) \prec_{\mathcal{R}} (1,0,0)$
$(1,1,0) \succ_{\mathcal{R}} (1,1,1), (1,1,0) \succ_{\mathcal{R}} (1,0,0), (1,1,0) \prec_{\mathcal{R}} (0,1,0)$
$(1,1,1) \prec_{\mathcal{R}} (0,1,0), (1,1,1) \prec_{\mathcal{R}} (1,0,0)$
$(0,1,0) \succ_{\mathcal{R}} (1,0,0).$

With the objects scheme $A_1 = \{\{H_1, H_2\}, \{H_3\}\}$ the social outcome $(0,0,0)$ is the unique global optimum, while with the objects scheme $A_2 = \{\{H_2, H_3\}, \{H_1\}\}$ the unique global optimum is $(0,1,1)$.

From now on A will be an objects scheme for a given social rule $\succ_{\mathcal{R}}$ on a social outcome space X that corresponds to the arrangement $\mathcal{A}_{n,m}$.

An interesting question is to what extent a local optimum x_i depends on the agenda. Let x_i be a local optimum for the agenda $\alpha \in \Lambda(A)$ and let $\beta \in \Lambda(A)$ be another agenda, then is x_i still a local optimum for the new agenda?

The answer to this question is yes. Indeed, we can prove the following theorem which demonstrates that object construction power is, in some sense, stronger than agenda power.

Theorem 1 *Let A be an objects scheme. A social outcome $z \in X$ is a local optimum for A, i.e. $\Psi(z, A) \neq \emptyset$ if and only if it is a local optimum for all possible agendas, i.e. $\Psi(z, A_\alpha) \neq \emptyset$ for all $\alpha \in \Lambda(A)$.*

In order to prove the theorem we need the following lemma:

Lemma 4.1 *Let A be an objects scheme and a social outcome $z \in X$ be a local optimum for A, then*

$$z \in \cap_{\alpha \in \Lambda(A)} \Psi(z, A_\alpha)$$

Proof. Let z be a local optimum for A then for all objects $\mathcal{A}_I \in A$ the set $\Phi(z, \mathcal{A}_I) = \emptyset$. It follows that $z \in \Psi(z, A_\alpha) \neq \emptyset$ for all $\alpha \in \Lambda(A)$ □

Proof of Theorem 1. It is a direct consequence of the above lemma. □

Notice, however, that the basin of attraction of a social outcome is, in general, different for different agendas and, therefore, even when starting from the same initial social outcome the choice processes may end up in different local optima for different agendas.

Another interesting question is whether and under which conditions, given a social outcome $z \in X$, it is possible to choose an objects scheme A such that the basin of attraction $\Psi(z, A)$ is not empty, i.e. such that z is a local optimum or a global one for some objects scheme and some agenda. The answers to these questions are given in the next two sections.

4.2 Objects and manipulability: local optimality

Suppose one of the agents has a preferred social outcome $z \in X$ and some form of power in the determination of the objects, e.g. because he or she is chairing a committee or he/she may somehow influence the cognitive framing of the choice, for instance through persuasion (Mullainathan, Schwartzstein, and Shleifer 2008). An interesting question is whether there exists an objects scheme $A = \{\mathcal{A}_{I_1}, \ldots, \mathcal{A}_{I_k}\}$ such that z is a local optimum for A, and, if this is the case, which combination of initial social outcome and agenda will produce z as social outcome.

Let z and x be two social outcomes, we say that they are *separated* by an hyperplane $H \in \mathcal{A}_{n,m}$, $z \mid H \mid x$, if H separates the chambers C_z and C_x. Moreover we say that z and x are *prominently separated* if there exist two hyperplanes $H_{i_1,j_1}, H_{i_2,j_2} \in \mathcal{A}_{n,m}$ with $i_1 \neq i_2$ and $z \mid H_{i_1,j_1} \mid x$, $z \mid H_{i_2,j_2} \mid x$.

Let us define the subset $\mathcal{H}_{x,z} \subset \mathcal{A}_{n,m}$ as follows:

$$\mathcal{H}_{x,z} = \{H \in \mathcal{A}_{n,m} \mid x \mid H \mid z\}.$$

Moreover we say that z and x are *prominently separated* if there exist two hyperplanes $H_{i_1,j_1}, H_{i_2,j_2} \in \mathcal{A}_{n,m}$ with $i_1 \neq i_2$ and $z \mid H_{i_1,j_1} \mid x$, $z \mid H_{i_2,j_2} \mid x$.

We define the *distance* between z and x as:

$$d(z, x) = \min \sharp\{H \in \mathcal{A}_{n,m} \text{ such that } z \mid H \mid x\}.$$

The *prominent distance* $d_p(z, x)$, is the minimum number of hyperplanes that prominently separate z and x. If $z \neq x$, we say that $d_p(x, z) = 1$ if z and x are not prominently separated.

Let us note that, according to our definition of an object $\mathcal{A}$, if $H_{i,j_1} \in \mathcal{A}$ then $H_{i,j} \in \mathcal{A}$ for all $0 \leq j \leq m-1$. Thus if $d_p(x, z) = 1$ and $d(x, z) > 1$ then all hyperplanes that separate z and x must be within the same object $\mathcal{A}$.

Now we can give the main result of this section, i.e. a necessary and sufficient condition for a social outcome z to be a local optimum for an appropriate set of objects. This condition requires any other social outcome that is preferred to z to have a prominent distance of at least 2 from z itself for that objects scheme.

