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Abstract

This paper demonstrates how both quantitative and qualitative results of general, analytically tractable asset-pricing model in which heterogeneous agents behave consistently with a constant relative risk aversion assumption can be applied to the particular case of "linear" investment choices. In this way it is shown how the framework developed in Anufriev and Bottazzi (2005) can be used inside the classical setting with demand derived from utility maximization. Consequently, some of the previous contributions of the agent-based literature are generalized.

In the course of the analysis of asymptotic market behavior the main attention is paid to a geometric approach which allows to visualize all possible equilibria by means of a simple one-dimensional curve referred as the Equilibrium Market Line. The case of linear (particularly, mean-variance) investment functions thoroughly analyzed in this paper allows to highlight those features of the asymptotic dynamics which are common to all types of the CRRA-investment behavior and those which are specific for the linear investment functions.

JEL codes: C62, D84, G12.

Keywords: Asset Pricing Model, CRRA Framework, Equilibrium Market Line, Rational Choice, Expected Utility Maximization, Mean-Variance Optimization, Linear Investment Functions.

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1 Introduction

In recent years a number of theoretical models exploring the consequences of heterogeneity of traders for the aggregate price dynamics of a speculative financial market have been developed. Among many examples, let us mention the models of Day and Huang (1990), DeLong, Shleifer, Summers, and Waldmann (1991), Chiarella (1992), Lux (1995), Brock and Hommes (1998) and Chiarella and He (2001). These and other "Heterogeneous Agent Models" (HAMs) have been recently reviewed in Hommes (2006). Inside the "agent-based" literature, HAMs can be seen as an important branch of studies supplementary to the numerous simulation models. Indeed, one of the problems with the simulation approach is that the systematic analysis of such models is made practically impossible by the enormous number of degrees of freedom. It is usually not clear which assumptions are responsible for generated patterns and, as a result, robustness of the models is difficult to investigate. HAMs have appeared as a response to this problem and, consequently, are built in such a way to make analytic investigation possible. It is not surprising, therefore, that heterogeneous agent models usually incorporate only few types of agents which differ in the ways they predict the future price but homogeneous in all other respects, i.e. in functional form of demand, available information, etc.

Even if analytic models have already answered a lot of theoretical questions concerning the consequences of behavioral heterogeneity for the market dynamics, they suffer some important drawbacks. First, most of the contributions are built inside the constant absolute risk aversion (CARA) framework, that is under the assumption that demand is independent of wealth. This leads to simplification in the analysis, because otherwise the wealth of each individual portfolio along the evolution of the economy has to be taken into account. However, this assumption is rather unrealistic if compared with other possible behavioral specification, e.g. with constant relative risk aversion (CRRA), see Levy, Levy, and Solomon (2000) or Campbell and Viceira (2002) for a discussion. Second, the majority of the models consider only few types (or classes) of behavior, thus substantially reducing the realistic level of heterogeneity[1]. Third, the tests for the robustness of the results with respect to the change of simple behavioral assumptions are very difficult to perform inside such models. For example, in order to understand the consequences of the entry of an agent with a new type of behavior in the market, one has to analyze completely new model from scratch. Summarizing, one can say that at this moment HAMs lack a general framework, flexible enough to incorporate different realistic agents' specifications.

An important step in the direction of a general framework has been made in Anufriev, Bottazzi, and Pancotto (2006) and Anufriev and Bottazzi (2005), where some analytic results are obtained for a market populated by an arbitrarily large number of technical traders whose possible demand functions belong to a relatively large set. The only imposed restriction on the individual demand functions is that they have to be proportional to the current wealth. This requirement is consistent with the *constant relative risk aversion* (CRRA) framework. Consequently, the price and agents' wealth are determined at the same time and both price and wealth dynamics are intertwined. To model the agents' behavior, Anufriev and Bottazzi (2005) introduce deterministic *investment functions* which map the past history of returns into the fraction of wealth which is invested into the risky security. These investment functions

[1]For instance, DeLong, Shleifer, Summers, and Waldmann (1991) consider two types of investors, the model of Day and Huang (1990) is populated by three types of traders, while Brock and Hommes (1998) provide a number of examples with two, three and four different types. One recent exception from this rule is the model of Brock, Hommes, and Wagener (2005) where the low-dimensional *Large Type Limit* with the number of types converging to infinity is introduced.

are left unspecified, so that the obtained results are very general.

The purpose of the current paper is to provide an illustration of how this general, analytically tractable agent-based model can be applied to important particular classes of investment behavior. Our main interest is focused on the functions which can be derived from optimization principle and, therefore, can be considered as "rational". According to conventional economic wisdom such optimizing behavior is a characteristic of the majority of the agents in financial markets, and therefore corresponding investment functions deserve a special analysis. We consider the investment functions derived from two types of rational choice procedures, expected utility (EU) maximization and mean-variance utility (MVU) maximization.

It is a well-known problem of the EU maximization framework with CRRA-traders that the resulting demand functions cannot be computed explicitly. In order to overcome this obstacle we will use the "Equilibrium Market Line" (EML), a geometric curve introduced in Anufriev and Bottazzi (2005) which allows to characterize both the location of all possible equilibria and (partially) the conditions of their stability *independent* of the specification of the traders' demands. In this way we obtain some predictions of equilibrium dynamics with EU maximizers even without explicit knowledge about their investment functions.

As opposite to the EU framework, the solution of the MVU optimization problem can be derived explicitly. The resulting demand depends on the agent's expectations about mean and variance of the return for the next period. It turns out that for some large class of these expectations, the investment functions become "linear" in the sense which will be clarified later. Since different types of expectations can still lead to different investment functions, we keep the discussion as general as possible and investigate the dynamics in the market with "linear" investment functions. In particular, we demonstrate that the phenomenon of multiple stable equilibria cannot emerge in such market. This is an important limitation of the "rational" framework with respect to the general case, especially if one believes that optimal behavior should not prevail in the market.

The analysis of the linear investment functions brings us to another goal of this paper. We show that one of the first analytic models developed in CRRA framework, namely the model of Chiarella and He (2001), can be easily understood and generalized, when considered inside the general framework of Anufriev and Bottazzi (2005). As a direct consequence, we can discuss the validity and limits of the "quasi-optimal selection principle" originally formulated by Chiarella and He. Through the re-consideration of the examples analyzed in that paper, we show that the so-called "quasi-optimal selection principle" introduced there was a consequence of a peculiar market ecology. For general behavior this principle does not hold and only local optimal selection principle formulated in Anufriev, Bottazzi, and Pancotto (2006) is valid.

