Laboratory of Economics and Management
Sant'Anna School of Advanced Studies
Piazza Martiri della Libertà, 33 - 56127 PISA (Italy)
Tel. +39-050-883-343 Fax +39-050-883-344
Email: lem@sssup.it Web Page: http://www.lem.sssup.it/

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# A Review of Nonfundamentalness and Identification in Structural VAR models

Lucia Alessi*
Matteo Barigozzi*
Marco Capasso*

*Scuola Superiore Sant'Anna

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Lucia Alessi*
Matteo Barigozzi†
Marco Capasso‡

Laboratory of Economics and Management
Sant'Anna School of Advanced Studies, Pisa

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**Abstract**

We review, under a historical perspective, the developement of the problem of nonfundamentalness of Moving Average (MA) representations of economic models, starting from the work by Hansen and Sargent [1980]. Nonfundamentalness typically arises when agents' information space is larger than the econometrican's one. Therefore it is impossible for the latter to use standard econometric techniques, as Vector AutoRegression (VAR), to estimate economic models. We re-state the conditions under which it is possible to invert an MA representation in order to get an ordinary VAR, and we consider how the latter is used in the literature to assess the validity of Dynamic Stochastic General Equilibrium models, providing some interesting examples. We believe that possible nonfundamental representations of considered models are too often neglected in the literature. We consider how factor models can be seen as an alternative to VAR for assessing the validity of an economic model without having to deal with the problem of nonfundamentalness. We then review the works by Lippi and Reichlin [1993] and Lippi and Reichlin [1994] which are the first attempts to give to nonfundamental representations the economic relevance that they deserve, and to outline a method to obtain such representations starting from an estimated VAR.

**Keywords:** Nonfundamentalness, Structural VAR, Dynamic Stochastic General Equilibrium Models, Factor Models.
**JEL-classification:** C32, C51, C52.

* *E-mail*: lucia.alessi@sssup.it
† *E-mail*: matteo.barigozzi@gmail.com
‡ *E-mail*: marco.capasso@gmail.com

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# 1 Introduction

Predictions from different structural models can be evaluated empirically using the VAR tool, where linear combinations of structural shocks are estimated as residuals of OLS regressions, being the structural shocks then identified by imposing a set of restrictions. If such restrictions are verified by a broad class of models, different predictions of models can be compared by looking at the estimated shocks and their coefficients (impulse response functions). However, if the structural model has a moving average (MA) component, the VAR representation is admissible only under some conditions which may not be verified in the structural model. In that case, there is no hope to recover the structural shocks from VAR estimation. This point was first made by Hansen and Sargent [1980], Hansen and Sargent [1980], Lippi and Reichlin [1993], and Lippi and Reichlin [1994] and recently brought back in the macroeconomic debate by Chari et al. [2005], Christiano et al. [2006] and Fernandez-Villaverde et al. [2005].

In the next section we give the main definitions of nonfundamentalness. In section 3 we illustrate the debate between Blanchard and Quah [1989] and Lippi and Reichlin [1993] as a textbook example of how an economically meaningful model can generate nonfundamental representations. In section 4 we look at another case of nonfundamentalness generated by rational expectations. In section 5 we consider briefly Dynamic Stochastic General Equilibrium (DSGE) models and a recent method proposed by Fernandez-Villaverde et al. [2005] to detect nonfundamentalness in these models. In section 6 we propose the Dynamic Factor model as an alternative tool for identification. First we introduce the model as a consequence of DSGE models with measurement errors and then we show how to deal with nonfundamentalness in this case. In section 7 we deal with a different method by Giannone and Reichlin [2006] to detect nonfundamentalness based on Granger causality that naturally leads to a factor representation. In Section 8 we recall Blanchard and Quah [1993] argument for nonfundamentalness in cointegrated models. In section 9 we explain how to generate nonfundamental representations by means of Blaschke matrices and we review the method proposed by Lippi and Reichlin [1994] to obtain such representations. Section 10 proposes a development for future research on nonfundamental representations.

# 2 Nonfundamentalness

Consider an $N$-dimensional covariance stationary zero-mean vector stochastic process $x_t$ of observable variables and a $q$-dimensional vector process $u_t$ of structural (i.e. with economic meaning) shocks that are not observable but that have impact on $x_t$. We can write

$$x_t = C(L)u_t\,, \tag{1}$$

where $C(L) = \sum_{k=0}^{\infty} C_k L^k$ is a one-sided polynomial in the lag operator $L$, in principle of infinite order. The shocks are orthogonal white noises: $u_t \sim \text{w.n.}(0, \Gamma_0^u)$, with $\Gamma_0^u$ diagonal, so that all possible serial correlation is captured by $C(L)$. In all what follows we assume that $x_t$ has rational spectral density and therefore the entries of $C(L)$ are rational functions of $L$. We define the $k$-th lag impulse response of the variable $x_{it}$ to the shock $u_{jt}$ as the $(i,j)$-th element of the matrix $C_k$. Whenever $u_t \in \overline{\text{span}}\,\{x_{t-k}, k \geq 0\}$, we say that $u_t$ is fundamental with respect to $x_t$. It is then clear that if $N < q$ then it is impossible to obtain $u_t$ from the present and past values of observed data, since we observe fewer series than the shocks that we want to recover. Thus a necessary condition for fundamentalness is that $N \geq q$.

The typical tool used to estimate (1) are VAR models where we always assume that $N = q$ and we estimate the equivalent representation

$$A(L)x_t = u_t\,, \tag{2}$$

where $A(L)$ is a one-sided polynomial of finite order. Clearly the obtained shocks $u_t$ are, by construction, fundamental and their identification as structural shocks is accomplished by imposing restrictions derived from economic theory. The old literature used to impose such restrictions directly on the lag coefficients, however Sims [1980] dubbed them as "incredible" and proposed to put weaker identifying restrictions generally on the covariance matrix of the residuals of a VAR, or on the impact multiplier $C(0)$ and on the long run multiplier $C(1)$ (for an introduction see Watson [1994]). A VAR with structural restrictions is usually called Structural VAR. In any case, whatever the identification scheme used, the identified shocks are still fundamental for the VAR representation given that they are simple rotations of the ones estimated in (2). There is no way to identify nonfundamental shocks by means of VAR techniques. However economic theory, in general, does not provide support for fundamentalness so that all representations that fulfill the same economic statements but are nonfundamental are ruled out by VAR with no justification. Typically nonfundamentalness can be restated as a case where the agents' information space is larger than the econometricians' one.

We now start by considering the square systems (i.e. $N = q$) and we provide the sufficient condition for fundamentalness.

**Definition 1 (Fundamentalness in square systems)** *Given a covariance stationary vector process $x_t$, the representation $x_t = C(L)u_t$ is fundamental if:*

1. *$u_t$ is a white noise vector;*
2. *$C(L)$ has no poles of modulus less or equal than unity, i.e. it has no poles inside the unit disc;*
3. *$\det C(z)$ has no roots of modulus less than unity, i.e. all its roots are outside the unit disc*

$$\det C(z) \neq 0 \quad \forall\, z \in \mathbb{C} \quad \text{s.t.}\ \ |z| < 1\,. \tag{3}$$

If the roots of $\det C(z)$ are outside the unit disc, we have invertibility in the past (i.e. the inverse representation of (2) depends only on non-negative powers of $L$) and therefore we have fundamentalness. If at least one of the roots of $\det C(z)$ is inside the unit disc, we still have invertibility and we have nonfundamentalness. However, since in this case the inverse representation of (2) depends also on negative powers of $L$, we speak of invertibility in the future. Finally if there is one root on the unit circle the representation is still fundamental but it is not invertible.

Hence, summarizing, if $\det C(z)$ has roots outside the unit disc we can estimate a VAR for $x_t$ and the residuals, once identified, are the real economic shocks. On the opposite, if at least one root is inside the unit circle, there is a problem of nonfundamentalness and we cannot use standard techniques as VAR to identify the model, due to the fact that different specifications might imply the same covariance structure. The problem of nonfundamentalness is a problem only for the estimation of Structural VAR models. When instead we use VAR models just for forecasting we are not concerned about nonfundamentalness since, in this case, we are not interested in recovering the structural shocks, but we just care of exploiting all the information available. Notice that that fundamental representations arise naturally with linear prediction, being the prediction error $u_t = x_t - \text{Proj}\ (x_t|x_{t-1}, x_{t-2}, \ldots)$, by construction, fundamental for $x_t$. Therefore when estimating a VARMA with forecasting purposes, the MA matrix polynomial is always chosen to be fundamental.