Theorem 2 *Let $\succeq_{\mathcal{R}}$ be a social decision rule over $X = \mathcal{S}_0(\mathcal{A}_{n,m})$ and $z \in X$ be a given social outcome. Then z is a local optimum for an objects scheme A_z if and only if for any social outcome x such that $x \succ_{\mathcal{R}} z$, the prominent distance is $d_p(x, z) > 1$.*

Proof. Given a social outcome z, let $x_{i_1}, \ldots, x_{i_k}$ be all the social outcomes such that $x_{i_j} \succ_{\mathcal{R}} z$. By hypothesis $d_p(x_{i_j}, z) > 1$, then x_{i_j} and z are prominently separated at least by two hyperplanes.

It follows that we can build an objects scheme A_z such that $\mathcal{H}_{x_{i_j},z} \nsubseteq \mathcal{A}$ for all $\mathcal{A} \in A_z$ and all $1 \leq j \leq k$. For example, if $H^1_{i_j}, H^2_{i_j} \in \mathcal{H}_{x_{i_j},z}$ are two hyperplanes related to different features for $1 \leq j \leq k$, then let us consider an objects scheme A_z such that for any x_{i_j} there exist two objects $\mathcal{A}^1_{i_j}, \mathcal{A}^2_{i_j}$ in A_z with $H^1_{i_j} \in \mathcal{A}^1_{i_j}$, $H^2_{i_j} \in \mathcal{A}^2_{i_j}$ and $\{H^1_{i_j}, H^2_{i_j}\} \nsubseteq \mathcal{A}$ for all $\mathcal{A} \in A_z$.

It is obvious that such an objects scheme exists. Moreover z is a local optimum for A_z. Indeed for all $x_{i_j} \succ_{\mathcal{R}} z$ and for all $\mathcal{A} \in A_z$ the chambers $C_{i_j}(\mathcal{A}^c)$ and $C_z(\mathcal{A}^c)$ are always separated by $H^{i_j}_1$ or $H^{i_j}_2$. That is $x_{i_j}(\mathcal{A}^c) \neq z(\mathcal{A}^c)$ and then $\Phi(z, \mathcal{A}) = \emptyset$ for all $\mathcal{A} \in A_z$. It follows that $z \in \Psi(z, A_z)$ and, by theorem 1, $z \in \Psi(z, A_{z,\alpha})$ for all agendas α, i.e. z is a local optimum.

On the other hand if x is a social outcome $x \succ_{\mathcal{R}} z$ such that $d_p(x, z) = 1$ then for any objects scheme A there is at least one object $\mathcal{A}$ such that all hyperplanes H separating x from z are in $\mathcal{A}$. Then, by definition, $x \in \Phi(z, \mathcal{A}) \neq \emptyset$. This concludes the proof. □

Remark 4.2 *Theorem 2 also gives a description of how to construct an objects scheme which makes social choice converge to a desired local optimum. Moreover, the independence of our construction from the* 1*-dimensional distance, i.e. the distance along the* 1*-dimensional family of hyperplanes* $\{H_{i,j}\}_{0 \leq j \leq m-1}$ *for a fixed* i*, is a consequence of the independence of the choice from the order in which a* 1*-dimensional list of objects is given.*

If a social outcome z does not meet the necessary and sufficient condition of theorem 2 it cannot be a local optimum. However, an agent with object construction power may nevertheless make choices converge to another social outcome close enough to z.

We say that a social outcome z is *free* with respect to a social decision rule $\succeq_{\mathcal{R}}$ if and only if for any social outcome x such that $x \succ_{\mathcal{R}} z$ then $d_p(x, z) > 1$. Thus, by the theorem 2, z is the local optimum for an objects scheme A_z if and only if z is free.

Moreover a social outcome $\overline{z}$ has *minimal distance* from z with respect to $\succeq_{\mathcal{R}}$ if and only if $\overline{z}$ is free with respect to a decision rule $\mathcal{R}$ and

$$d(z, \overline{z}) = \texttt{min}\{d(z, x) \mid x \text{ is free }\}$$

If z is free then it coincides with its social outcome of minimal distance.

Thus, as a direct consequence of theorem 2 we have the following:

Corollary 4.3 *Given a decision rule* $\succeq_{\mathcal{R}}$ *and a social outcome* z *it is always possible to build an objects scheme* $A_{\overline{z}}$ *such that the social outcome* $\overline{z}$ *of minimal distance from* z *with respect to* $\succeq_{\mathcal{R}}$ *is a local optimum.*

Remark 4.4 *If we consider the classical* 1*-dimensional problem, then the prominent distance between two social outcomes* x *and* z *is always* $d_p(x, z) =$ 1*. It follows:*

- *the social outcome z is free if and only if z is an optimum, i.e. for any social outcome x, $z \succ_{\mathcal{R}} x$;*

- *if the social outcome of minimal distance from z with respect to $\succeq_{\mathcal{R}}$ exists then it is the only optimum;*

- *if the social outcome of minimal distance from z with respect to $\succeq_{\mathcal{R}}$ does not exist, then our theorem simply recovers the usual intransitive cycles.*

4.3 Objects and manipulability: global optimality

We now analyze under which condition a local optimum z is a global optimum for an agenda A, i.e. when there exists an agenda α of A such that the basin of attraction $\Psi(z, A_\alpha) = X$ and when this is true for all agendas $\alpha\ in \Lambda(A)$.