The rest of the paper is organized as follows. In the next Section we give a brief description of the general model of a speculative market in the CRRA framework together with the most important results. In Section 3 we introduce two important special cases of investment behavior: one which is based on expected utility maximization, and another which is derived from mean-variance optimization. While in the former case only qualitative results can be discussed, in the latter case a rigorous analysis is feasible. We start this analysis in Section 4, where the case of a single agent with "linear" investment function is considered in detail. In Section 5 we come back to the mean-variance type of behavior and consider those investment behaviors which were introduced in Chiarella and He (2001). Some final remarks are given in Section 6.

2 Equilibria in a Market with Generic Traders

In this Section we present the general analytic model of a speculative market in which the individual demand functions are proportional to wealth. For the derivation and complete discussion of the results provided below the reader is referred to Anufriev and Bottazzi (2005). We stress that all the results are obtained without a precise specification of the demand functions. It implies, of course, that the results are incomplete. For example, we find that all equilibria belong to a one-dimensional curve, but the precise location of the equilibria still depends on the unknown investment functions. We start this Section with a brief review of the general framework and then present the results of equilibrium and local stability analysis.

2.1 General Setup

We consider a simple pure exchange economy, populated by a fixed number N of traders, where trading activities take place in discrete time. The economy is composed of a riskless asset yielding in each period a constant interest rate $r_f > 0$ and a risky asset paying a random dividend D_t at the beginning of each period t. The riskless asset is considered the numéraire of the economy and its price is fixed to 1. The ex-dividend price P_t of the risky asset is determined at each period through a market-clearing condition, where the outside supply of the asset is constant and normalized to 1.

Let $x_{t,n}$ stand for the fraction of the wealth $W_{t,n}$ which, at time t, agent n ($n \in \{1, \ldots, N\}$) invests in the risky asset. We assume that individual demand is proportional to the current wealth, which means that $x_{t,n}$ is independent of $W_{t,n}$. Thus, we confine ourselves to the CRRA framework. Furthermore, $x_{t,n}$ is also independent of the current price, since investment decision at time t has to be made before fixing the price. The evolution of economy is described by the following system containing the individual wealth dynamics and market-clearing condition:

$$
\begin{aligned}
W_{t,n} &= (1 - x_{t-1,n})\, W_{t-1,n}\, (1 + r_f) + \frac{x_{t-1,n}\, W_{t-1,n}}{P_{t-1}}\, (P_t + D_t) \quad , \\
P_t &= \sum_{n=1}^{N} x_{t,n}\, W_{t,n} \quad .
\end{aligned}
\tag{2.1}
$$

Since price and wealth are determined at the same time, these equations give the evolution of the state variables $W_{t,n}$ and P_t only implicitly. Under suitable conditions this implicit dynamics can be made explicit. Let us, first, introduce $\langle a \rangle_t$ as a notation for the *wealth weighted average* at time t of some agent-specific variable a_n

$$
\langle a \rangle_t = \sum_n a_n\, \varphi_{t,n} \quad , \qquad \text{where} \quad \varphi_{t,n} = \frac{W_{t,n}}{\sum_m W_{t,m}} \quad . \tag{2.2}
$$

In order to eliminate from the dynamics (2.1) an exogenous expansion due to continuous injection of new shares of the riskless asset we can rescale variables as follows

$$
w_{t,n} = \frac{W_{t,n}}{(1 + r_f)^t} \quad , \qquad p_t = \frac{P_t}{(1 + r_f)^t} \quad , \qquad e_t = \frac{D_t}{P_{t-1}\,(1 + r_f)} \quad . \tag{2.3}
$$

Then, it is easy to show that the price growth rate $r_{t+1} = p_{t+1}/p_t - 1$ evolves as

$$
r_{t+1} = \frac{\langle x_{t+1} - x_t \rangle_t + e_{t+1}\, \langle x_t\, x_{t+1} \rangle_t}{\langle x_t\, (1 - x_{t+1}) \rangle_t} \quad , \tag{2.4}
$$

while the evolution of the wealth shares $\varphi_{t,n}$ reads:

$$\varphi_{t+1,n} = \varphi_{t,n} \frac{1 + (r_{t+1} + e_{t+1})\, x_{t,n}}{1 + (r_{t+1} + e_{t+1}) \left\langle x_t \right\rangle_t} \qquad \forall n \in \{1, \dots, N\} \quad . \tag{2.5}$$

Notice that the dynamics in (2.4) and (2.5) do not depend on the price level directly, but, instead, are defined in terms of price return and dividend yield. In compliance with intuition, in the CRRA framework the equilibria can be identified as states of steady expansion (or contraction) of the economy.

In order to close the system defined by (2.4) and (2.5) one has to provide the stochastic (due to random dividend payment D_t) yield process $\{e_t\}$ and specify the set of investment shares $\{x_{t,n}\}$. Concerning the former we make the following

Assumption 1. The dividend yields e_t are i.i.d. random variables obtained from a common distribution with positive support and mean value $\bar{e} \in (0, 1)$.

This assumption is common to a number of studies in the literature, see e.g. Chiarella and He (2001, 2002), and also roughly consistent with the real data. We assume that the structure of the yield process is known to everybody. Consequently, the information set available to traders at round t reduces to the sequence of past realized returns $\mathbb{I}_{t-1} = \{r_{t-1}, r_{t-2}, \dots\}$. On the basis of this set agents determine the investment shares, as we describe in the following

Assumption 2. For each agent n there exists a finite memory time span L (which, without loss of generality, can be assumed to be the same for all the agents), and differentiable *investment function* f_n which maps the present information set consisting of the past L available returns into an investment share of the agent:

$$x_{t,n} = f_n(r_{t-1}, \dots, r_{t-L}) \quad . \tag{2.6}$$

The function f_n on the right-hand side of (2.6) gives a complete description of the investment decision of agent n. The knowledge about the fundamental dividend process is not inserted into the information set but embedded in the function f_n itself. Investment decisions of each agent $x_{t,n}$ evolve taking into consideration past market performance.

To summarize, we have considered the evolution of a speculative market where demand is proportional to agent's wealth. The proportionality coefficient, the investment share, is agent-specific and modeled in a general way by means of the investment function. The corresponding dynamics is derived in terms of price returns and agents' wealth shares. It is important to keep in mind that all analysis below will be performed in terms of the rescaled variables (2.3). The return of the *unscaled* price $R_t = P_t/P_{t-1} - 1$ is linked with the return r_t of the rescaled price through the relation $1 + R_t = (1 + r_t)(1 + r_f)$. Thus, zero rescaled price return $r_t = 0$ corresponds to the risk free interest rate in terms of unscaled price return $R_t = r_f$.