We illustrate definition 1 with a simple example. Consider the two representations for $x_t$

$$\text{A)} \quad x_t = (1 - cL)u_t \quad u_t \sim \text{i.i.d}(0, \sigma_u^2)\,,$$

$$\text{B)} \quad x_t = (1 - \tfrac{1}{c}L)\tilde{u}_t \quad \tilde{u}_t \sim \text{i.i.d}(0, \sigma_{\tilde{u}}^2)\,,$$

with $|\,c| > 1$ and $\sigma_{\tilde{u}}^2 = c^2\sigma_u^2$, so that in both cases the variance of $x_t$ is $\sigma_u^2(1 + c^2)$. Representation A is not invertible but the first two moments of $x_t$ are not enough to discriminate between this model and model B which instead is invertible. Suppose model A is the true one, a researcher using the VAR representation will be forced to estimate B and will then recover $\tilde{u}_t$ as the structural shocks and not the true $u_t$.

Nonfundamentalness appears in the literature in two ways: endogenously or exogenously. In the first case the model is by definition nonfundamental, this is the case of permanent income models (see Blanchard and Quah [1993] and Fernandez-Villaverde et al. [2005]), rational expectations (see Hansen and Sargent [1980]) and heterogenous beliefs. While in the exogenous case it is the way in which the dynamics of exogenous variables is specified which makes the model fundamental or not. We start with an example of this latter case by Blanchard and Quah [1989] and by Lippi and Reichlin [1993].

# 3 Why do nonfundamental representations matter?

Lippi and Reichlin [1993] in a comment to the well known VAR model by Blanchard and Quah [1989] clearly highlight the possible existence of nonfundamental representations that, although not recoverable with a VAR, may still give rise to economic meaningful representations. Both these works take, as a starting point, the following model based on Fischer [1977]:

$$\begin{aligned} y_t &= m_t - p_t + a\theta_t\,, \\ y_t &= n_t + \theta_t\,, \\ p_t &= w_t - \theta_t\,, \\ w_t &= w|\left[E_{t-1}(n_t = \overline{n})\right]\,, \end{aligned}$$

where $y$, $n$, and $\theta$ denote the logs of output, employment, and productivity; $\overline{n}$ is full employment; $w$, $p$ and $m$ are the logs of nominal wage, price level, and money supply; $a\theta$ is investment demand with $a > 0$. In the last equation nominal wages at $t$ are set so that the expectation at $t-1$ of employment at $t$ equals full employment. The evolution of money supply and productivity is given by:

$$\begin{aligned} m_t &= m_{t-1} + u_t^d\,, \\ \theta_t &= \theta_{t-1} + d(L)u_t^s\,. \end{aligned}$$

There are two types of uncorrelated shocks, one that has a permanent effect on output through productivity, while the other has not. The former can be interpreted as supply disturbances ($u_t^s$) while the latter as demand disturbances ($u_t^d$). This model for output growth rate ($(1-L)y_t$) and unemployment ($U_t$) has the structural form

$$\begin{bmatrix} \Delta y_t \\ U_t \end{bmatrix} = \begin{bmatrix} (1-L) & d(L)+(1-L)a \\ -1 & -a \end{bmatrix} \begin{bmatrix} u_t^d \\ u_t^s \end{bmatrix} = C(L) \begin{bmatrix} u_t^d \\ u_t^s \end{bmatrix}. \tag{4}$$

The only difference between the models by Blanchard and Quah [1989] and the model by Lippi and Reichlin [1993] is on the impact of the supply shock on output growth rate. The model by Blanchard and Quah [1989] assumes no dynamics in productivity except for the istantaneous response to the supply shock, therefore they implicitly assume $d(L)=1$. The model by Lippi and Reichlin [1993] assumes a learning-by-doing dynamics such that $d(1)=1$, therefore in their model the rate of increase of productivity at time $t+k$ is $d_k u_t^s$.
We now review in detail the implications of these two choices.

## Fundamental representations

Blanchard and Quah [1989] first estimate the VAR $\Phi(L)x_t = e_t$, where $x_t = (\Delta y_t \ U_t)'$, $\Phi(0)=I$, and $\mathrm{E}(e_t e_t') = \Gamma_0^e$. This can be interpreted as the reduced form of the following VAR

$$A(L) \begin{bmatrix} \Delta y_t \\ \tilde{n}_t \end{bmatrix} = \begin{bmatrix} u_t^d \\ u_t^s \end{bmatrix}, \tag{5}$$

where we impose $\mathrm{E}(u_t u_t') = I$. By comparing structural and reduced forms we immediately see that $u_t = A(0)e_t$. Therefore, the structural shocks $u_t$ can be recovered from the estimated VAR innovations $e_t$, provided that we impose restrictions in order to identify $A(0)$. The inverse representation of (5) is given in (4) with $d(L)=1$. Blanchard and Quah [1989] impose long-run neutrality of the demand shock on $y_t$, i.e. $C_{11}(1)=0$, moreover $C(0)C(0)' = \Gamma_0^e$. These conditions are enough to achieve identification of the structural shocks. The procedure is the following: first estimate the reduced form VAR and get $\Phi(L)$, $e_t$ and their covariance $\Gamma_0^e$; from these together with the identification conditions obtain $C(0)$, $C(L) = A(L)C(0)$, and $u_t$, thus all the elements of (4). By estimating the model with real data the following impulse responses $C(L)$ are obtained: the effect of the demand shock is hump-shaped for both variables, while the effect of the supply shock on output increases steadily over time before reaching a plateau.
Note that the issue of nonfundamentalness is always present when dealing with VAR models, even when it is not explicitly mentioned as in the work by Blanchard and Quah [1989]. Indeed all their procedure is correct provided that $C(L)$ is invertible in the past. From (4), with the condition $d(L)=1$, we have that $\det C(z) = 1$, and definition 1 is trivially satisfied. Therefore the VAR of equation (5) is a correct representation of the model. Note that if this were not the case, then the estimated innovations $e_t$ would not be a simple linear combination of $u_t$ since the latter ones would be nonfundamental for $x_t$. Therefore, the econometrician would estimate nonfundamental shocks as if they were fundamental, thus committing a possibly fatal error.

## Nonfundamental representations

As mentioned above, Lippi and Reichlin [1993] assume a non trivial dynamics for the productivity and this simple and very realistic assumption generates a variety of other possible

impulse responses. Indeed in this case $\det C(z) = d(z)$, therefore invertibility of (4) (i.e. fundamentalness of $u_t$) is no more automatically guaranteed unless we impose additional restrictions on the process of learning-by-doing. However, economic theory does not provide sufficent restrictions for $\theta_t$ in order to satisfy definition 1. For instance, the typical case of learning-by-doing characterizing the diffusion of technological innovations can be modelled by assuming a bell-shaped pattern for the coefficients $d_k$, which generates an S-shaped long-run impulse response of the output growth rate to a supply shock. Lippi and Reichlin [1993] show that such a choice may imply that some roots of $\det C(z)$ are inside the unit disc. In their specification of the model the property of long-run neutrality of the demand shock is still imposed as in Blanchard and Quah [1989], but in addition a non-trivial learning-by-doing process of diffusion of technical change is assumed. The bottom line of the work by Lippi and Reichlin [1993] consists in the possibility of producing economically sensible models in which the standard assumption of fundamentalness is violated. In fact we can still estimate a VAR for such a model but we will face two problems: the usual problem of determining the matrix $A(0)$ through identification restrictions, plus the problem of establishing the position of the zeroes of the representation (4). The key point of the whole procedure lies in the fact that by inverting the estimated VAR we will obtain a fundamental representation, but it is possible to obtain many other nonfundamental representations that we cannot rule out since some of them may have meaningful economic interpretation as the learning-by-doing example.
To show how this can happen, Lippi and Reichlin [1993] use the same data as in Blanchard and Quah [1989] and first estimate a VAR, then they invert it to get its MA representation, and starting from its roots, that are by definition outside the unit disc, they generate many different nonfundamental representations and their impulse responses by just inverting some of the roots. Some of the impulse responses are immediately rejected as implausible, while others can be interpreted as responses to a technology shock which does not have an instantaneous one-to-one impact on the variables of interest.