From now on, given a social outcome $z \in X$ we will denote by A_z an objects scheme such that $\Phi(z, A_z) = \emptyset$.

Let us remark that, given a social outcome $z \in X$, if z is a global optimum for an agenda $\alpha \in \Lambda(A_z)$ then the following two conditions hold:

1. if $x \in X$ is a social outcome such that $x \succ_{\mathcal{R}} z$ then there exist social outcomes $\{x_{i_1}, \dots, x_{i_k}\}$ such that

$$z \prec_{\mathcal{R}} x \prec_{\mathcal{R}} x_{i_1} \prec_{\mathcal{R}} \dots \prec_{\mathcal{R}} x_{i_k} \prec_{\mathcal{R}} z$$

   Indeed if this is not the case, all paths $P(x, A)$ starting from x would never end up in z for all possible objects schemes A; i.e. $x \notin \Psi(z, A)$ for any objects scheme A.

2. $\Phi(x, A_z) \neq \emptyset$ for all social outcomes $x \neq z$. Otherwise if $\Phi(x, A_z) \neq \emptyset$, then $x \in \Psi(x, A_{z,\alpha})$ for all agendas $\alpha \in \Lambda(A_z)$.

Obviously, these conditions also hold for a global optimum for an objects scheme A.

The first condition is simple to verify. The second one is true for all social outcomes $x \in X$ that are not free (see theorem 2), while free social outcomes must satisfy the following conditions.

Theorem 3 *Let $x \succ_{\mathcal{R}} z$ be two free social outcomes in X with respect to a social rule $\succ_{\mathcal{R}}$. Then there exists an objects scheme A_z such that $\Phi(z, A_z) = \emptyset$ and $\Phi(x, A_z) \neq \emptyset$ if and only if the following condition holds:*

$$\exists y \succ_{\mathcal{R}} x \text{ such that } \mathcal{H}_{w,z} \nsubseteq \mathcal{H}_{x,y} \quad \forall w \succ_{\mathcal{R}} z. \tag{6}$$

In order to prove this theorem we need the following:

Lemma 4.5 *Let $\succ_{\mathcal{R}}$ be a social rule over a space X and $x \succ_{\mathcal{R}} z$ be two social outcomes. Let us assume that for all social outcomes $y \succ_{\mathcal{R}} x$ there exists $w \succ_{\mathcal{R}} z$ such that $\mathcal{H}_{w,z} \subseteq \mathcal{H}_{x,y}$. Under these hypothesis if z is a local optimum for an objects scheme A, then also x is a local optimum for A.*

Proof of Lemma 4.5: Let A be an objects scheme such that z is a local optimum for A, then $\Phi(z, A) = \emptyset$.
Let us suppose that x is not a local optimum for A, then $\Phi(x, A) \neq \emptyset$. Let y be a social outcome in $\Phi(x, A)$, then $y \succ_{\mathcal{R}} x$ and, by hypothesis, there exists a social outcome $w \succ_{\mathcal{R}} z$ such that for all $H \in \mathcal{A}_{n,m}$ which separate w from z then H separates x from y.
Recall that $y \in \Phi(x, A)$ iff $\exists \mathcal{A} \in A$ such that $y(\mathcal{A}) \neq x(\mathcal{A})$ and $y(\mathcal{A}^c) = x(\mathcal{A}^c)$. Then, by construction, all hyperplanes H which separate y and x must be in $\mathcal{A}$, i.e. $\mathcal{H}_{x,y} \subset \mathcal{A}$. By hypothesis it follows that also $\mathcal{H}_{w,z} \subset \mathcal{A}$, i.e. $w(\mathcal{A}) \neq z(\mathcal{A})$ and $w(\mathcal{A}^c) = z(\mathcal{A}^c)$. Then $w \in \Phi(z, \mathcal{A})$ and z is not a local optimum anymore. But this is absurd. Then $\Phi(x, A)$ is empty and x is a local optimum for A □

We are now ready to provide the:

Proof of Theorem 3: Sufficiency trivially follows from Lemma 4.5. As to necessity, let z be a local optimum for an objects scheme A_z obtained as in theorem 2. Let $G(z, A_z) \subset X$ be the set

$$G(z, A_z) = \{x \in X \mid x \succ_{\mathcal{R}} z \text{ and } \Phi(x, A_z) = \emptyset\}$$

i.e. if $x \in G(z, A_z)$ then it is a local optimum for A_z. If $G(z, A_z) = \emptyset$ we have proven necessity, otherwise given $x \in G(z, A_z)$ we can build a new objects scheme A'_z as follows. Let $y \succ_{\mathcal{R}} x$ as in the hypothesis, then we can consider the new objects scheme

$$A'_z = A_z \cup \mathcal{H}_{x,y}$$

Clearly $y \in \Phi(x, \mathcal{H}_{x,y})$, i.e. x is not a local optimum for A'_z while z is still a local optimum for A'_z. Indeed $\Phi(z, \mathcal{A}) = \emptyset$ for all $\mathcal{A} \in A_z$ by construction while $\Phi(z, \mathcal{H}_{x,y}) = \emptyset$ by hypothesis:

let w be a social outcome in $\Phi(z, \mathcal{H}_{x,y}) \neq \emptyset$. Then $w \succ_{\mathcal{R}} z$ and $\mathcal{H}_{w,z} \subset \mathcal{H}_{x,y}$, but this is not possible as y is such that $\mathcal{H}_{w,z} \not\subseteq \mathcal{H}_{x,y}$ for all $w \succ_{\mathcal{R}} z$.