2.2 Location of Equilibria

Anufriev and Bottazzi (2005) study the asymptotic properties of the dynamics introduced above for arbitrary investment functions f_n. They substitute the realization of the yield process by its mean value $\bar{e}$ and consider the *deterministic* skeleton of a multi-dimensional system composed by (2.4), (2.5) and (2.6). Let us denote the fixed point of the skeleton as $\boldsymbol{x}^*$. It is composed of the (rescaled) price return r^*, the equilibrium investment shares of all the agents x_n^*, and the relative wealth shares of the agents φ_n^*.

Before presentation of the formal results we introduce three definitions. First of all, fixed point of the system should generate *positive* prices. It motivates the following

Definition 2.1. The equilibrium $\boldsymbol{x}^*$ with positive initial price p_0 and return $r^* > -1$ is called *feasible*. Otherwise, $\boldsymbol{x}^*$ is an *unfeasible* equilibrium.

Second, we need the deterministic version of the concepts of survival and dominance used in DeLong, Shleifer, Summers, and Waldmann (1991):

Definition 2.2. Agent n is said to *"survive"* in $\boldsymbol{x}^*$ if his wealth share is strictly positive, $\varphi_n^* > 0$. Agent n is said to *"dominate"* the economy, if he is the only survivor, so that $\varphi_n^* = 1$.

Finally, let us introduce the special geometric locus which will play the main role for the characterization of the equilibria.

Definition 2.3. The *Equilibrium Market Line* (EML) is the function $l(r)$ defined as

$$l(r) = \frac{r}{\bar{e} + r} \quad . \tag{2.7}$$

The following statement characterizes all possible equilibria of the underlying system

Proposition 2.1. *Let $\boldsymbol{x}^*$ be a fixed point of the deterministic skeleton of the system defined by* (2.4), (2.5) *and* (2.6). *Then*

$$x_n^* = f_n(r^*, \ldots, r^*) \qquad \forall n \in \{1, \ldots, N\} \quad , \tag{2.8}$$

and the following three mutually exclusive cases are possible:

(i) ***Survival of a single agent.*** *In $\boldsymbol{x}^*$ only one agent survives and, therefore, dominates the economy. Without loss of generality we can assume this agent to be agent* 1 *so that $\varphi_1^* = 1$, while all other equilibrium wealth shares are zero. The equilibrium return r^* satisfies the following equation*

$$l(r^*) = f_1\big(r^*, \ldots, r^*\big) \quad . \tag{2.9}$$

(ii) ***Survival of many agents.*** *In $\boldsymbol{x}^*$ more than one agent survives. Without loss of generality we can assume that the survivors are the first k agents (with $k > 1$) so that the equilibrium wealth shares satisfy*

$$\begin{cases} \varphi_n^* \in (0,1) & \text{if } n \leq k \,, \\ \varphi_n^* = 0 & \text{if } n > k \end{cases} \quad , \qquad \sum_{n=1}^{k} \varphi_n^* = 1 \quad . \tag{2.10}$$

The equilibrium return r^* satisfies the following k equations

$$l(r^*) = f_n\big(r^*, \ldots, r^*\big) \qquad \forall n \in \{1, \ldots, k\} \quad , \tag{2.11}$$

so that the first k agents possess, at equilibrium, the same investment share $x_{1\diamond k}^* = l(r^*)$.

(iii) ***"No risk premium" with many survivors.*** *In $\boldsymbol{x}^*$ the investment shares and wealth shares of the agents satisfy*

$$\sum_{n=1}^{N} x_n^* \, \varphi_n^* = 0 \qquad \text{and} \qquad \sum_{n=1}^{N} \varphi_n^* = 1 \quad , \tag{2.12}$$

while equilibrium return $r^* = -\bar{e}$.

This Proposition shows that depending on the investment functions present in the market, three types of equilibria are possible. At the aggregate level, the main feature of the last type is that the investment and wealth shares of the agents are balanced in such a way that the capital gain and the dividend yield of the risky asset offset each other. Consequently, we call such equilibria "no risk premium", since the risky asset has the same return as the riskless one. In other types of equilibria the returns of the two assets are different. The EML allows to identify all such equilibria, and in this respect the first and second equilibria types are similar[2]. However, they are different, at least, in two aspects: in geometric nature and in generality.

From a geometric point of view, while the equilibrium $\boldsymbol{x}^*$ with single survivor identifies a precise value for each component of $\boldsymbol{x}^*$, when many agents survive there is a residual degree of freedom in the definition of the equilibrium, since the only requirement on the equilibrium wealth shares of the survivors is the fulfillment of the second equality in (2.10). Therefore, if one equilibrium with k survivors exists, then there exist a $k-1$-simplex of equilibria with k survivors. In other words, Proposition 2.1*(ii)* does not define a single equilibrium point, but an infinite set of equilibria. Another difference between the first two types of equilibria is due to the fact that in the second type all the investment shares $x_1^*, \ldots, x_k^*$ must *at the same time* be equal to a single value $x_{1\diamond k}^*$, while in the first case no requirements are imposed. Thus, the equilibrium with $k > 1$ survivors exists only in the particular case in which k investment functions $f_1, \ldots, f_k$ satisfy this restriction. Thus, many survivors equilibria are non-generic.

Instead, "no risk premium" equilibria are generic equilibria with many survivors. It is easy to check that the only requirement for the existence of such equilibrium is the co-existence of two agents in the market, one with positive and one with negative investment share. Consequently, "no risk premium" equilibria do not exist in the market with a single agent.

The geometric interpretation of the market equilibria can be presented on a single two-dimensional plot with the aid of the EML. The key reason for the existence of such a simple illustration is that the agent's memory span L is irrelevant for the question of the existence and location of equilibria: only the restriction of the investment function f on the plane $r_{t-1} = \cdots = r_{t-L}$ is relevant. As an example, let us consider the left panel in Fig. 1. Two functions shown there as the thick curves constitute investment functions (or, more precise, "symmetrizations" of some investment functions), while the hyperbolic curve shown as the thin line represents the EML defined in (2.7). Notice that the EML consists of two branches separated by a vertical asymptote at $-\bar{e}$. This asymptote corresponds to the "no-risk-premium" equilibria described in Proposition 2.1*(iii)*. In this example there exist one such equilibrium which can be represented by the two points A_1 and A_2 showing the corresponding investment shares of the agents. The equilibrium wealth shares can be derived from (2.12).