In general the literature does not provide support for fundamentalness, so that all representations that fulfill the same economic statements but are nonfundamental are ruled out with no justification. Although skeptical about the economic usefulness of nonfundamental representations, Blanchard and Quah [1993] recognize that we cannot neglect this problem just by assuming that it is not present. As another example of nonfundamentalness, they consider the model of permanent income by Friedman-Muth where income $y_t$ is decomposed in a permanent part $y_{1t}$ and a transitory part $y_{0t}$ which are independently affected by uncorrelated shocks

$$\begin{aligned} \Delta y_{1t} &= u_{1t}\,, \\ y_{0t} &= u_{0t}\,. \end{aligned}$$

If consumption follows the permanent income hypothesis as in Hall [1978] we have: $\Delta c_t = u_{1t} - (1-\beta)u_{0t}$ where $\beta$ is the agent discount factor. Therefore we have

$$\begin{bmatrix} \Delta y_t \\ \Delta c_t \end{bmatrix} = \begin{bmatrix} 1 & 1-L \\ 1 & 1-\beta \end{bmatrix} \begin{bmatrix} u_{1t} \\ u_{0t} \end{bmatrix} = C(L)u_t\,.$$

In this case $\det C(z) = (z-\beta)$ and hence it has the only root in $\beta$, which by definition is inside the unit disc. The representation is nonfundamental. Permanent and transitory components of income are not recoverable just by considering only income and consumption as in a VAR. This is a typical case of endogenous nonfundamentalness, in that this property does not depend on any exogenous variable, it is instead a real property of the model that cannot be eliminated. The model by Lippi and Reichlin [1993] is instead a case in which nonfundamentalness is exogenously generated by the way in which the technological shock hits the economy. However, exogeneity is not a good reason for considering nonfundamentalness as an innocuous problem. Indeed, as we just showed, we can generate nonfundamental but meaningful economic models, that Structural VAR models cannot identify. Unless we knew the real economic model, we must take into account all the possible representations including the nonfundamental ones.

Both examples in this section show how nonfundamental representations can arise even in very simple models where no expectations are present, as it is instead the case for the models that we will consider in the next sections. Since evidence of economic meaningful nonfundamental representations is accumulating, it is useful to find a way for considering such representations every time that we have to deal with identification issues.

# 4 A first example of nonfundamentalness: rational expectations

Hansen and Sargent [1980] introduced the problem of nonfundamentalness while trying to set up a method for formulating and estimating dynamic linear econometric models with rational expectations. Estimation is usually run by estimating agents' decision rules jointly with the model of the stochastic process they face, subject to the restrictions implied by the rational expectations rules. These in turn imply that agents observe and respond to more data than the econometrician possess, i.e. the agents' information space is larger than the econometrician's one. Hansen and Sargent [1980] express the problem as follows:

> "[...]the dynamic economic theory implies that agents' decison rules are *exact* (nonstochastic) functions of the information they possess about the relevant state variables governing the dynamic process they wish to control. The econometrician must resort to *some* device to convert the exact equations delivered by economic theory into inexact (stochastic) equations susceptible to econometric analysis."

As an example, Hansen and Sargent [1980] develop a model for the error terms in the estimated decison rules by considering the simple case of a firm devising a contingency plan for the employment of a single factor of production $L_t$ (e.g. labour) subject to quadratic costs of adjustment and uncertain technology $Z_t$ and factor rental (e.g. wage) processes $W_t$. Firms will have to solve the maximization problem

$$\max_{L_t} \ \mathrm{E}_0 \left[ \sum_{t=0}^{\infty} \beta^t \left( (\gamma_0 + Z_t - W_t L_t) - (\gamma_1/2)^2 L_t^2 - (\delta/2)(L_t - L_{t-1})^2 \right) \right] , \tag{6}$$

for some parameters $\gamma_0$, $\gamma_1$, $\delta$ and $\beta$. The stochastic process $Z_t$ is seen by the firm but not by the econometrician and it is an AR(p) with structural innovations $u_t$, while $W_t$ is observed by both the firm and the econometrician. Using dynamic programming or alternative techniques to solve (6), it is possible to obtain a closed formula for the decision rule as a function of the shocks. Given that $Z_t$ is not observed by the econometrician, she will introduce an error term $e_t$ and, by use of the computed decision rule, she can always write

$$a(L)e_t = b(L)u_t \, ,$$

where $u_t$ are the structural innovations. The parameters of this representation are completely determined by the model thus we are not free to choose them, and, in general, the roots of $b(z)$ can lie on both sides of the unit circle. Therefore there may be a problem of nonfundamentalness and, even though $u_t$ is fundamental for $Z_t$, it may not be fundamental for $e_t$. If this is the case, the econometrician that estimates $e_t$ will not be able to recover the true technological innovations $u_t$.

Hansen and Sargent [1991] provide another simple example where nonfundamentalness may arise as a consequence of rational expectations. Suppose that one set of economic variables $w_t$, representing the true process, is generated by an invertible moving average process, while another set $x_t$, representing the estimated process, is made of expectational variables. Namely,

$$\begin{aligned} w_t &= u_t - \theta u_{t-1} = \tilde{C}(L) u_t \,, \\ x_t &= E_0 \left[ \sum_{t=0}^{\infty} \beta^t w_t \right] = (1 - \beta\theta) u_t - \theta u_{t-1} = C(L) u_t \,. \end{aligned}$$

The only root of $C(z)$ is $(1 - \beta\theta)/\theta$ which can be inside the unit disc even if $\tilde{C}(z)$ has its root outside the unit disc. If only $x_t$ are available to the econometrician then she may not be able to recover the structural shocks $u_t$ that generate $w_t$.

Brock et al. [2006] analyse the role of rational expectations in the framework of frequency domain analysis of linear systems with feedback control rules. Indeed, forward-looking systems with rational expectations give origin to non invertible representations. The authors show that by means of an appropriate choice of the control, e.g. monetary policy, it is possible to take the roots of the characteristic polynomial inside or outside the unit circle.
Formally, a forward-looking system with rational expectations is written as

$$A_0 x_t = \beta E_t[x_{t+1}] + A(L) x_{t-1} + B(L) c_t + \varepsilon_t \,,$$

where $x_t$ are the state variables, $c_t$ are the control variables and $\varepsilon_t = W(L) u_t$ with $u_t$ being the fundamental shocks. A generic linear feedback rule is written as

$$c_t = K(L) x_{t-1} \,.$$

Finally, we denote with $x_t = C(L) u_t$ the equilibrium moving average representation of the system. The key point is that $C(L)$ depends on the choice of the control rule, i.e on the polynomial matrix $K(L)$. The choice of different control rules has an impact on the spectral density matrix of the state variable $x_t$, which is

$$f_x(\theta) = \frac{1}{2\pi} C(e^{-i\theta}) \Sigma_u(\theta) C(e^{-i\theta})' \,,$$

$\Sigma_u(\theta)$ being the variance covariance matrix of the fundamental shocks. Indeed, the control enters the expression for $C(e^{-i\theta})$:

$$C(e^{-i\theta}) = A_0 - (A(e^{-i\theta}) + B(e^{-i\theta}) K(e^{-i\theta}) e^{-i\theta})^{-1} W(e^{-i\theta}) \,.$$

It is possible to show that the application of a given control can have an impact on the value of $C(L)$ and on the location of the zeroes of its determinant. This is crucial in the case of forward-looking systems when the fundamental shocks cannot be recovered by current and past values of the state variables. These latter constitute the policymaker information set, while the agents also observe the fundamental disturbances and know their process $W(L)u_t$. However, with an appropriate choice of the feedback control, the policymaker is able to turn a nonrevealing equilibrium into a revealing one, and vice versa. In appendix to their paper, Brock et al. [2006] provide an example in the univariate case.

Finally, notice that the usual procedure used to get an equilibrium formula for rational expectations is valid only in the cases when the structural shocks are independent over time, and this limits the possibility of analysing the effect of different policies which are made of non necessarily independent shocks. Futia [1981] generalizes rational expectations equilibria to the case of general exogenous stochastic shocks $v_t$, by using the so-called $z$-tranforms which are functions $f(z)$ such that

$$x_t = f(L)v_t\,,$$

with the condition that $f$ has no poles inside the unit disc. Once again this representation can be inverted only if $f(z) \neq 0$ for $|z| < 1$. This is simply the generalization of the problem of nonfundamentalness in the case of rational expectations models with non-independent shocks.