We have a new objects scheme A'_z such that z is a local optimum for A'_z and $G(A'_z, z) \subset G(z, A_z) \setminus \{x\}$. We can iterate until we obtain an objects scheme A such that z is a local optimum and $G(A, z) = \emptyset$. $\square$

Theorem 4 *Let $x \prec_{\mathcal{R}} z$ be free social outcomes in X with respect to a social rule $\succ_{\mathcal{R}}$. Then there exists an objects scheme A_z such that $\Phi(z, A_z) = \emptyset$ and $\Phi(x, A_z) \neq \emptyset$ if and only if the following condition holds:*

$$\exists y \succ_{\mathcal{R}} x \text{ such that } \mathcal{H}_{w,z} \not\subseteq \mathcal{H}_{x,y} \quad \forall w \succ_{\mathcal{R}} z. \tag{7}$$

Proof: *Sufficiency:* if $\Phi(x, A_z) \neq \emptyset$ then there exists a social outcome y in $\Phi(x, A_z)$. By construction y satisfies the following two conditions:

1. $y \succ_{\mathcal{R}} x$
2. there exists $\mathcal{A} \in A_z$ such that $\mathcal{H}_{x,y} \subset \mathcal{A}$ or, equivalently, $\mathcal{H}^c_{x,y} \supset \mathcal{A}^c$.

Moreover $\Phi(z, A_z) = \emptyset$ implies that for all objects $\mathcal{A} \in A_z$ and for all $w \succ_{\mathcal{R}} z$ the intersection $\mathcal{A}^c \cap \mathcal{H}_{w,z} \neq \emptyset$, then $\mathcal{H}^c_{x,y} \cap \mathcal{H}_{w,z} \neq \emptyset$, i.e. $\mathcal{H}_{w,z} \not\subseteq \mathcal{H}_{x,y}$ for all $w \succ_{\mathcal{R}} z$.
Necessity: let $L(z, A_z) \subset X$ be the set defined as:

$$L(z, A_z) = \{x \in X \mid x \prec_{\mathcal{R}} z \text{ and } \Phi(x, A_z) = \emptyset\}.$$

By hypothesis if $x \in L(z, A_z)$ then there exists a social outcome $y \succ_{\mathcal{R}} x$ such that $\mathcal{H}^c_{x,y} \cap \mathcal{H}_{w,z} \neq \emptyset$ for all $w \succ_{\mathcal{R}} z$. We define a new objects scheme $A'_z = A_z \cup \mathcal{H}_{x,y}$.

Then $y \in \Phi(x, A'_z) \neq \emptyset$ while z is still a local optimum for A'_z, i.e. $\Phi(z, A'_z) = \emptyset$:

let w be a social outcome in $\Phi(z, A'_z)$ then $w \in \Phi(z, \mathcal{H}_{x,y})$, i.e. $\mathcal{H}_{w,z} \subset \mathcal{H}_{x,y}$ which is impossible by hypothesis.

Moreover $L(z, A'_z) \subset L(z, A_z) \setminus \{x\}$, indeed if $t \in X$ such that $\Phi(t, A_z) \neq \emptyset$ then, obviously, $\Phi(t, A'_z) \neq \emptyset$. It follows that we can iterate the above construction until we get an objects scheme A such that $L(z, A)$ is empty. $\square$

The above propositions allow us to construct an objects scheme A_z such that for all possible agenda and all possible starting social outcomes $x \in X$ the voting process ends up in z or in a cycle.
Moreover, given a starting social outcome $x \in X$, conditions (6) and (7) allow us to construct an objects scheme A_z and an agenda α in A such that $x \in \Psi(z, A_{z,\alpha})$.

4.4 How to get there from here

So far we have analyzed cases in which objects, agenda and initial status quo may all be manipulated in order to obtain a desired social outcome. In this section instead we assume that the initial condition is exogenously given and only objects and agenda are subject to manipulation. In theorem 5 we give a necessary and sufficient condition that a status quo social outcome x needs to be satisfied in order to belong to the basin of attraction of a chosen social outcome z with respect to an objects scheme A. In other words, given an agent with preferred free social outcome z, we find the conditions under which he/she may obtain z from the given initial condition.

Let $\Pi(\mathcal{A}_{n,m})$ be the set of all possible objects schemes in $\mathcal{A}_{n,m}$. We call the *universal basin of attraction* of a social outcome $z \in X$ the set

$$\Psi(z) = \cup_{A \in \Pi(\mathcal{A}_{n,m})} \Psi(z, A)$$

i.e. the set of all social outcomes x such that there exists an objects scheme delivering a path from social outcome x and ending in social outcome z.

By theorem 2 the universal basin is nonempty, $\Psi(z) \neq \emptyset$, if and only if z is a *free* social outcome.