According to Proposition 2.1 all other possible equilibria can be found as the intersections of the investment functions with the EML (cf. (2.9) and (2.11)). Moreover, since these two functions do not possess common intersections with the EML, the equilibria of the second type, identified in item *(ii)*, are impossible in this example. There exist, however, four equilibria with a single survivor. In two of them (S_1 and U_1) the first agent, with non-linear investment function, survives, so that $\varphi_1^* = 1$. In the other two equilibria (S_2 and U_2) the second agent, with linear investment function, survives, and $\varphi_1^* = 0$. In each equilibrium, the intersection of the investment function of the surviving agent with the EML gives both the equilibrium return and the equilibrium investment share of the survivor. The equilibrium investment share of the non-surviving agent can be found, in accordance with (2.8), as the intersection of his own investment function with the vertical line passing through the equilibrium return.

[2]Moreover, Proposition 2.1*(i)* can be considered as a particular case of Proposition 2.1*(ii)*.

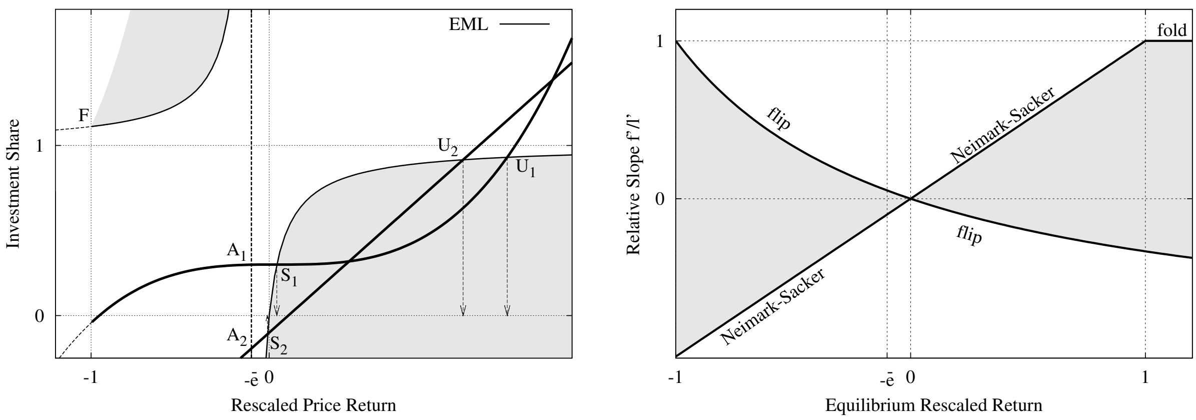

Figure 1: Equilibria and their stability for the many agent system. **Left panel**: Generic situation with 2 agents operating in the market. Four single survivor equilibria S_1, S_2, U_1 and U_2 are defined as the intersections of the "symmetrizations" of the investment functions (thick lines) with the EML (thin line). The only "no risk premium" equilibrium is represented by points A_1 and A_2. Region where investment shares of non-surviving agents satisfy to the stability conditions (2.15) is shown in gray. **Right panel:** Equilibrium stability region (gray) and the bifurcation types for the single agent case with $L = 1$ in coordinates r^* and $f'(r^*)/l'(r^*)$.

In order to confine the analysis only to feasible equilibria, one has to exclude from the EML those points where the equilibrium return $r^* \leq -1$. Corresponding part of the EML is separated by point F in the left panel of Fig. 1 and shown by the dashed line. We remind that the analysis is performed with respect to the rescaled variables as defined in (2.3). Thus, negative return r^* corresponds to the return less than r_f in terms of the unscaled variables.

2.3 Stability Analysis of Equilibria

This Section completes the presentation of the general results with the stability conditions of the equilibria of the system defined in Section 2.1. We present these conditions for the non-generic equilibria with many survivors found in Proposition 2.1*(ii)*. As a particular case, we also find the stability conditions for the generic case of one single survivor described in Proposition 2.1*(i)*. Finally, we discuss some general implications of the results. The proof of the proposition, more detailed discussion, and also the stability conditions for "no risk premium" equilibria can be found in Anufriev and Bottazzi (2005).

The stability conditions are derived from the analysis of the roots of the characteristic polynomial associated with the Jacobian of the corresponding system computed at equilibrium. The characteristic polynomial, in general, depends on the behavior of the individual investment function f in an infinitesimal neighborhood of the equilibrium $\boldsymbol{x}^*$. This dependence can be summarized with the help of the following

Definition 2.4. The *stability polynomial* $P(\mu)$ of the investment function f in $\boldsymbol{x}^*$ is

$$P_f(\mu) = \frac{\partial f}{\partial r_{t-1}}\,\mu^{L-1} + \frac{\partial f}{\partial r_{t-2}}\,\mu^{L-2} + \cdots + \frac{\partial f}{\partial r_{t-L+1}}\,\mu + \frac{\partial f}{\partial r_{t-L}} \quad , \tag{2.13}$$

where all the derivatives of f are computed in point $(r^*, \ldots, r^*)$.

Using the previous definition, the stability conditions can be formulated in terms of the equilibrium return r^*, and of the slope of the EML in equilibrium, $l'(r^*) = \bar{e}/(\bar{e}+r^*)^2$.

Proposition 2.2. *Let* $\boldsymbol{x}^*$ *be a fixed point of the deterministic skeleton of system* (2.4)*,* (2.5) *and* (2.6)*, found in Proposition 2.1(ii), where the first* k *agents survive, so that* (2.10) *holds.*

Let $P_{f_n}(\mu)$ *be the stability polynomial of investment function* f_n*. The equilibrium* $\boldsymbol{x}^*$ *is (locally) stable if the two following conditions are met:*

1) *all the roots of polynomial*

$$Q_{1\diamond k}(\mu) = \mu^{L+1} - \frac{(1+r^*)\,\mu - 1}{r^*\,l'(r^*)} \sum_{n=1}^{k} \varphi_n^*\, P_{f_n}(\mu) \quad , \tag{2.14}$$

are inside the unit circle.

2) *the equilibrium investment shares of the non-surviving agents satisfy the relations*

$$-2 - r^* < x_n^*\,(r^* + \bar{e}) < r^* \quad , \qquad \textit{for} \qquad k < n \leq N \; . \tag{2.15}$$

In particular, if $k = 1$ *and equilibrium* $\boldsymbol{x}^*$ *is described by item (i) of Proposition 2.1, then these two conditions are sufficient for (local)* asymptotic *stability.*

If $k > 1$ *and the equilibrium* $\boldsymbol{x}^*$ *is described by item (ii) of Proposition 2.1, then the fixed point* $\boldsymbol{x}^*$ *is never hyperbolic, and, consequently, is never (locally) asymptotically stable. Its non-hyperbolic submanifold is the* $k-1$*-simplex defined by the second part of* (2.10).