# 5 Dynamic Stochastic General Equilibrium models

A large portion of macroeconomic literature focuses on Dynamic Stochastic General Equilibrium (DSGE) models in order to micro-found business cycle models, which are usually generated by serially uncorrelated and orthogonal stochastic structural shocks. The validation procedure, used for example in a couple of recent papers by Chari et al. [2005] and by Christiano et al. [2006], in order to assess the reliability of VAR as an instrument to discriminate among competing models, is the following:

1. consider a DSGE model (e.g a real business cycle model or a nominal rigidities model);
2. reformulate it in a state space form usually obtained by (log)-linearizing about the non-stochastic steady state;
3. estimate the parameters of the state space form (e.g. by Maximum Likelihood or with Bayesian methods);
4. compute the impulse response functions of the DSGE variables to the economic shocks as given in the state space form;
5. generate new data from the state space model, using parameters estimated at step 3 (this and the following steps are repeated thousands of times in a Monte Carlo experiment);
6. using the data generated in the previous step, estimate a VAR jointly with economically meaningful identification restrictions, and compute the same impulse responses, together with their confidence intervals;
7. compare these simulated VAR impulse responses with the ones obtained from the estimated VAR at step 4.

The last step is crucial since, if there is no bias in the estimated impulse responses and in their confidence intervals, we can say that VARs are indeed a useful tool for discriminating

among different models, i.e. we can estimate the VAR with real data and, from its impulse responses, we can say which is the more correct economic model.
Let us now show how the problem of nonfundamentalness arises in dealing with DSGEs. When using real data to estimate the impulse responses, observations for many state variables (usually stocks as e.g. capital) are typically not available. Therefore, it is not possible to estimate the same impulse responses as the simulated ones since some of the variables of the DSGE are omitted when using real data. Whenever we omit a variable we do not have anymore a VAR representation but we typically end up writing a VARMA representation of the linearized DSGE. When estimating a VARMA we must always consider the possibility of having a nonfundamental MA part before transforming it in a VAR.

Consider the example by Pagan [2007] of a fiscal policy case where $x_t$ is the primary deficit and the level of debt is defined as a gap relative to its desired equilibrium value. Debt accumulates as $\Delta d_t = x_t$ where we set the interest rate on past debt to zero. In order to stabilize debt we need a fiscal rule that relates to the past debt level and responds to an output gap $y_t$ i.e.

$$\Delta d_t = x_t = a d_{t-1} + c y_t + u_t \quad \text{with} \quad a < 0\,.$$

Typically we drop debt from the VAR thus we need to solve the previous equation for $d_t$ and substitute it in the fiscal policy equation, obtaining

$$x_t = (1-a)x_{t-1} + c\Delta y_t + \Delta u_t\,.$$

This is no more a VAR but a VARMA where the MA part $\Delta u_t = (1-L)u_t$ has its root in $z = 1$, thus it is not invertible.

Nonfundamentalness generated by omitted variables is often considered innocuous provided that we estimate a VAR with enough lags. However, the feasibility of writing a VAR representation of a particular DSGE model is never seriously considered. Indeed, given the presence of expectations in such models, it is not unlikely to face a problem of nonfundamentalness already when solving and linearizing the DSGE. When this happens, the entire procedure of validation of a DSGE model through a VAR is invalid, given that it will recover fundamental representations of a nonfundamental structural model.
Fernandez-Villaverde et al. [2005] state the conditions under which we can write a VAR as a linearized solution of a DSGE model. Let us consider a DSGE model with an $N$-dimensional vector of observable variables $x_t$ and a $q$-dimensional vector of economic shocks $u_t$ such that $u_t \sim \text{w.n.}\,(0, I_q)$. We can obtain its state space form by log-linearizing about a non-stochastic steady state. Once we define a $k$-dimensional vector of state variables $f_t$ then the state equation is

$$f_t = A f_{t-1} + B u_t\,, \tag{7}$$

and the measurement equation is

$$x_t = C f_t + D w_t\,. \tag{8}$$

It is then possible to find conditions on $A$, $B$, $C$, and $D$ that allow for the existence of a VAR representation for $x_t$. We consider here only the simplest case when $q = N$. Indeed, since we are in the square case, we can invert (8) and we can rewrite (7) as

$$\left[I - (A - BD^{-1}C)L\right] f_t = BD^{-1} x_{t-1}\,.$$

We now need to invert this equation in order to have $f_t$ as a function of $x_t$. The usual condition for invertibility in the past (i.e. for fundamentalness) is that the eigenvalues of $(A - BD^{-1}C)$ are all inside the unit circle so that we can invert the previous equation by using the sum of a geometric series. If this is the case, then

$$f_t = \left[\sum_{k=0}^{\infty}(A - BD^{-1}C)^k L^k\right](BD^{-1}x_t)\,,$$

and from (8)

$$\left[I - C\sum_{k=0}^{\infty}(A - BD^{-1}C)^k L^k\right] x_t = Du_t\,.$$

If we now compare this theoretical expression with the VAR that an economtrician will estimate, say for example $A(L)x_t = e_t$, we realize that the VAR representation that an econometrician will estimate is consistent with the theory only if the eigenvalues of $(A - BD^{-1}C)$ lie all inside the unit circle. This condition stated by Fernandez-Villaverde et al. [2005] gives us a practical way to check for fundamentalness of the economic shocks $u_t$. Such a criterion might be useful in the case we have a state space form of our model but we do not have a structural representation for it as in (1), so that we cannot check directly definition 1.

# 6 Factor models for structural identification

Although the literature often considers nonfundamentalness as a minor problem at least in all practical cases, we tried to convince the reader that ruling out nonfundamental representations might hide the econometrician a large number of alternative possible meaningful representations of a given model. We would like to find econometric models that do not have to bother with the problem, but still are able to achieve identification of structural shocks. Dynamic factor models are a good tool for this latter purpose. In this section we outline how these models are built and how they deal with nonfundamentalness.

## Dynamic factor models as representations of DSGE models

Giannone et al. [2006] and Boivin and Giannoni [2006] provide the motivation for considering a factor structure in validating DSGE models. Typical theoretical macroeconomic models have few shocks driving the business cycle, e.g. only one technology shock in first generation real business cycle models, two or three in second generation ones. We say that these models have reduced stochastic rank. Usually in DSGE models also measurements errors are considered and in this case it can be shown that the model can have a factor structure, since factor models separate out measurement errors by their own nature. Indeed, in these models the spectral density matrix of the observed variables is decomposed into two orthogonal parts: the spectral density of the common component, of reduced rank, that contains all the relevant information of covariances (at all leads and lags), and the spectral density of the idiosyncratic component, of full rank, that represents non correlated or mildly correlated measurement errors. This approach wipes away measurement errors, which heavily affect VAR impulse responses. Therefore factor models seem to be a good alternative tool to validate DSGE models, as will be formally discussed in this and the following sections.

The most general DSGE model is formulated as follows:

$$\begin{aligned}\max_{Y_t} \quad & \mathrm{E}_0\left[\sum_{t=0}^{\infty}\beta^t U(Y_t)\right]\\ \text{s.t.} \quad & g(Y_t, Y_{t-1}, \dots, S_t, S_{t-1}) \leq 0\,.\end{aligned}$$

The model includes $n$ endogenous predetermined variables $Y_t$ and $q$ exogenous variables $S_t$. These exogenous variables are usually modelled as functions of $q$ serially uncorrelated orthonormal structural shocks $u_t$. Therefore the system contains $N = n + q$ variables $X_t = (Y_t\ S_t)'$. Let us indicate with small letters the difference between the log of the variables and their non-stochastic steady state. We have the linearization of the model

$$\begin{aligned} y_t &= \Theta(L)s_t\,,\\ \Psi(L)s_t &= u_t\,. \end{aligned}$$

The system can be transformed into a state space form by defining the state variables as $f_t = (s'_t, \dots, s'_{t-s})$ where $s$ is the maximum degree between $\Psi(L)$ and $\Theta(L)$. Therefore we have

$$\begin{aligned} x_t &= \Lambda f_t\,, \\ A(L)f_t &= Bu_t\,. \end{aligned} \tag{9}$$