As above, let $x \in X$ be a starting social outcome which satisfies the conditions of theorems 3 and 4 with respect to a preferred social outcome $z \in X$. Let us define the following set:

$$G_x^z = \{y \succ_{\mathcal{R}} x \mid \mathcal{H}_{w,z} \nsubseteq \mathcal{H}_{x,y} \quad \forall w \succ_{\mathcal{R}} z \text{ and } B(x, \mathcal{H}_{x,y}) \neq \emptyset\}. \qquad (8)$$

Proposition 4.6 *Given two social outcomes $x, z \in X$, if x is in the universal basin of attraction $\Psi(z)$ then $G_x^z \neq \emptyset$.*

Proof. Let us assume that G_x^z is empty then either:

i) neither condition 6 nor 7 is met by any of the social outcomes $y \in X$. Therefore, by theorems 3 and 4, it follows that $\Phi(x, A_z) = \emptyset$ for all A_z such that $\Phi(z, A_z) = \emptyset$, i.e. if x is the starting social outcome then all maximal paths in A_z end up in x for any agenda α;

or:

ii) $\Phi(x, A_z) \neq \emptyset$ but $B(x, A_z) = \emptyset$ for all objects schemes A_z such that $\Phi(z, A_z) = \emptyset$, i.e. if x is the starting social outcome then all voting processes end up in a cycle.

In both cases $x \notin \Psi(z)$ and this concludes the proof $\square$

Let $x \in X$ be a social outcome such that $G_x^z \neq \emptyset$. Clearly if $B(x, \mathcal{H}_{x,y}) \neq \emptyset$ then its cardinality is one, i.e. $B(x, \mathcal{H}_{x,y}) = \{b_{x,y}\}$ and we can consider the set:

$$BG_x^z = \{b_{x,y} \mid y \in G_x^z\}.$$

It is worth remarking that, with the above notations, BG_x^z is a subset of G_x^z; i.e. $BG_x^z \subseteq G_x^z$. Indeed if $b_{x,y} \in BG_x^z$, then $b_{x,y} \succ_{\mathcal{R}} x$ and $\mathcal{H}_{x,b_{x,y}} \subseteq \mathcal{H}_{x,y}$ implies that condition $\mathcal{H}_{w,z} \nsubseteq \mathcal{H}_{x,b_{x,y}} \quad \forall w \succ_{\mathcal{R}} z$ is satisfied. Moreover if $\mathcal{A}_1 \subseteq \mathcal{A}_2$ are two objects such that $B(x, A_i) = w_i$ for $i = 1, 2$, then $w_1 = w_2$ or $w_1 \prec_{\mathcal{R}} w_2$; it follows that $B(x, \mathcal{H}_{x,b_{x,y}}) = \{b_{x,y}\} \neq \emptyset$ and $b_{x,y} \in G_x^z$.

Then we can consider the following finite subsets in X:

$$\begin{aligned}
&E_0^z = \{z\};\\
&E_1^z = \{x \in X \setminus \{z\} \mid z \in BG_x^z\};\\
&E_2^z = \{x \in X \setminus \cup_{i=0}^{1} E_i^z \mid E_1^z \cap BG_x^z \neq \emptyset\};\\
&\vdots\\
&E_h^z = \{x \in X \setminus \cup_{i=0}^{h-1} E_i^z \mid E_{h-1}^z \cap BG_x^z \neq \emptyset\};\\
&E_{h+1}^z = \{x \in X \setminus \cup_{i=0}^{h} E_i^z \mid E_h^z \cap BG_x^z \neq \emptyset\} = \emptyset.
\end{aligned}$$

It is a simple remark that the number $h+1$ of subsets in the above definition depends only on z.
Let us define:

$$E^z = \cup_{i=1}^{h} E_i^z. \tag{9}$$

We have the following:

Theorem 5 *Given two social outcomes $x, z \in X$, where z is free, then x is in the basin of attraction $\Psi(z)$ if and only if $x \in E^z$; i.e.*

$$\Psi(z) = E^z. \tag{10}$$

Proof. If $x \in \Psi(z)$ is a social outcome, then there exists an objects scheme $A = \{\mathcal{A}_{I_1}, \ldots, \mathcal{A}_{I_t}\}$ such that $x \in \Psi(z, A)$ and an agenda $\alpha = (h_k, \ldots, h_1)$ for A such that the ordered path along α

$$x = x_k \prec_{\mathcal{R}} x_{k-1} \prec_{\mathcal{R}} \ldots \prec_{\mathcal{R}} x_1 \prec_{\mathcal{R}} x_0 = z$$

is maximal.
Since we have $x_0 = z \in E^z$, then, by induction, let us assume that $x_j \in E^z$, then $x_j \in E^z_i$ for some i. By construction $x_j \in B(x_{j+1}, \mathcal{A}_{h_{j+1}})$ then $x_j \in B(x_{j+1}, \mathcal{H}_{x_{j+1},x_j}) \neq \emptyset$. Moreover $\mathcal{H}_{w,z} \nsubseteq \mathcal{H}_{x_{j+1},x_j} \quad \forall w \succ_{\mathcal{R}} z$, indeed if $w \succ_{\mathcal{R}} z$ is a social outcome such that $\mathcal{H}_{w,z} \subseteq \mathcal{H}_{x_{j+1},x_j} \subseteq \mathcal{A}_{h_j}$ then $z \notin \Psi(z, A)$ which is absurd.
It follows that $x_j \in BG^z_{x_{j+1}}$ and, by induction, $x_j \in E^z_i$, i.e. x_{j+1} is such that $x_j \in E^z_i \cap BG^z_{x_{j+1}} \neq \emptyset$. Then either $x_{j+1} \in E^z_{i+1}$ or $x_{j+1} \in \cup_{s=0}^{i-1} E^z_s$, i.e. $x_{j+1} \in E^z$.