To understand this result notice that condition ***1)*** is independent of the behavior of non-survivors. Thus, it says that the equilibrium should be "self-consistent", i.e. it should remain stable even if any non-surviving agent would be removed from the economy. This is however not enough. A further requirement comes from the inequalities in (2.15). In particular, in those equilibria where $r^* > -\bar{e}$ the surviving agents must be the most aggressive, i.e. invest the highest wealth share among all agents in the risky asset. In those equilibria where $r^* < -\bar{e}$ the survivors have to be the least aggressive investors.

The stability of the single survivor equilibrium is the special case of the last Proposition. In this case the polynomial (2.14) can be simplified and reads:

$$Q_1(\mu) = \mu^{L+1} - \frac{(1+r^*)\,\mu - 1}{r^*\,l'(r^*)}\, P_{f_1}(\mu) \quad . \tag{2.16}$$

If this agent with investment function f_1 is alone in the market, the condition ***2)*** disappears and the only requirement for local *asymptotical* stability is the fulfillment of condition ***1)*** with polynomial (2.16). In the market with many agents, item ***2)*** plays a role and condition (2.15) is necessary for stability.

In the left panel of Fig. 1 we report in gray those regions where condition (2.15) is satisfied. Let us assume that S_1 and S_2 are stable equilibria when the first and second agents, respectively, are present alone in the market (i.e. that condition ***1)*** in Proposition 2.2 holds). Then, S_1 is the only stable equilibrium of the system with two agents. Notice, indeed, that in the abscissa of S_1, i.e. for the equilibrium return, the linear investment function of the non-surviving agent passes below the investment function of the surviving agent and belongs to the gray area. On the contrary, in the abscissa of S_2, the investment function of the non-surviving agent has greater value and does not belong to the gray area. Consequently, this equilibrium is unstable.

The stability conditions for equilibria with many survivors have a similar interpretation. The only difference is due to the weighted structure in the polynomial $Q_{1\diamond k}(\mu)$. The non hyperbolic nature of such equilibria is a direct consequence of their non-unique specification in Proposition 2.1*(ii)*. The motion of the system along the $k-1$ dimensional subspace consisting of the continuum of equilibria leaves the aggregate properties of the system unaltered, however.

Case of $L = 1$. The analysis of the roots of $Q(\mu)$ can be used to reveal the role of the different parameters in stabilizing or destabilizing a given equilibrium. For illustrative purposes, let us consider the simplest case[3] in which $L = 1$. The expression in (2.16) reduces to a polynomial of second degree and one can easily derive the following inequalities sufficient to satisfy the conditions ***1)*** of Proposition 2.2:

$$\frac{f'(r^*)}{l'(r^*)}\,\frac{1}{r^*} < 1\,, \qquad \frac{f'(r^*)}{l'(r^*)} < 1 \qquad \text{and} \qquad \frac{f'(r^*)}{l'(r^*)}\,\frac{2+r^*}{r^*} > -1 \quad . \tag{2.17}$$

The region where these three inequalities are satisfied is shown in the right panel of Fig. 1 in coordinates r^* and $f'(r^*)/l'(r^*)$. The second coordinate is the relative slope of the investment function at equilibrium with respect to the slope of the EML. If the slope of f at the equilibrium increases, the system tends to lose its stability. In particular, in the stable equilibrium the slope of investment function is smaller than the slope of the EML. Let us consider, for example, the left panel of Fig. 1 and suppose for the moment that these are the functions of the agents with memory span equal to 1. One can immediately see that equilibria U_1 and U_2 are unstable, due to the violation of the second inequality in (2.17). On the contrary, the slope of the nonlinear investment function in S_1 is very small, so that, presumably, this equilibrium is stable.

Selection in the Equilibrium. Let us briefly discuss another important implication of Proposition 2.2. We consider a stable many agents equilibrium with price return r^*. According to the results of the stability analysis, the wealth return of all survivors is equal to r^*, so that r^* is also the asymptotic growth rate of the total wealth. At the same time, the wealth growth rates of the non surviving agents are lower than r^*. Then, if they were surviving, and consequently were affecting the dynamics of the total wealth, the whole economy would grow at a lower rate. To put the same statement in negative terms, the economy will never end up in an equilibrium where its growth rate is lower than it would be if the survivor(s) were substituted by some other agent(s). One could see in this result an *optimal selection principle* since it suggests that the market endogenously selects the best aggregate outcome.

This result is in line with the intuitive idea that in a model with endogenous wealth dynamics, the agent who invests more in the growing asset increases his influence and, eventually, dominates those traders who invest less. Our analysis confirms in part this intuition, but also highlights two important limitations. First, the optimal selection principle does not apply to the whole set of equilibria, but only to the subset formed by the equilibria associated with stable fixed points in the single agent case. For instance, the market shown in the left panel of Fig. 1 will never end up in U_1 or U_2, even if these are the equilibria with the highest returns. Second, the possibility of having *multiple stable equilibria*, even with one single trader, implies that the optimal selection principle has only a local character: the economy does not necessarily converge to the stable equilibrium with the highest possible return.

[3]One of possible interpretations of this example is that an agent has a *naïve* forecast of the future return and uses his investment function to transform this forecast to the investment share. The more general case, when agents use the exponential weighted moving averages to estimate the mean and variance of the next period return is discussed in Anufriev, Bottazzi, and Pancotto (2006).

3 Investment Functions of Optimizing Traders

All the results presented in the previous Section are stated in terms of general investment functions. Here we provide some examples of the agents' behavior which can be modeled by means of such functions. These examples originate from a classical, utility-maximization choice theory. Therefore, all investment functions which are derived in this Section can be referred as "rational". We stress that this agents' rationality has nothing to do with rationality in expectations[4]. Using terminology of Herbert Simon (see e.g. Simon (1976)), the traders which we have modeled in Assumption 2 were *procedurally* rational, while now we will confine our attention only on the *substantively* rational traders.

3.1 Expected Utility Maximization

3.1.1 Expected Utility of the Wealth

It is well known that expected utility (EU) maximization with *power* utility function of wealth

$$U(W;\gamma_n) = \frac{W^{1-\gamma_n}-1}{1-\gamma_n} \quad , \qquad \gamma_n > 0 \quad , \tag{3.1}$$

results in the demand function proportional to wealth. The equivalent optimization problem in terms of the rescaled variables reads

$$\max_{x_t} \mathrm{E}[U(w_{t+1};\gamma)] \qquad \text{s.t.} \quad w_{t+1} = w_t\left(1 + x_t\left(r_{t+1}+e_{t+1}\right)\right) \quad , \tag{3.2}$$

and it is straight-forward to see that solution x_t^* of this problem is independent of the agent's wealth. This solution will depend on the risk aversion coefficient γ_n and also on the agent's expectation about distribution of future total return

$$y_{t+1} = r_{t+1} + e_{t+1} \quad .$$

The agent's expectation should be based either upon the commonly available distribution of the dividend yield, or upon the previous return history, or both. Thus, independent of how an agent forms his expectations, a "rational" investment function can be defined and the analysis of the previous Section can be applied.