Note that the dimension of $f_t$ is $r = q(s + 1)$ and the static rank of the system (i.e. the rank of the covariance of $x_t$) is at most $r$ and it is given by the restrictions imposed on the VAR (the $q$ shocks) and on the number of lags included in the model $s$, therefore it depends on the structure of the economy. In most DSGE models we have reduced static rank i.e. $r < N$, which is also empirically found in the form of common cycles. From (9) we obtain the MA representation

$$x_t = \Lambda(I - A(L))^{-1}Bu_t = C(L)u_t\,. \tag{10}$$

From this equation is clear that the dynamic rank of $x_t$ (i.e. the rank of its spectral density matrix) is $q$, and therefore it depends on the number of exogenous forces. In general for macroeconomic datasets $q < N$, which means that there is collinearity among the $N$ variables.[1] The reduced static and the dynamic ranks are restrictions that come from the theory and that could be tested. In principle we could now estimate the VAR $D(L)x_t = \varepsilon_t$ where $\varepsilon_t = Bu_t$. However, to estimate this VAR we need that $r = N$ in order to invert $\Gamma^x_0$, which is almost never the case. Thus VAR estimation is not possible due to the reduced static rank of macroeconomic datasets. There are two alternatives: either estimate a VAR only on blocks of $r$ variables, or add measurement errors. In the latter case we eliminate the collinearity among variables and we can estimate the full system, thus either we estimate a VARMA on the whole system, or we estimate a dynamic factor model. Thus, the last case is the one that we are interested in (see Giannone et al. [2006] for details on all the cases).

[1] About the ranks note that

$$\Sigma^x(\theta) = C(e^{-i\theta})\Gamma^u_0 C(e^{i\theta})'\,,$$

and since $\operatorname{rank} C(L) = q$ the dynamic rank is $q$, while

$$\Gamma^x_0 = \Lambda F_t F'_t \Lambda'\,.$$

Therefore the maximum static rank is $r$.

## Introducing measurement errors

When adding orthogonal measurement errors $\xi_t$, we lose collinearity of the variables and we can write (10) for a covariance stationary process $x_t$ as a dynamic factor model

$$x_t = C(L)u_t + \xi_t = \chi_t + \xi_t \,, \tag{11}$$

where $u_t$ is the $q$-dimensional vector of common shocks s.t. $u_t \sim$ w.n.$(0, I_q)$, and $\xi_t$ is an idiosyncratic $N$-dimensional process of measurement errors s.t. $\xi_{it-k}$ is orthogonal to $u_{jt}$ for any $i$, $j$, and $k$. Two assumptions are made for the factor model: the $q$ largest eigenvalues of the spectral density matrix of $x_t$ diverge as $N \to \infty$, while the $(q+1)$-th is bounded almost everywhere for all frequencies $\theta \in [-\pi, \pi]$. These assumptions are reasonable since measurement errors are supposed to vanish when considering linear combinations of many collinear variables. As a consequence, the common component $\chi_t$ has reduced dynamic rank $q < N$, while $\xi_t$ has full dynamic rank: this is how we break collinearity. Notice that the need of large cross sections to apply the factor model is perfectly consistent with the standard practice of central banks, which use all the available information when making decisions.
We can also add measurement errors to the state space form (9)

$$\begin{aligned} x_t &= \Lambda f_t + \xi_t \,, \\ A(L) f_t &= B u_t \,. \end{aligned} \tag{12}$$

Once again, given the previous assumptions, we have a common part with reduced static rank and an idiosyncratic part with asymptotically vanishing covariance that has full static rank. Therefore, when dealing with large cross sections we still have reduced dynamic and static rank of the whole dataset $x_t$. We can estimate a factor structure on every model with reduced static and dynamic ranks, which are typical properties of macroeconomic datasets. Hereafter we will call $f_t$ the static factors while $u_t$ will be the dynamic factors that correspond to the structural shocks of the economy. We want to identify $u_t$ and the impulse responses that they generate.

The most general factor model is the Generalized Dynamic Factor Model (GDFM) by Forni et al. [2000], where some cross-correlation between the elements of $\xi_t$ is allowed. In this case, by using the one-sided estimator proposed by Forni et al. [2005] it is possible to recover the static factors $f_t$ as the $r$ largest generalized principal components. Then, the dynamic factors $u_t$ are estimated by inverting the second of (12), where usually $A(L)$ is of order one.[2] Actually, this procedure recovers $e_t = Ru_t$ with $RR' = I_q$, therefore once the parameters of (12) are estimated, the impulse responses are $\Lambda(I - A(L))^{-1}BR$. Once again, $R$ is determined by imposing economic restriction.

To sum up, the two main advantages from imposing a factor structure on the linearized solution of a DSGE model are the following:

1. given the properties of the estimator by Forni et al. [2000] we need a large cross section ($N \to \infty$) and to have a good estimation of the spectral density we require also a large

[2] The first $r$ generalized eigenvectors $z_i$ correspond to the $r$ largest generalized eigenvalues $\lambda_i$ that satisfy

$$\Gamma_0^{\chi} z_i = \Gamma_0^{\xi} \lambda_i z_i \quad \text{for } i = 1, \ldots, r \,.$$

The principal components are the projection of $x_t$ onto the space spanned by $(z_1 \ldots z_r)$.

time dimension. This seems a perfectly realistic requirement in agreement with the practice followed by central banks, where usually DSGE models are applied;

2. $x_t$ contains the observed variables of the DSGE model and some proxies of the state variables which are often unobserved and can be estimated as the latent static factors $f_t$. Indeed, the typical macroeconomic variables included in the panel are indicators of economic activity built by aggregation, which can be seen as linear combinations of unobserved state variables (and their lags) plus some measurement errors. It is possible to impose structural relations between the observed $x_t$ and the unobserved $f_t$, i.e. to impose restrictions on $\Lambda$. The two-step procedure for estimating the restricted model is the following: (i) carry out a non-parametric estimation of $f_t$ as in Forni et al. [2000]; (ii) apply a Quasi-ML Kalman filter estimator as the one proposed by Doz et al. [2006].

## Fundamentalness in dynamic factor models

Why in the previous section, when considering factor models as a tool for validating DSGE models, have we not raised the issue of fundamentalness, that is pervasive when dealing with VAR? Because we can show that actually nonfundamentalness is not a generic problem in factor models, and, under reasonable assumptions, we can always guarantee that the dynamic factors $u_t$ are fundamental for $x_t$ (see Forni et al. [2006]). In factor models we always have $N > q$, therefore we first need a definition of nonfundamentalness that generalizes definition 1 to the case of singular systems. It is indeed the singularity of dynamic factor models that makes the property of nonfundamentalness non generic and therefore makes us ask "Why should we care about it?".

**Definition 2 (Fundamentalness in singular models)** *Given a covariance stationary vector process $x_t$, the representation $x_t = C(L)u_t$ is fundamental if:*

1. *$u_t$ is a white noise vector;*

2. *$C(L)$ has no poles of modulus less or equal than unity, i.e. it has no poles inside the unit disc;*

3. *$C(L)$ has full rank inside the unit disc*

$$\text{rank } C(z) = q \quad \forall\, z \in \mathbb{C} \quad \text{s.t. } |z| < 1\,.$$

*Alternatively we can restate this condition in terms of the roots of* $\det C(z)$*. We ask that the determinants of all the $q \times q$ submatrices of $C(z)$ have no common roots inside the unit disc. More precisely, if we call $C_j(L)$ the submatrices contained in $C(L)$ and we define the set of indexes $\mathbb{I} = \left\{ j \in \mathbb{N} \text{ s.t. } j = 1, \ldots, \binom{N}{q} \right\}$, the definition of nonfundamentalness requires that*

$$\nexists\, z \in \mathbb{C} \ \text{ s.t.} \begin{cases} |z| < 1 \\ \\ \det C_j(z) = 0 \quad \forall\, j \in \mathbb{I}\,. \end{cases}$$

As an example, consider the case $q = 1$. If $N = 1$ we are back to definition 1 and we require that no root of $C(z)$ is smaller than one. If instead $N > 1$ we have $N$ polynomials $C_j(z)$ and from definiton 2 the representation is nonfundamental if they have a common root smaller than one. Thus, if $N = q$, nonfundamentalness is generic since if it holds in a point then, for continuity of the roots of $C(z)$, it holds also in its neighborhood; while if $N > q$ nonfundamentalness is non-generic because to have a common root we must satisfy $\binom{N}{q} - 1$ equality constraints. In singular models we usually have highly heterogeneous impulse responses of the variables to the few structural shocks, therefore it is highly improbable to have a common root for all of them, although it is not unlikely to have common roots for some submatrices of $C(L)$. Roughly speaking, although in principle the econometrician has a smaller information set than the agents' one (i.e. there is nonfundamentalness), he can supply the lack of information by observing additional series, and if dynamic heterogeneity is guaranteed then these series contain useful information. In macroeconomic datasets this is very likely to happen, thus fundamentalness in factor models is a reasonable property.

