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# A comment on the relationship between firms' size and growth rate

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**Abstract**

Since the seminal work of Pareto, many empirical analyses suggested that the distribution of firms size is characterized by an asymptotic power like behavior. At the same time, recent investigations show that the distribution of annual growth rates of business firms displays a remarkable double-exponential shape. A recent letter propose a possible connection between these two empirical regularities. By assuming a bivariate Marshall-Olkin power-like distribution for the size of firms in subsequent time steps, and performing a qualitative asymptotic analysis, it is suggested that the implied growth rates distribution takes a Laplace shape. By performing a complete analytical investigation, I show that this statement is not correct. The implied distribution does in general possess a non-continuous component and becomes degenerate when perfect correlation is assumed between size levels at different time steps. Essentially, the approach is faulty as it treats firm size levels as stationary stochastic variables and neglects the integrated nature of the growth process.

*Key words:* Firm Growth, Size distribution, Power law, Laplace distribution.
*JEL Codes*: C16,D21,D30

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## 1 Introduction

Since the seminal work of Pareto on wealth distribution of households, many empirical studies suggested that, at least above some minimal size threshold, the distribution of a number of economic relevant variables (like wealth, income, revenues or capital assets) follow a power like behavior (see Kleiber and Kotz (2003) for an extended review). This statistical fact is commonly named, after his early discoverer, "Pareto Law". At the same time, recent investigations show that the distribution of annual growth rates of business firms displays a remarkable double-exponential shape, called "Laplace" distribution (see Bottazzi and Secchi (2006) and reference therein). This result is robust to different levels of aggregation and to the use of different variables (revenues, number of employees, value added, etc.) to proxy firm size.

The question naturally arises if these two distinct "stylized facts" can be somehow reconciled so that a more "unifying" theoretical view of the dynamics of firms is recovered. This is certainly a relevant task, since, as vigorously pointed out in Brock (1999), moving the analysis from "unconditional" quantities, like distributions of random variables, to more "conditional" (if not causal) relationships between the different regularities will obviously improve our understanding of the underlying drivers of the observed dynamics.

A recent letter (Palestrini, 2007) tries to directly address the question of unifying different regularities by proposing a possible relationship between the power-like nature of the distribution of firm sizes and the Laplace character of the distribution of firm growth rates. The issue raised is relevant, and the letter by Professor Palestrini represents, without any doubt, a valuable effort. His approach falls however short in two respects. First, he treats the logarithm of firm size at different time steps like a stationary random process, while it is constantly observed on real data that these variables are in general characterized by a so called "integrated" nature: it is the difference [1] of these logarithms, rather than their levels, which displays, at least some degree of, stationarity. Second, assuming a bivariate Pareto distribution for the size of firms in two different time steps, he derives the asymptotic behavior of the implied distribution of growth rates. In the hope to reconcile the two stylized facts mentioned above, he wrongly concludes, from the asymptotic analysis, that this distribution possesses a Laplace shape. In what follows, I will start by addressing the latter issue. Namely, I will show how to derive, under

[1] Or some other more complicate manipulation of the original variables in the case of fractional integration. The discussion of this point is outside the scope of the present note.

the same hypothesis, the *exact* distribution of growth rates. As my computation reveals, the implied distribution is not, in general, Laplacian nor everywhere continuous, having a finite atomic contribution for null growth rates. Using this result, I will show that the failure of the proposed model to reconcile Pareto and Laplace Laws is ultimately due to the integrated nature of the growth process.