Unfortunately, the explicit functional shape of the solution x^* of (3.2) with power utility (3.1) cannot be derived for all reasonable (e.g. log-normal) continuous distributions. Consequently, the investment functions for EU maximization are not defined explicitly. The most common way to resolve this problem is to analyze some approximation of the solution. We discuss this approach below, and then demonstrate that our general framework allows to get some conclusions without relying on the approximation.

Approximation of the Solution. Since an analytic derivation of the investment function resulting from the EU maximization with power utility is impossible, different approximations of the solution can be considered. For instance, Chiarella and He (2001) use the continuous-time approximation and derive[5] the following approximated solution of (3.2):

$$x_t = \frac{1}{\gamma\,(1+r_f)}\,\frac{\mathrm{E}_{t-1}\left[r_{t+1}+e_{t+1}\right]}{\mathrm{V}_{t-1}\left[r_{t+1}+e_{t+1}\right]} \quad , \tag{3.3}$$

[4]Indeed among our "rational" agents some will be fundamentalists, and some will be chartists.

[5]The corresponding derivation in terms of unscaled price return can be found in Appendix A.1 of Chiarella and He (2001). Applying definitions (2.3) of the rescaled variables we get the result reproduced here.

where $\mathrm{E}_{t-1}[r_{t+1}+e_{t+1}]$ and $\mathrm{V}_{t-1}[r_{t+1}+e_{t+1}]$ stand for the agent's expectations about total return and its variance, respectively. The subscript $t-1$ stresses the fact that the expectations are based on the information set $\mathbb{I}_{t-1}$ available before period t. Approximation (3.3) is derived under the assumption of normal distribution of the return, whose first two moments should be substituted into the above expression.

Many other approximations are also possible. It leads us to the problem of precision of any particular approximation and, therefore, to the reliability of such an approach. Usually, no estimation of the error incurred in the approximation is provided. However, even if the approximation (3.3) should be precise in the limit when the time unit converges to zero, the error of the approximation for actual time scale can be large. Incidentally notice that Campbell and Viceira (2002) derive another continuous-time approximation which differs from (3.3) on the constant term[6]. Furthermore, it may happen that the additional specific assumptions imposed on the return distribution (necessary to derive the approximation) are in contradiction with the realized dynamics. In this case the approximation is not justified anymore.

Properties of the Actual Solution. The framework reviewed in Section 2 allows to overcome the abovementioned problems with the approximate solutions. The EU maximization problem (3.2) with power utility (3.1) leads to an investment function which depends on the risk aversion coefficient γ_n and on the agent's belief about distribution of future return y_{t+1}. Let us denote this perceived distribution as $g(y)$, the expected value of the total return as $\bar{y}$, and the corresponding investment function as $f^{EP}\big(\gamma, g(y)\big)$. Then the following applies

Proposition 3.1. *Let f^{EP}_γ stand for the partial derivative of the investment function f^{EP} with respect to the risk aversion coefficient γ. Then it is:*

$$\textit{If} \quad \bar{y} \gtreqless 0\,, \quad \textit{then} \quad f^{EP} \gtreqless 0 \quad \textit{and} \quad f^{EP}_\gamma \lesseqgtr 0 \quad .$$

Proof. See appendix A. □

This Proposition together with the general results of Section 2 allows us to discuss some of the equilibrium properties of the market with EU maximizers even without complete knowledge of their investment functions.

Let us consider, for example, the population of EU maximizers with homogeneous expectations about the next period return and assume that these expectations do not depend on past returns. This assumption implies that the investment functions are horizontal. We call such investors *fundamentalists*, since their demand is unaffected by the past market history. Notice that condition ***1)*** of Proposition 2.2 is always satisfied for fundamentalists.

We assume that agents expect a positive *total* rescaled return, i.e. $\bar{y} > 0$. Combining the results of Proposition 3.1 with general analysis, one goes to the conclusion that in such market three different scenarios are possible. First, if all the agents are moderately risk-averse so that their investment shares are less than 1, then only the agents with the smallest risk aversion survive in the stable equilibrium S_p belonging to the lower-right branch of the EML. This case is illustrated in the left panel of Fig. 2. Second, if the agents have very small risk-aversion coefficients so that their investment shares are greater than 1, then only those with the highest risk aversion coefficient survive. The equilibrium in this case belongs to the upper-left branch of the EML. Third, if some agents have investment share greater than 1, and some not, then

[6]See formula (2.25) in Campbell and Viceira (2002), which can be written as $x'_t = x_t + 1/(2\gamma)$, where x_t is given by (3.3).

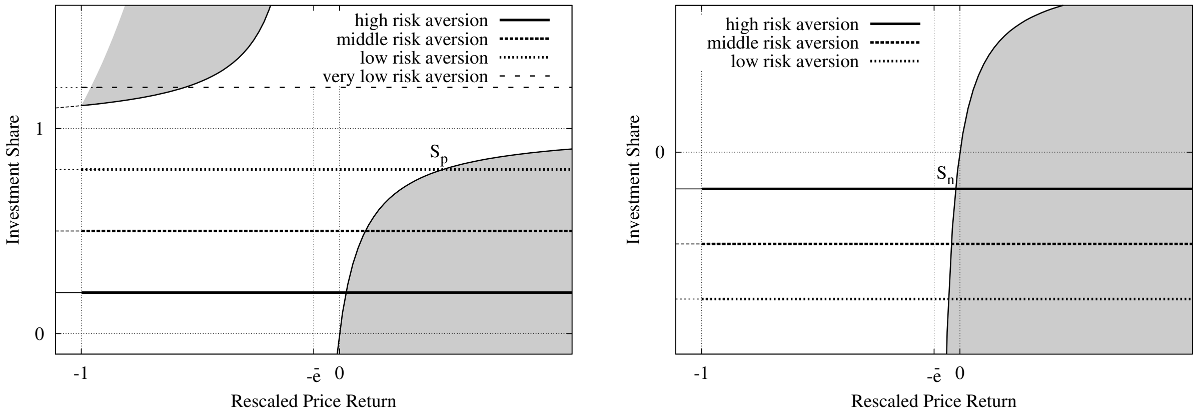

Figure 2: Equilibria in the market with EU maximizing fundamentalists. **Left panel**: If the positive return is expected, either the agents with the smallest risk aversion survive in the only stable equilibrium S_p, or stable equilibrium does not exist. **Right panel:** If the negative return is expected, the agents with the highest risk aversion survive in the only stable equilibrium S_n.

there is no stable equilibrium in the market. Notice, however, that assumption of positive $\bar{y}$ implies that $\mathrm{E}_{t-1}[r_{t+1}] > -\bar{e}$ and, therefore, only the first scenario is consistent with the *sign* of the agents' expectations.