Viceversa, if $x \in E^z$ then $x \in E^z_i$ and we can construct a path P as follows:

$$x = x_i \prec_{\mathcal{R}} x_{i-1} \prec_{\mathcal{R}} \ldots \prec_{\mathcal{R}} x_1 \prec_{\mathcal{R}} x_0 = z$$

such that $x_j \in E^z_j \cap BG_{x_{j+1}}$ for $0 \leq j \leq i-1$.
Let us consider an objects scheme A_z such that $z \in \Psi(z, A_z)$, then

$$A = \{\mathcal{H}_{x_j,x_{j+1}}\}_{0 \leq j \leq i-1} \cup A_z$$

still satisfies $z \in \Psi(z, A)$.
Moreover let $\alpha = (h_0, \ldots, h_k)$ be an agenda for A such that the h_t-object in the ordered scheme A_α is $\mathcal{H}_{x_{i-t},x_{i-t-1}}$ for $0 \leq t \leq i-1$. Then, by construction, P is exactly the maximal path $P(x, z, A_\alpha)$ and $x \in \Psi(z)$. $\square$

4.5 Breaking intransitive cycles

In this section we show that also intransitive cycles which, as well known, can be generated by any social aggregation rule are subject to manipulation through object construction. The main result of this section, theorem 6, demonstrates that almost any intransitive cycle may be broken by appropriate modification of the objects scheme. It turns out that cycles may be broken by introducing new local optima. Thus, we observe a trade-off between decidability, i.e. the possibility of reaching some social optimum in a feasible time, and non-manipulability, that is, the convergence of the social decision process to a unique global outcome that does not depend on initial condition and agenda. The balance in this trade-off is struck by the objects scheme. In particular, we will show that coarse objects containing many features tend to produce many cycles and few local optima, whereas fine objects containing only one or a few features are much less likely to induce cycles but tend to generate many local optima and, therefore, greatly increase the opportunity for manipulation.

We first prove that given a set Γ of free cycles it is always possible to choose an objects scheme A such that social choice according to some rule $\succeq_{\mathcal{R}}$ will never enter into any cycle belonging to Γ. Suppose that for some objects scheme A' social preferences may encounter a set of cycles Γ. We will say that an objects scheme $A \neq A'$ *breaks* the cycles Γ if social choice does not enter into any cycle belonging to Γ with the new objects scheme.

Given a cycle $\gamma : x_{i_1} \prec_{\mathcal{R}} \ldots \prec_{\mathcal{R}} x_{i_k} \prec_{\mathcal{R}} x_{i_{k+1}} = x_{i_1}$, let us define the following subset of hyperplanes in $\mathcal{A}_{n,m}$:

$$\mathcal{H}_\gamma = \bigcup_{1 \leq j \leq k+1} \mathcal{H}_{x_{i_j}, x_{i_{j+1}}}. \tag{11}$$

We will say that a cycle $\gamma : x_{i_1} \prec_{\mathcal{R}} \ldots \prec_{\mathcal{R}} x_{i_k} \prec_{\mathcal{R}} x_{i_{k+1}} = x_{i_1}$ is *free* if there are at least two consecutive indices i_j, i_{j+1} such that $d_p(x_{i_j}, x_{i_{j+1}}) > 1$.

Then we have the following:

Theorem 6 *Let $\mathcal{Y}_{\succ_{\mathcal{R}}}$ be the oriented graph for a given social rule $\succ_{\mathcal{R}}$ and $\mathcal{A}_{n,m}$ the arrangement associated to the space of social outcomes. Then for any given set $\Gamma = \{\gamma_1, \ldots, \gamma_h\}$ of free cycles in $\mathcal{Y}_{\succ_{\mathcal{R}}}$ there exists an objects scheme A such that all voting process will never end up in any cycle of Γ. Moreover this scheme can be chosen with maximum size.*

In order to prove this theorem we need the following:

Lemma 4.7 *Let $\gamma \in \mathcal{Y}_{\succ_{\mathcal{R}}}$ be a cycle and A an objects scheme such that:*

1. *$\mathcal{H}_\gamma \nsubseteq \mathcal{A}$ for any object $\mathcal{A} \in A$;*
2. *for any agenda α and any starting configurations x the path $P(x, A_\alpha)$ does not end up in γ.*

then γ disappears in A.

Proof. The proof is obvious. □

Lemma 4.8 *Let $\gamma \in \mathcal{Y}_{\succ_{\mathcal{R}}}$ be a cycle and A an objects scheme such that at least two consecutive outcomes $x_{i_j} \prec_{\mathcal{R}} x_{i_{j+1}}$ in the cycle satisfy $\mathcal{H}_{x_{i_j}, x_{i_{j+1}}} \nsubseteq \mathcal{A}$ for any object $\mathcal{A} \in A$. Then γ disappears in A.*

Proof. Condition 1 of Lemma 4.7 is trivially satisfied. Moreover, by hypothesis, let $x_{i_j} \prec_{\mathcal{R}} x_{i_{j+1}}$ be two consecutive outcomes in γ such that $\mathcal{H}_{x_{i_j},x_{i_{j+1}}} \nsubseteq \mathcal{A}$ for any object $\mathcal{A} \in A$. Then, by definition, $x_{i_{j+1}} \notin B(x_{i_j}, \mathcal{A})$ for all objects $\mathcal{A} \in A$, i.e. it is not possible to move from x_{i_j} to $x_{i_{j+1}}$ directly and then condition 2 of Lemma 4.7 holds. This conclude the proof. □