## 2 The Marshall-Olkin bivariate distribution

Abstracting from precise economic definitions, let $\mathbf{S}_1$ and $\mathbf{S}_2$ be the size of a firm in two successive time steps. If the distribution of firm size is Paretian, the distribution of its logarithm follows an exponential distribution. Then, taking $\mathbf{s}_1 = \log(\mathbf{S}_1)$ and $\mathbf{s}_2 = \log(\mathbf{S}_2)$, the validity of the Pareto laws implies that $\mathbf{s}_1$ and $\mathbf{s}_2$ are exponentially distributed

$$\log\left(\text{Prob}\left\{\mathbf{s}_i > x\right\}\right) \sim x \quad i = 1, 2 \; .$$

Consider the joint distribution of the couple of random variables $(\mathbf{s}_1, \mathbf{s}_2)$. For the Pareto Law to be valid, this distribution should possesses exponential marginals. A natural candidate for a distribution of this kind, and the one considered in Palestrini (2007), is constituted by the multivariate exponential distribution proposed in Marshall and Olkin (1967), which in the bivariate case reads

$$\begin{aligned} \text{Prob}\left\{\mathbf{s}_1 > s_1, \mathbf{s}_2 > s_2\right\} &= 1 - F(s_1, s_2) \\ &= \exp\left\{-\lambda_1 s_1 - \lambda_2 s_2 - \lambda_{12} \max\{s_1, s_2\}\right\} \; . \end{aligned} \tag{1}$$

It is immediate to see that, according to (1), the marginal distribution for the random variables $\mathbf{s}_1$ and $\mathbf{s}_2$ are exponential with parameter $\bar{\lambda}_1 = \lambda_1 + \lambda_{12}$ and $\bar{\lambda}_2 = \lambda_2 + \lambda_{12}$, respectively. Hence, the expression for expected vale and the variance of the two random variables immediately follow. The covariance between $\mathbf{s}_1$ and $\mathbf{s}_2$ is given by

$$\text{Cov}(\mathbf{s}_1, \mathbf{s}_2) = \frac{\lambda_{12}}{\lambda \bar{\lambda}_1 \bar{\lambda}_2} \; , \tag{2}$$

where $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$. Thus, the correlation coefficient becomes equal to $\lambda_{12}/\lambda$ and is different from zero as long as $\lambda_{12} \neq 0$. As discussed in Marshall and Olkin (1967), the distribution $F$ defined in (1) is not everywhere continuous. The atomic contribution comes from the presence of the discontinuous "maximum" term. The presence of this term will play a major role in the subsequent analysis. For easy of reference I report here the expression for the generating function (Laplace transform) of the distribution $F$ derived in Marshall and Olkin (1967). I will use it in the first Theorem of the next Section. It reads

$$\begin{aligned}\psi(\omega_1,\omega_2) &= \int_0^{+\infty}\int_0^{+\infty} e^{-s_1\omega_1-s_2\omega_2}dF(s_1,s_2) \\ &= \frac{\bar{\lambda}_1\bar{\lambda}_2(\lambda+\omega_1+\omega_2)+\omega_1\omega_2\lambda_{12}}{(\lambda+\omega_1+\omega_2)(\bar{\lambda}_1+\omega_1)(\bar{\lambda}_2+\omega_2)}\ .\end{aligned} \tag{3}$$

## 3 The distribution of growth rates

In this section we derive the analytical expression of the distribution of the (logarithmic) growth rate under the assumption that the log size of the firms is distributed, in two successive time steps, according to the bivariate Mashall-Olkin distribution defined in (1). The firm growth rate $r$ over a given period of time is, by definition, the difference of the logarithm of the size at the end and at the beginning of said period. Then, using the notation introduced above, one has $\mathbf{r} = \mathbf{s}_2 - \mathbf{s}_1$. Let $G(r) = \text{Prob}\{\mathbf{r} \le r\}$ be the probability distribution of growth rates and $\tilde{g}(k) = E[\exp i\, k\mathbf{r}]$ the associated characteristic function. Using the generating function in (3) one can easily derive the expression for $\tilde{g}(k)$. One has the following

**Theorem 1** *If $\mathbf{s}_1$ and $\mathbf{s}_2$ follow a bivariate Marhsall-Olkin distribution the characteristic function $\tilde{g}(k)$ of their difference $\mathbf{r} = \mathbf{s}_2 - \mathbf{s}_1$ is given by*

$$\tilde{g}(k) = \frac{\lambda\bar{\lambda}_1\bar{\lambda}_2 + \lambda_1 2k^2}{\lambda(\bar{\lambda}_1 - ik)(\bar{\lambda}_2 + ik)}\ . \tag{4}$$