Analogously, if homogeneous fundamentalists believe that $\bar{y} < 0$, the agents with the highest risk aversion will survive in the stable equilibrium S_n as we show in the right panel of Fig. 2. However, in this equilibrium $r^* > -\bar{e}$, which is inconsistent with the expectations. Finally, the "no risk premium" equilibria can emerge in such a market if and only if some of the agents expect positive total return and some of the agents expect the negative return. Since the actual return is zero in "no risk premium" equilibrium, all the investors would make systematic mistakes in the simple forecast of the average return.

Proposition 3.1 can also be applied to the cases with more sophisticated EU maximizers, even if such application can be more involved due to the complexity of the stability conditions. In any case, Proposition 3.1 says that, *independently* of the agent's perceived distribution of the next period return, an increase in the risk aversion will result in a downward movement of those parts of the investment functions which lie above the horizontal axes and in an upward movement of those parts which are below the axes.

3.1.2 Expected Utility of the Return

Our general framework can also be applied to the setting in which agents solve the EU maximization problem with respect to the *return* of their wealth with *exponential* utility function:

$$U(\rho_{t+1,n};\beta_n) = -e^{-\beta_n\,\rho_{t+1,n}} = -e^{-\beta_n\,x_{t,n}\,(r_{t+1}+e_{t+1})} \quad ,$$

where $\rho_{t+1,n} = w_{t+1,n}/w_{t,n} - 1$ denotes the return of the agent's (rescaled) wealth. Straightforward computations show that if perception of the agents is such that the total return $y_{t+1} = r_{t+1} + e_{t+1}$ is normally distributed with expected value $\mathrm{E}_{t-1}[r_{t+1} + e_{t+1}]$ and variance $\mathrm{V}_{t-1}[r_{t+1} + e_{t+1}]$, then the solution of the corresponding EU maximization reads:

$$x_t = \frac{1}{2\,\beta}\,\frac{\mathrm{E}_{t-1}\left[r_{t+1} + e_{t+1}\right]}{\mathrm{V}_{t-1}\left[r_{t+1} + e_{t+1}\right]} \quad . \tag{3.4}$$

As usually, the substitution of any rule of the expectation formation in the right-hand side of this expression leads to the properly defined investment function.

3.2 Mean-Variance Utility Maximization

Another class of the investment functions in the CRRA framework can be obtained through the solution of the following mean-variance (MV) optimization problem:

$$\max_{x_t} \left\{ \mathrm{E}_{t-1}[w_{t+1}] - \frac{\gamma}{2\,w_t}\,\mathrm{V}_{t-1}[w_{t+1}] \right\} \qquad \text{s.t.} \quad w_{t+1} = w_t \left(1 + x_t \left(r_{t+1} + e_{t+1}\right)\right) \quad , \tag{3.5}$$

where γ is some positive constant. As opposite to the standard MV framework, the coefficient which measures the sensitivity of the agent's utility to the risk decreases with wealth. Simple computation shows that the solution of (3.5) is provided by

$$x_t = \frac{1}{\gamma}\,\frac{\mathrm{E}_{t-1}[r_{t+1} + e_{t+1}]}{\mathrm{V}_{t-1}[r_{t+1} + e_{t+1}]} \quad , \tag{3.6}$$

and does not depend on the current wealth. Notice that the derivation is performed without any specific assumption about the distribution of the price return and dividend yield. When such assumption is made, the beliefs of the agent about expected return and its variance can be plugged into the right-hand side of (3.6). The resulting expression will represent some investment function.

Approximated solution (3.3) of the EU maximization of the wealth with power utility function, exact solution (3.4) of the EU maximization of the return with exponential utility function and, finally, exact solution (3.6) of the MV optimization are identical up to constant factor in the risk aversion coefficient. It is easy to see that all the corresponding investment functions satisfy the same property which we derived in Proposition 3.1 for the exact solution of the EU maximization with power utility of wealth. In particular, when the expected mean of the total return is positive, the investment functions of the agents with smaller risk aversion are above the investment functions of the agents with higher risk aversion.

In order to investigate the properties of these investment functions further, we have to specify the agents' beliefs about the first two moments. Since, the framework of Chiarella and He (2001) is identical to the one outlined in Section 2.1 (in particular, the market structure is the same and Assumption 1 holds), it will be informative and useful for illustrative purposes to work with the same specification of the expectations as in that paper. Chiarella and He work with unscaled variables and consider the following specification of the expectations with respect to the total (unscaled) return $R^T_{t+1} = (P_{t+1} - P_t + D_{t+1})/P_t$:

$$\mathrm{E}_{t-1}[R^T_{t+1}] = r_f + \delta + d\,m_t \quad , \tag{3.7}$$

$$\mathrm{V}_{t-1}[R^T_{t+1}] = \sigma^2 \left(1 + b\,(1 - (1 + v_t)^{-\xi})\right) \quad , \tag{3.8}$$

where m_t and v_t denote the sample estimates of the average return and its variance computed as equally weighted averages of the previous L observations

$$m_t = \frac{1}{L}\sum_{k=1}^{L} R^T_{t-k} \qquad \text{and} \qquad v_t = \frac{1}{L}\sum_{k=1}^{L} (R^T_{t-k} - m_t)^2 \quad . \tag{3.9}$$

Chiarella and He refer the reader to the contribution of Franke and Sethi (1998) for the justification of the choice (3.8) for the variance forecast. However, this choice and, in particular, positive parameters b and ξ are irrelevant for the equilibrium analysis, as we will see below. The specification of the expected conditional return (3.7) is important, however. It is defined as the risk free rate r_f plus the excess return. The latter is composed of a constant component representing a risk premium, $\delta \geq 0$, and a variable component, $d\,m_t$. The parameter d represents the way in which agents react to variations in the history of realized returns and can be used to distinguish between different classes of investors. A trader with $d = 0$ will ignore past realized returns and, consequently, can be thought as a *fundamentalist*[7]. If $d > 0$ the agent can be considered a *trend follower*, if $d < 0$ he can be considered a *contrarian*.