Fernandez-Villaverde et al. [2005] provide an economic example, used also by Forni et al. [2006], that clarifies this point. Let us consider the permanent income consumption model

$$\begin{aligned} c_t &= c_{t-1} + \sigma_u(1-\rho^{-1})u_t\,, \\ s_t &= y_t - c_t = -c_{t-1} + \sigma_u\rho^{-1}u_t\,, \end{aligned}$$

where $c_t$ is consumption, $y_t$ is labour income, $u_t$ a white noise process and $\rho$ the gross interest rate. Fernandez-Villaverde et al. [2005] assume that $s_t$ is observable while $c_t$ is not. From equations above, we have

$$s_t - s_{t-1} = \sigma_u\rho^{-1}(1-\rho L)u_t = d(L)u_t\,.$$

Thus since $d(z) = 0$ for $z = \rho^{-1} < 1$, $u_t$ is nonfundamental for $s_t$. Thus a VAR(1) estimated by the econometrician would produce innovations which are not the structural shocks. However, if the econometrician observes also some additional variables $z_{it} = b_i(L)u_t$, then $u_t$ is fundamental for the whole system $(s_t\, z_t)'$ unless $d(z)$ and $b_i(z)$ have the same root, i.e. unless $b_i(\rho^{-1}) = 0$, $\forall\, i$, which is extremely unlikely.

In what follows we formalize the ideas shown in this example. When considering dynamic factor models, we make the assumption that only the largest $r$ eigenvalues of $\Gamma_0^x$ diverge as $N \to \infty$ the others being bounded. This in turn implies that $\text{rank}\,(\Lambda'\Lambda)/N = \text{rank}\,\Gamma_0^\chi/N = r$ for large $N$. Such a condition can be guaranteed if no restrictions are imposed on the entries of $C(L)$ which are the elements of $\Lambda$. Therefore this is equivalent to ask for heterogeneity of the impulse responses, which in turn requires $s > 0$, i.e. $r > q$. Dynamic heterogeneity is reasonable in a factor model with large cross sectional dimension $N$ as economic variables react differently to structural shocks: this is precisely what we need for considering nonfundamentalness a non-generic problem. It is thus reasonable to assume fundamentalness of dynamic factor models. Notice that the dynamic (11) and static (12) representations are equivalent if there exists a squared-summable one-sided $r \times q$ filter $N(L)$ such that

$$C(L) = \Lambda N(L) \qquad \text{and} \qquad f_t = N(L)u_t\,. \tag{13}$$

Forni et al. [2006] prove that, under the assumptions of the dynamic factor model, fundamentalness of $u_t$ for $\chi_t$ is equivalent to left invertibility of $N(L)$, i.e. to the existence of a $q \times r$ filter $G(L)$ such that $G(L)N(L) = I_q$. It is enough to take $S(L) = G(L)(\Lambda'\Lambda)^{-1}\Lambda'$, and we have

$$S(L)x_t = G(L)(\Lambda'\Lambda)^{-1}\Lambda'\Lambda F_t + S(L)\xi_t \overset{m.s.}{\longrightarrow} G(L)N(L)u_t = u_t \qquad \text{for } N \to \infty\,.$$

Therefore $u_t$ lies in the space spanned by the present and past values of $\chi_t$. Usually, as in the previous section, we choose $N(L) = (I_q\,(I_q L)\ldots(I_q L^s))'$ and the following identities hold

$$\begin{aligned} f_t &= (u_t'\,u_{t-1}'\ldots u_{t-s}')' \,, \\ \Lambda &= (C_0\ \ldots C_s)\,, \\ r &= q(s+1)\,. \end{aligned}$$

Representations (11) and (12) are indeed equivalent for a given lag length $s$ and with this choice $G(L) = (I_q\,0_q\ldots 0_q)$.

The whole reasoning suggests that, given a large cross section, we will have dynamic heterogeneity which, through the assumptions of pervasiveness of the static and dynamic factors and reduced dynamic rank and of fundamentalness of dynamic factors with respect to the static ones, in turn implies that $u_t$ is fundamental for the whole $\chi_t$, but it may not be fundamental for a subsample. However, in such cases we can always use additional cross sectional information from other series to recover the dynamic factors, therefore supplying the missing information due to local nonfundamentalness. Formally, let us consider the projection

$$f_t = \text{Proj}\ (f_t|f_{t-1}, f_{t-2}, \ldots, f_{t-m}) + w_t\,, \tag{14}$$

where the prediction error $w_t$ is fundamental by construction. From the assumption of fundamentalness, $u_t$ is fundamental for $f_t$, therefore the representation $f_t = N(L)u_t$ has an equivalent VAR representation $A(L)f_t = Bu_t$ where $B$ is $r \times q$ and $A(L)$ is $r \times r$. By comparing this last representation with (14) we get $w_t = Bu_t$. In many cases when there is dynamic heterogeneity $(r > q)$, the information contained in the lagged values of $f_t$ can be substituted by using cross sectional information, therefore one lag for $A(L)$ seems to be enough and we have the VAR(1) specification

$$f_t = Af_{t-1} + Bu_t\,. \tag{15}$$

We already know that, once the factor model is estimated, the dynamic factors (or structural shocks) are identified only up to a rotation $R$. Given that fundamentalness can be assumed in this framework, identification is then reduced to the choice of the matrix $R$ such that economically motivated restrictions on $C(L)R$ are satisfied. The number of restrictions that we have to impose is just $q(q-1)/2$, which is a big advantage since we need to impose few restrictions but we do not have any limitation on the size of the panel.

A simple example is taken from Forni et al. [2006]. Consider the case with only one dynamic factor loaded with one lag, therefore $q = 1$ and $s = 1$. The common part of the $i$-th series is

$$\chi_{it} = (1 - c_i L)u_t = \Lambda_{i\cdot} f_t\,.$$

If we had homogeneous responses to the static factors $f_t$ we would have $c_i = c\ \forall i$. In this case, we can easily see that

$$\text{rank}\,(\Lambda'\Lambda) = \text{rank}\begin{bmatrix} N & Nc \\ Nc & Nc^2 \end{bmatrix} = 1\,,$$

hence $r = 1$. Since $N(L) = (1 - cL)$, fundamentalness is guaranteed only if we impose $|c| < 1$. In this case the problem of nonfundamentalness is pervasive.
In order to have fundamentalness without any additional restrictions we need heterogeneity

in the dynamics of the responses, i.e. $c_i \neq c_j$ for $i \neq j$. In this case rank $(\Lambda'\Lambda) = 2$ and $r = 2$ since

$$\text{rank}\,(\Lambda'\Lambda) = \text{rank}\begin{bmatrix} N & \sum_i c_i \\ \sum_i c_i & \sum_i (c_i)^2 \end{bmatrix} = 2\,.$$

Moreover, now

$$f_t = \begin{bmatrix} 1 \\ L \end{bmatrix} u_t = N(L)u_t\,,$$

hence fundamentalness is always satisfied with $G(L) = (1\ 0)$. In this case indeed we can recover $u_t$ from any couple of series as

$$u_t = \frac{c_j\chi_{it} - c_i\chi_{jt}}{c_j - c_i}\,.$$

Therefore $u_t$ is fundamental for $(\chi_{it}, \chi_{jt})$ even if $c_i > 1\ \forall\, i$, i.e. even if $u_t$ is not fundamental for $\chi_{it}$.