Lemma 4.9 *A given free cycle $\gamma \in \mathcal{Y}_{\succ_{\mathcal{R}}}$ disappears for the trivial object scheme*

$$A = \{\{H_{i,j}\}_{\substack{1\le i\le n \\ 0\le j\le m-1}}\} \quad .$$

Proof. The proof is a direct consequence of Lemma 4.8. □

Proof of Theorem 6. Let us consider the finite set $\mathbb{A}_\Gamma$ of all possible objects schemes A of $\mathcal{A}_{n,m}$ such that all voting process in A will never end up in any cycle in Γ. By Lemma 4.9 this set is not empty, indeed the trivial objects scheme

$$A = \{\{H_{i,j}\}_{\substack{1\le i\le n \\ 0\le j\le m-1}}\}$$

is such that any social choice process will never end up in any cycle in Γ. Moreover the set of sizes of elements $A \in \mathbb{A}_\Gamma$ is a finite non-empty set of natural numbers $\mathbb{N}$ and then it admits a maximum.
This conclude the proof. □

Example 10 *In the example 8 the objects schemes $A = \{\{H_1, H_3\}, \{H_2\}\}$ and $A = \{\{H_2, H_3\}, \{H_1\}\}$ break at once all cycles in the graph $\mathcal{Y}_{\succ_{\mathcal{R}}}$ of example 3.*

Remark 4.10 *A cycle γ which is not free can also be broken in most cases, but not always. An example is given by the cycle $\gamma : 00 \prec_{\mathcal{R}} 01 \prec_{\mathcal{R}} 11 \prec_{\mathcal{R}} 10 \prec_{\mathcal{R}} 00$ in the case $\mathcal{A}_{2,1}$. Indeed, in this case, the only two objects schemes are $A = \{\{H_1, H_2\}\}$ and $A = \{\{H_1\}, \{H_2\}\}$ and none of them breaks γ.*

Cycles which are not free are very few and far between with respect to free one, but nevertheless they deserve further study.

The above theorem proves the existence of an objects scheme A such that all the cycles in a given set $\Gamma = \{\gamma_1, \dots, \gamma_h\}$ are *broken*, while Lemma 4.8 shows how to construct it.

All in all, a cycle γ may be *broken*, i.e. it disappears in a given objects scheme A, if there exist consecutive configurations x, w involved in the cycle

γ such that $\mathcal{H}_{x,w} \nsubseteq \mathcal{A}$ for all $\mathcal{A}$ belonging to the scheme A. Thus, the number of cycles broken by a given objects scheme A increases when the size of A, i.e. the cardinality of its largest object, decreases.

Symmetrically, the number of local optima increases when the size of the objects scheme A decreases. Indeed by theorem 2, a free configuration z is a local optimum for a scheme A if and only if for all configurations $x \succeq_{\mathcal{R}} z$ $\mathcal{H}_{x,z} \nsubseteq \mathcal{A}$ for all $\mathcal{A}$ belonging to the scheme A. It follows that if the objects scheme A is composed by k objects, then we have at least $(k-1)!$ different local optima. Thus we obtain a decidability vs. non-manipulability trade-off. Schemes composed of few large objects tend to produce cycles, while schemes of many small objects tend to produce considerably fewer cycles but increasingly many local optima.

4.6 On the median voter theorem

In addition to intransitive cycles and agenda power, also another classic result of social choice theory, the median voter theorem, is subject to manipulability through object construction. By appropriately modifying the objects scheme, the outcome of a social decision may be made as distant as one wishes from the median voter's preferred configuration.

Recall that the median voter theorem in its strong version (Black 1958, Downs 1957) states that if a voter with median preferences exists, her/his most preferred outcome will beat any other alternative in pairwise majority voting.

In our framework we can easily prove the following:

Remark 4.11 *Given a rule $\succ_{\mathcal{R}}$ with free configurations, if a median voter m exists, theorem 2 proves that it is always possible to find objects scheme A for which the preferred outcome is different from the one of the median voter. Moreover, if z is the opposite configuration with respect to the median voter's preferred one and z is free, then it is possible to find a scheme with preferred outcome z.*

This is equivalent to saying that in our construction it is always possible to manipulate the scheme in such a way that the median voter theorem does not apply.

5 Conclusions

Economic theory usually reduces decision problems to choice problems, by assuming that alternatives are given and agents have to simply pick the one they prefer according to some criterion. However a considerable amount of time and resources in many political and economic organizations and institutions are devoted to figuring out these alternatives. We have proposed a model that analyzes the situations in which alternatives are constructed through bundling of constituent elements and have shown that different bundles of the same constituent elements lead to very different social outcomes. By controlling the formation of these bundles an authority may strongly manipulate social decisions, even when the latter are taken by majority voting or any other kind of free and democratic selection criterion.

At least two issues have been left unexplored in this paper which need further investigation. First, we assumed that agents sincerely express their preferences, but they could strategically misrepresent their preferences in order to countervail the object construction power. The extent to which object construction may be offset or mitigated by strategic behaviors is an interesting issue.