**PROOF.** Considering the distribution function $G(r)$ one has

$$\begin{aligned}G(r) &= \text{Prob}\{\mathbf{r} \le r\} = \text{Prob}\{\mathbf{s}_2 - \mathbf{s}_1 \le r\} \\ &= \int_0^{+\infty}\int_0^{+\infty} \theta(r - s_1 + s_2)dF(s_1,s_2)\end{aligned}$$

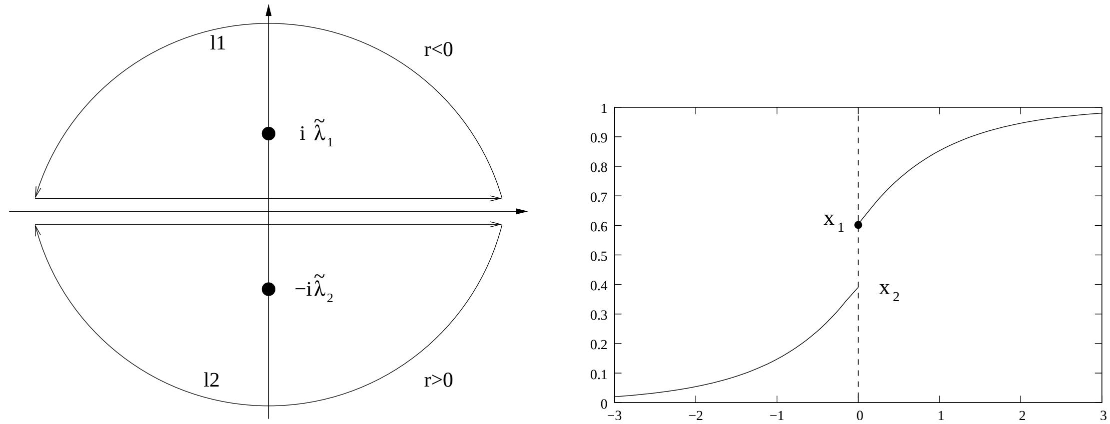


Fig. 1. **Left:** Complex plane for the anti-transform of the characteristic function. The poles are shown together with the paths of the Cauchy integral $l1$ and $l2$ used when $r < 0$ and $r > 0$, respectively. **Right:** Growth rates distribution function $G$ for $\lambda_1 = 1$, $\lambda_2 = 1$ and $\lambda_{12} = .5$.

where $\theta$ is the right continuous Heaviside theta function, which is equal to 1 if its argument is positive or null and zero otherwise. From the definition of characteristic function it is

$$\tilde{g}(k) = \int_{-\infty}^{+\infty} e^{ikr}\, dG(r) \ ,$$

substituting the previous expression and inverting the order of integrations one has

$$\tilde{g}(k) = \int_0^{+\infty}\int_0^{+\infty} e^{ik(s_1-s_2)}\, dF(s_1, s_2)$$

which, remembering (3), gives $\tilde{g}(k) = \psi(-ik, ik)$. Finally, direct substitution of (3) proves the assertion. □

The term proportional to $k^2$ in the numerator of $\tilde{g}(k)$ in (4) is what makes the implied growth rate distribution not Laplacian. Indeed, if $\lambda_{12} = 0$, then the expression in (4) would reduce to the characteristic function of an asymmetric (or symmetric, if $\lambda_1$=$\lambda_2$) Laplace distribution (c.f. Kotz et al. (2003) p.141). As shown in the next Theorem, the effect of this term is to introduce a finite probability for the occurrence of zero growth rates, that is an atomic component in the point $r = 0$. In general, one has the following