# 7 Granger causality and nonfundamentalness

Giannone and Reichlin [2006] propose a criterion to detect nonfundamentalness in VAR representations that is based on the concept of Granger causality. Once again, this approach leads naturally to a factor structure for the data: indeed, while fundamentalness cannot be tested in a VAR, it can be tested in a factor model. They consider the well known VAR firstly estimated by Gali [1999], which can be derived from very different DSGE models such as real business cycle models or New-Keynesian models

$$\begin{bmatrix} \Delta a_t \\ \Delta l_t \end{bmatrix} = C(L) \begin{bmatrix} u_t^z \\ u_t^d \end{bmatrix}, \tag{16}$$

where $a_t$ is the log of aggregate labour productivity and $l_t$ is the log of aggregate labour supply. There are two structural shocks: a tehnological shock $u_T^z$ and a shock $u_t^d$ which is neutral for productivity in the log-run, being thus interpretable as a labour income (or demand) shock or a monetary shock. Let us call $x_t^* = (\Delta a_t\ \Delta l_t)'$ the vector of observable variables which we augment with other variables $x_t$, so that (16) for the larger system becomes

$$\begin{bmatrix} x_t^* \\ x_t \end{bmatrix} = \begin{bmatrix} C(L) & 0 \\ D(L) & \Psi(L) \end{bmatrix} \begin{bmatrix} u_t \\ v_t \end{bmatrix},$$

with $v_t$ as additional structural shocks orthogonal to $u_t$. If $u_t$ is fundamental for $x_t^*$ then there exists a one-sided filter $N(L)$ such that $u_t = N(L)x_t^*$, therefore

$$x_{it} = D_{i\cdot}(L)N(L)x_t^* + \Psi_{i\cdot}(L)v_t \qquad \text{for } i = 1, \ldots, N\,.$$

Hence, each $x_{it}$ depends only on the past of $x_t^*$ and does not incorporate any further information useful for forecasting $x_t^*$, i.e. none of the $x_{it}$ Granger causes $x_t^*$. This result was firstly introduced by Forni and Reichlin [1996]. It follows that nonfundamentalness can be detected empirically by checking whether the variables of interest $x_t^*$ are weakly exogenous with respect to potentially relevant additional blocks of variables that are likely to be driven by shocks which are common to the variables belonging to the block of interest. In the model above, Giannone and Reichlin [2006] consider as additional variables labour productivity and labour input at sectoral level and they indeed reject the hypothesis of weak exogeneity, thus giving a clue for nonfundamentalness.

Adding larger information is useful to detect nonfundamentalness but does not solve the problem, although it can help. To show how this is possible, we consider finite MA representations, i.e. $D(L)$ is of order $s$. Therefore we can write that

$$x_t = D_0 u_t + \ldots + D_s u_{t-s} + \Psi(L) v_t \,.$$

It is possible to show that we can asymptotically recover $u_t$ if

1. $\text{rank}(D'D)/N \to q(s+1)$ as $N \to \infty$, which implies that the shocks $u_t$ are pervasive;

   and

2. the largest eigenvalue of the covariance matrix of $\Psi(L)v_t$ is bounded for $N \to \infty$, i.e. the idiosyncratic shocks are either measurement errors or sectoral shocks as in the example above.

From the first assumption we have that $x_t$ Granger causes $x_t^*$. As shown in the previous section (see Forni et al. [2006]), these assumptions are satisfied if the data can be represented by an approximate factor model. Finally, if the time dimension is smaller than the cross dimension we face a problem of dimensionality that can be solved by considering a dynamic factor model that satisfies assumptions 1 and 2 and where the sources of variation are no more $q(s+1)$ but simply $q$, and the following representation holds:

$$\begin{bmatrix} x_t^* \\ x_t \end{bmatrix} = \begin{bmatrix} \Lambda^* \\ \Lambda \end{bmatrix} f_t + \Psi(L) v_t \,,$$

where $A(L)f_t = Bu_t$ and $f_t$ is $q(s+1)$-dimensional while $u_t$ is $q$-dimensional. Once again we turn to factor models as tools for identification where the problem of nonfundamentalness is no more pervasive. It is worth to note that, by using this method, Giannone and Reichlin [2006] find that the shock estimated for the model (16) are actually non-structural shocks. Therefore, nothing can be said about the dispute between real business cycle models and models with nominal rigidities by looking only at labour productivity and labour input as it is usually done in the literature (e.g. see Gali [1999]).

# 8 Nonfundamentalness and cointegration

There is one last reason for which nonfundamental representations can arise, which is explained in Blanchard and Quah [1993] and has to do with cointegrated models. Assume to have a bi-dimensional vector $x_t = (x_{1t}\, x_{2t})'$ of integrated time series which has a fundamental MA representation in first difference: $\Delta x_t = C(L)u_t$ where $u_t$ are structural shocks. We can apply the Beveridge-Nelson decomposition into trend and cycle, i.e. into long- and short-run dynamics

$$\Delta x_t = D(1)e_t + (I - L)D^*(L)e_t = \Delta \text{Trend}_t + \text{Cycle}_t \,, \tag{17}$$

where $(I - L)D^*(L) = D(L) - D(1)$. If $\text{rank} D(1) = 1$ then the two components of $x_t$ are cointegrated, therefore they have a common trend and its enough to include a sufficient number of lags of $\Delta x_t$ in the empirical analysis in order to identify the short-run dynamics, provided that $D^*(L)$ is invertible. Actually, if we consider decomposition (17) for integrated variables we are sure that $\det D^*(z)$ has no roots for $|z| = 1$ since we obtain it by differentiation. However, it remains the possibility to have roots for $|z| < 1$ as illustrated in a numerical example by Blanchard and Quah [1993]. If indeed some roots of $\det D^*(z)$ happen to be inside the unit disc, then there is no way to recover the fundamental shocks $u_t$ form the estimated $e_t$.

# 9 The search for nonfundamental representations

## MA representations and Blaschke matrices

Nonfundamental representations can be generated by means of Blaschke matrices which are the main subject of this section, at the end of which we will review in detail the procedure used by Lippi and Reichlin [1993] for actually finding such representations.
Once again consider an MA representation $x_t = C(L)u_t$. If either definition 2 or 1 is satisfied, we say that $u_t$ is fundamental with respect to $x_t$. On the other hand, nonfundamentalness implies that, although $u_t$ belongs to the agent information space, it is not contained in the econometrician information space. This happens in many DSGE models, and such cases pose a serious problem of identification of the shocks $u_t$. It follows that it is correct to choose the fundamental representation only if the agent information space is equal to the econometrician's one, since in this case economic meaning is guaranteed. Indeed, only in this case it is possible to invert the MA and obtain a well defined VAR that, once identified, will allow to recover the actual structural shocks.
However, there are many economically sensible theories that may generate nonfundamental representations. Therefore, we believe that the search of the impulse responses that characterize a given economic model should not be limited to fundamental representations but should instead include also the nonfundamental ones. This is why we are interested in how to switch from one group to the other. This task is accomplished by means of Blaschke matrices, which take the zeroes of a representation from outside to inside the unit disc thus generating a nonfundamental representation from a fundamental one.
We have the following definition: [3]

**Definition 3 (Blaschke matrix)** *A complex-valued matrix $B(z)$ is a Blaschke matrix if:*

1. *it has no poles inside the unit disc;*
2. $B(z)^{-1} = \overline{B'}(z^{-1})$*, where the bar indicates the matrix obtained by taking conjugate coefficients.*

Whenever we apply a Blaschke matrix to an MA process we get the new nonfundamental representation defined as

$$x_t = D(L)v_t = C(L)B(L)B(L)^{-1}u_t\,. \tag{18}$$

The main property of Blaschke transformations is that if $u_t$ is an orthonormal white noise then $v_t = B(L)u_t$ is an orthonormal white noise if and only if $B(L)$ is a Blaschke matrix. This ensures also for nonfundamental representations the requirement of uncorrelated structural shocks which is necessary in all structural models. Thus (18) together with usual identification restrictions is still a valid structural model with new impulse responses that are not recoverable with an ordinary VAR.

[3] This section is entirely drawn on Lippi and Reichlin [1994].

As examples of Blaschke matrices we have the orthogonal matrices and the matrices with a Blaschke factor as one of the entries. A generic Blaschke matrix can be always written as the product of these two.