Second, we only compared the properties of different objects schemes under the implicit assumption that objects schemes could be constructed and modified at will and object construction power can be fully exerted because the authority knows everybody's preferences. In many real-life situations object construction is strongly path-dependent and objects can be only modified through a process of adaptive small changes that allow a boundedly rational authority to learn the preferences of the other agents. An evolutionary model of learning and adaptation of objects could account for these dynamics.

A Appendix: Some basic notions of hyperplanes arrangements

In this appendix we recall some basic notions from the theory of hyperplanes arrangements. The interested reader can refer to, for instance, Orlik and Terao (1992) for a much more detailed and extended survey.

In geometry and combinatorics, an **arrangement of hyperplanes** is a finite set $\mathcal{A}$ of hyperplanes in a linear, affine, or projective space S. Questions about hyperplane arrangements $\mathcal{A}$ generally concern geometrical, topological, or other properties of the complement, $M(\mathcal{A})$, that is the set that is left when hyperplanes are removed from the space. One may ask how these properties are related to the arrangement and its intersection semilattice.

The intersection semilattice of $\mathcal{A}$, written $L(\mathcal{A})$, is the set of all subspaces that are obtained by intersecting some of the hyperplanes. Among these subspaces are S itself, all the individual hyperplanes, all the intersections of pairs of hyperplanes, etc. (excluding, in the affine case, the empty set). These subspaces are called the **flats** of $\mathcal{A}$. $L(\mathcal{A})$ is partially ordered by reverse inclusion.

If the space S is 2-dimensional, the hyperplanes are lines[2], if S is 3-dimensional we have arrangements of planes.

More precisely, let $\mathbb{K}$ be a field and let $V_{\mathbb{K}}$ be a vector space of dimension n. A **hyperplane** H in $V_{\mathbb{K}}$ is an affine subspace of dimension $(n-1)$. A hyperplane arrangement

$$\mathcal{A}_{\mathbb{K}} = (\mathcal{A}_{\mathbb{K}}, V_{\mathbb{K}})$$

is a finite set of hyperplanes in $V_{\mathbb{K}}$.

One is normally interested both in the real and the complex cases, hence $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $V = \mathbb{R}^n, \mathbb{C}^n$. Thus, given the canonical base $\{e_1, \dots, e_n\}$ in V, each hyperplane $H \in \mathcal{A}$ is the kernel of a polynomial $\alpha_H \in \mathbb{K}[x_1, \dots, x_n]$ of degree 1 defined up to a costant. The product:

$$\mathcal{Q}(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$$

is called the **defining polynomial** of $\mathcal{A}$.

The **cardinality** $\mid \mathcal{A} \mid$ of the arrangement $\mathcal{A}$ is the number of hyperplanes in $\mathcal{A}$.

If $\mathcal{B} \subset \mathcal{A}$ is a subset of $\mathcal{A}$, then it is called a **subarrangement** of $\mathcal{A}$. We define the set of all nonempty intersections of elements of $\mathcal{A}$ as:

$$L(\mathcal{A}) = \{\cap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}\}$$

and the **complement** of $\mathcal{A}$ by:

$$M(\mathcal{A}) = V \setminus \cup_{H \in \mathcal{A}} H.$$

The complement of an arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is clearly disconnected: it is made up of separate pieces called **chambers** or **regions**, each of which may be either a bounded or an unbounded region.

Each flat of $\mathcal{A}$ is also divided into sections by the hyperplanes that do not contain the flat; these sections are called the **faces** of $\mathcal{A}$. Chambers are

[2] Historically, real arrangements of lines were the first to be investigated.

faces because the whole space is a flat. The faces of codimension 1 may be called the **facets** of A.

The **face semilattice** of an arrangement is the set of all faces, ordered by inclusion.

Example 11 *Let us give some examples:*

- *if the arrangement consists of three parallel lines, the intersection semilattice consists of the plane and the three lines, but not the empty set. There are four regions, none of them bounded (panel a) of figure 5 below).*

- *If we add a line crossing the three parallels, then the intersection semilattice consists of the plane, the four lines, and the three points of intersection. There are eight regions, still none of them bounded (panel b) of figure 5 below).*

- *If we add one more line, parallel to the last, then there are 12 regions, of which two are bounded parallelograms (panel c) of figure 5 below).*

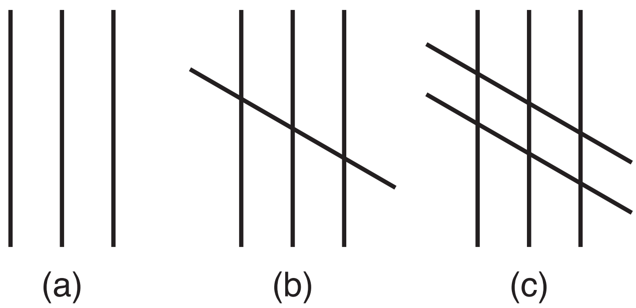

Figure 5: A graphical illustration of example 11

Every arrangement $(\mathcal{A}_{\mathbb{R}}, \mathbb{R}^n)$ also generates an arrangement over $\mathbb{C}$. Let $(\mathcal{A}_{\mathbb{R}}, \mathbb{R}^n)$ be an arrangement with defining polynomial $\mathcal{Q}(\mathcal{A}_{\mathbb{R}})$. The $\mathbb{C}$-extended arrangement is in $\mathbb{C}^n$. It consists of the hyperplanes which are the kernel of the polynomial α_H in $\mathbb{C}^n$ instead of $\mathbb{R}^n$.

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