**Theorem 2** *If* $\mathbf{s}_1$ *and* $\mathbf{s}_2$ *follow a bivariate Marhsall-Olkin distribution the distribution function* $G(r)$ *of their difference* $\mathbf{r} = \mathbf{s}_2 - \mathbf{s}_1$ *is given by*

$$G(r) = \begin{cases} e^{\bar{\lambda}_1 r} \frac{\lambda_2}{\lambda} & r < 0 \\ 1 - e^{-\bar{\lambda}_2 r} \frac{\lambda_1}{\lambda} & r \geq 0 \end{cases} . \tag{5}$$

**PROOF.** Consider the formal definition of the density $g(r) = G'(r)$ as the anti-Fourier transform of the characteristic function

$$g(r) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \, e^{-ikr} \, \tilde{g}(k) \; .$$

Since $\lim_{k\to\infty} |\tilde{g}(k)| = 0$ on the whole complex plane, for Jordan's lemma the previous integral, and consequently the density function, exists for any $r \neq 0$. The characteristic function $\tilde{g}(k)$ possess two simple poles on the imaginary axis, in $i\bar{\lambda}_1$ and $-i\bar{\lambda}_2$. Then, the previous expression can be written as a Cauchy integral on the upper or lower half plane when the value of $r$ is respectively lower or greater then zero. The two closed curves are depicted in Fig. 3 (left panel). According to Cauchy integral theorem, in each case the value of the integral is proportional to the residue of the function computed in the internal pole. After a little algebra one has

$$g(r) = \begin{cases} e^{\bar{\lambda}_1 r} \frac{\bar{\lambda}_1 \lambda_2}{\lambda} & r < 0 \\ e^{-\bar{\lambda}_2 r} \frac{\bar{\lambda}_2 \lambda_1}{\lambda} & r \geq 0 \end{cases} .$$

Using the previous expression, the distribution function $G$ can be computed as

$$G(r) = \int_{-\infty}^{r} dr' \, g(r') \quad \text{if} \quad r < 0$$

or

$$G(r) = 1 - \int_{r}^{+\infty} dr' \, g(r') \quad \text{if} \quad r > 0 \; .$$

Since the distribution function is by definition right continuous, the assertion follows. $\square$

An example of the shape of the distribution $G$ isnreported in Fig. 3 (right panel). Notice that the finite weight at $r = 0$ can be easily computed using (5). Indeed one has

$$\text{Prob}\{\mathbf{r} = 0\} = \lim_{\delta \to 0^+} G(\delta) - G(-\delta) = \frac{\lambda_{12}}{\lambda} ,$$

that is the discontinuity in zero of the distribution function is proportional to the correlation coefficient of the two variables $\mathbf{s}_1$ and $\mathbf{s}_2$.

## 4 Conclusions

I have explicitly obtained the distribution function of the growth rates of firms under the assumption that their sizes, in two subsequent time steps, followa bivariate Marshall-Olkin distribution. This assumption was introduced in Palestrini (2007) to account for the validity of the Pareto Law in both time steps. However, by extending the analysis there, I have shown that this assumption implies a growth rates distribution which is not only incompatible with a Laplace (symmetric or asymmetric) shape but is not even everywhere continuous. Moreover, the discontinuity at the origin is proportional to the cross correlation coefficient between the size levels $\mathbf{s}_1$ and $\mathbf{s}_2$ in two subsequent time steps. As many empirical studies have shown, this coefficient is always very near (and often statistically equal) to one. Consequently, in real cases, the growth rates distribution implied by Marshall-Olkin distributed logarithmic sizes is almost degenerate. This fact greatly reduces the possibility of this distribution to ever provide an effective statistical description of empirical data. On the theoretical side, it should be noted that the ultimate responsible of the failure of the unifying approach proposed in Palestrini (2007) is the fact that the integrated nature of the growth process of firms, even if largely confirmed by empirical data, was not taken into account. The lesson to be learned is that in order to obtain a reliable phenomenological description of the growth dynamics of firms, or, to that extent, of any economic process, one has to start from data and carefully investigate their regularities while, at the same time, abstaining himself, as far as possible, from introducing untested theoretical hypothesis.

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