**Theorem 1** *Let $B(z)$ be an $N \times N$ Blaschke matrix then $\exists\, m \in \mathbb{N}$ and $\exists\, \alpha_i \in \mathbb{C}$ s.t. $|\alpha_i| < 1$ for $i = 1, \ldots, m$ and*

$$B(z) = \prod_{i=1}^{m} K(\alpha_i, L)\, R_i = \prod_{i=1}^{m} \begin{pmatrix} \frac{z-\alpha_i}{1-\overline{\alpha}_i z} & 0 \\ 0 & I_{N-1} \end{pmatrix} R_i \,, \tag{19}$$

*where $R_i \overline{R}_i' = I_N$.*

Note that $B(z)$ has poles in $(\overline{\alpha}_i)^{-1}$, i.e. outside the unit disc as required from definition 3.
With reference to (18), given a fundamental representation $x_t = C(L)u_t$, let us consider the zeroes of $\det C(z)$, which by definition are all outside the unit disc, and call them $\gamma_i$. We can build a nonfundamental representation just by applying a Blaschke matrix $B(L)$ to $C(L)$ with $\alpha_i = (\overline{\gamma_i})^{-1}$ for $i = 1, \ldots, m$ and $1 \leq m \leq N$. Theorem 1 tells us that $B(L)$ is taking zeroes of $C(L)$, that are outside the unit disc ($|\gamma_i| > 1$), into zeroes of $D(L)$ which are inside the unit disc ($|\alpha_i| = |(\overline{\gamma_i})^{-1}| < 1$).
Finally, note that $x_t = C(L)B(L)v_t$, therefore $B(L)^{-1}C(L)^{-1}x_t = v_t$, but, although $C(L)$ is invertible in the past (i.e. is fundamental) by construction, the inverse of a Blaschke matrix requires the use of $L^{-1}$ (the forward operator), therefore it is impossible to recover $v_t$ only from the past of $x_t$: this is nonfundamentalness.

## ARMA representations

We now move to ARMA representations $M(L)x_t = C(L)u_t$, where $\det M(z)$ has no zeroes inside the unit disc in order to guarantee stationarity and causality for the AR part. The ARMA representation is fundamental if its MA part, $C(L)u_t$, is fundamental. Lippi and Reichlin [1994] look for different ARMA specifications where, while the AR part is completely identified, the MA part is identified up to a Blaschke matrix transformation. They point out how many examples of intertemporal maximization under rational expectations produce indeed such a situation, as discussed in section 4. If $C(L)$ is fundamental then its determinant has all $h \leq N$ roots $\alpha_i$ outside the unit disc, hence we can build nonfundamental representations $D(L)$ just by moving one or more roots of $\det C(z)$ from outside to inside the unit circle by means of a Blaschke matrix.
In order to do so, first define the subset $\Omega \in \mathbb{R}^h$ such that $\Omega = \{\omega = (\omega_1 \ldots \omega_h)\, \text{s.t.}\, \omega_i = \pm 1\}$.
We have the following theorem:

**Theorem 2** *For any possible $\omega \in \Omega$ there exist representations $M(L)x_t = P(L)v_t$ such that $\det P(z)$ has $h$ roots $\beta_i$ defined as*

$$\begin{aligned} \beta_i &= \alpha_i \quad &\text{if} \;\; \omega_i = 1\,, \\ \beta_i &= (\overline{\alpha_i})^{-1} \quad &\text{if} \;\; \omega_i = -1\,. \end{aligned}$$

*Moreover, if $P(L)$ and $Q(L)$ correspond to the same $\omega$, then $P(L) = KQ(L)$ with $K$ orthogonal, i.e. the two representations are unique up to a rotation.*

Note that if at least one of the elements of $\omega$ is $-1$ then $P(L)$ will be a nonfundamental representation. All the nonfundamental representations obtained in this way are called basic.

They come from an ARMA just by transforming the MA part while leaving untouched the AR part. Moreover, if we start from an ARMA(p,q) then all its basic representations are ARMA(p,q). Non-basic representations are obtained by multiplying the MA part $C(L)$ by an arbitrary Blaschke matrix. By doing so we increase the order of the MA and AR matrices and if $\gamma$ is a nonfundamental root of the MA, then $(\overline{\gamma})^{-1}$ is a root of the AR part. Both common sense and literature suggest that this latter case is not likely to occur, thus it makes sense to search only for basic nonfundamental representations.

## VAR representations

In general we always start from an estimated VAR, and, once inverted, we get an MA representation that by definition will be fundamental. However, from the latter we can always get nonfundamental representations that generate the impulse responses of our alternative theoretical model. This is the procedure followed by Lippi and Reichlin [1993] to generate impulse responses that represent technological diffusion under learning-by-doing dynamics. Such method is clearly explained by Lippi and Reichlin [1994]. If the true fundamental MA representation $x_t = C(L)u_t$ were known then all its nonfundamental counterparts would easily be recovered just by applying a Blaschke matrix as in (18). However, from an estimated VAR, $A(L)x_t = u_t$, we can only get the approximate ARMA representation as

$$(\det A(L))\, x_t = A_{ad}(L)u_t\,.$$

Its associated approximate MA representation is $x_t = T(L)u_t$ with $T(L) = (\det A(L))^{-1}\, A_{ad}(L)$. We have approximations because these are all finite order representations, although in theory they should have an infinite MA part or, viceversa, if the true MA were of finite order, then we should estimate an infinite VAR.

As an example, Lippi and Reichlin [1994] consider the following two-dimensional MA representation:

$$x_t = C(L)v_t = (I - CL)u_t\,.$$

They assume that $\det(I - Cz)$ has two roots $\alpha_1$ and $\alpha_2$, which by fundamentalness are both outside the unit disc ($|\alpha_i| > 1$). The VAR representation that we estimate is only the order $p$ approximation

$$A(L) = I + \sum_{k=1}^{p} C^k L^k \simeq (I - CL)^{-1}\,.$$

It is possible to show that the $2p$ complex roots of $\det A(z)$ are

$$\alpha_i \exp\left(k\frac{2\pi i}{p+1}\right) \quad \text{for } i = 1, 2 \ \text{ and } \ k = 1, \ldots, p\,.$$

Therefore, the roots of the VAR are all on circles of radius $|\alpha_i| > 1$.[4]
Actually, we are able only to get an estimate of $A(L)$, thus we cannot estimate directly the roots of $C(L)$. But we can determine the radius $\rho$ of the circle where the roots of $A(L)$ lie. For every complex $\beta$ such that $|\beta| = \rho$, we proceed as though $\beta$ were a root of $T(L)$, which is only an approximation of $C(L)$. We therefore apply theorems 1 and 2 by multiplying $T(L)$, which

[4] If the roots of the MA are complex we have only one circle of roots, if instead they are real we have two circles. Hereafter we consider the case of two complex conjugate roots $\alpha_1 = \overline{\alpha}_2$.

is by construction a fundamental representation, by a Blaschke matrix in order to obtain a nonfundamental representation. First, we look for a rotation $R$ such that $T(z)R$ has in its first column the factor $(z-\beta)$. $R$ has to satisfy

$$[T(z)R]e_1 = (z-\beta)e_1 \tag{20}$$

where $e_1 = (1\ \ 0)'$. Note that (20) is a condition only on the first column of $R$, while the second column is obtained just by using the orthogonality condition: $R\overline{R}' = I$.[5] After rotating $T(z)$, we can move the root, that now is in the first column, from $\beta$ to $(\overline{\beta})^{-1}$ with

$$K\left((\overline{\beta})^{-1}, L\right) = \begin{bmatrix} \frac{z-(\overline{\beta})^{-1}}{1-\beta^{-1}z} & 0 \\ 0 & 1 \end{bmatrix}. \tag{21}$$

We thus obtain a nonfundamental representation

$$x_t = T(L)B(L)B(L)^{-1}u_t \quad \text{where} \quad B(L) = RK\left((\overline{\beta})^{-1}, L\right). \tag{22}$$

Actually, since we know only $\rho$, we need to repeat this procedure $n$ times in order to explore all the circle of roots of $A(L)$. We choose $\beta = \rho\exp(ik\theta)$ with $\theta = \pi/n$, and $k = 1, \ldots, n-1$, $n$ being the number of roots. Note however that since we consider all $\beta$ on the circle we are taking in account not only the roots of $C(L)$ but also other values, therefore we are looking also for non-basic representations. This in turn implies that no uniqueness result as in theorem 2 holds in this case. Finally, we can study the impulse responses of the nonfundamental representations, see if some of them are economically sensible and possibly assess differences with the fundamental impulse responses $T(L)$. Although this is only an approximate procedure, it has delivered promising results in Lippi and Reichlin [1993].

# 10 Further research

Given a moving average representation of an economic model, we would like to identify a correspondence between its roots and specified impulse responses. The same method would allow us to find theoretical impulse responses which may derive also from nonfundamental representations and are consistent both with the data and with the structural model. This is the subject of our current research.

[5] If the system were $N$-dimensional we would determine unambigously only the first column of $R$ while no rule exists for fixing all other columns besides the orthogonality condition.

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