Estimates m_t and v_t depend on the previous L total unscaled returns or, equivalently, on the past L rescaled price returns. Let f^{CH} stand for the investment function obtained after corresponding substitutions of (3.7), (3.8) and (3.9) into the right-hand side of (3.6). This function reads:

$$f^{CH}\big(r_{t-1}, \ldots, r_{t-L}\big) = \frac{1}{\gamma}\,\frac{\tilde{\delta} + \tilde{d}\,m_t}{1 + b\,\big(1 - (1 + v_t)^{-\xi}\big)}\ , \quad \text{with} \quad \tilde{\delta} = \frac{\delta}{\sigma^2}\ , \quad \tilde{d} = \frac{d}{\sigma^2}\ . \qquad (3.10)$$

Investment function f^{CH} represents one of numerous special examples of the optimizing investment behavior compatible with our framework. This example is quite specific, however, because investment function f^{CH} becomes linear for the constant return history. Indeed, when the restriction $r = r_{t-1} = \cdots = r_{t-L}$ is imposed, one gets $m_t = R^T$ and $v_t = 0$. Thus, such "symmetrization" is linear with respect to R^T. Applying the rescaling (2.3), one gets linearity of the "symmetrization" also with respect to r.

In the previous Section it was shown that the long-run outcome in the framework with CRRA trading behaviors crucially depends on the ecology of the traders present in the market. Therefore, the property of the linearity of the investment function under the constant return history can somehow limit the range of possible market dynamics. We will investigate this question in the next Section for the investment functions with linear "symmetrization" and without any additional restriction. This analysis will, first, prepare the ground for the re-investigation of the Chiarella and He model, and, second, provide quite extensive illustration of how our geometric machinery can be applied.

4 Equilibria for the Linear Investment Functions

Let us assume that investment function f is such that its restriction to the subspace defined as $r = r_t = r_{t-1} = \cdots = r_{t-L+1}$, is a *linear* function of r. With some abuse of language we will refer on the investment functions with such "linear" symmetrization as on the "linear investment functions". It will be convenient to use the following parameterization of them:

$$f(r, \ldots, r) = \big(A + 1\big) + B\,\big(r + \bar{e}\big) \quad . \qquad (4.1)$$

Two parameters are involved in the description of this class of investment functions. B stands for the *slope* of the function, while $A + 1$ gives the value of this function in the point $-\bar{e}$. This parameterization is illustrated in the left panel of Fig. 3. Obviously, any investment function whose "symmetrization" is linear can be represented according to (4.1) with some A and B.

[7]This definition is consistent with our own definition of fundamentalists introduced in Section 3.1.1.

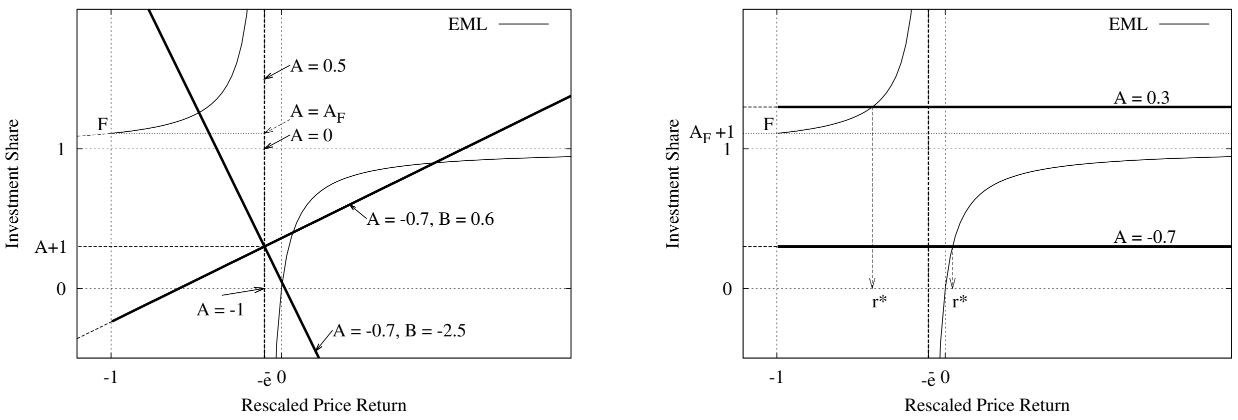

Figure 3: **Left Panel:** Examples of the linear investment functions (4.1) for different parameter pairs (A, B). **Right Panel:** Equilibria for constant investment functions.

In particular, the investment behavior (3.10) considered in Chiarella and He (2001) can be described by (4.1) with coefficients defined as

$$A^{CH} = \frac{\tilde{\delta} + \tilde{d}\,r_f}{\gamma} - 1 \qquad \text{and} \qquad B^{CH} = \frac{\tilde{d}}{\gamma}\,(1 + r_f) \quad . \tag{4.2}$$

4.1 Location of Equilibria for a Single Linear Investment Function

The equilibrium analysis performed in Section 2.2 can be easily applied for the linear investment functions (4.1). We consider here the case when one single agent operates in the market. It is clear from Proposition 2.1 that the properties of all multi-agent equilibria (except "no risk premium") can be easily understood from studying the single agent case.

From the geometric plot of the EML it is clear that depending on the values of A and B there exist at most two equilibria for any linear investment function. Simple computations confirm this inference. One has the following

Proposition 4.1. *Consider equilibria of the market with single survivor possessing the investment function with linear "symmetrization"* (4.1)*. Then the following cases are possible:*

(i) ***Constant function:*** $B = 0$. *For* $A = 0$ *there are no equilibria. If* $A \neq 0$ *there exist one equilibrium with return*

$$r^* = -\frac{\bar{e}}{A} - \bar{e} \quad . \tag{4.3}$$

which is feasible, i.e. it generates positive price, when $A < 0$ *and when* $A > A_F = \frac{\bar{e}}{1-\bar{e}}$.

(ii) ***Non-constant function:*** $B \neq 0$.

Consider $D = A^2 - 4B\bar{e}$. *Then if* $D < 0$*, then there are no equilibria. Otherwise, when* $D \geq 0$*, there are two equilibria (coinciding when* $D = 0$*) with the following returns:*

$$r_1^* = \frac{-A - \sqrt{A^2 - 4B\bar{e}}}{2B} - \bar{e} \qquad , \qquad r_2^* = \frac{-A + \sqrt{A^2 - 4B\bar{e}}}{2B} - \bar{e} \quad . \tag{4.4}$$

The equilibrium is feasible, i.e. it generates positive price, if the return exceeds -1.

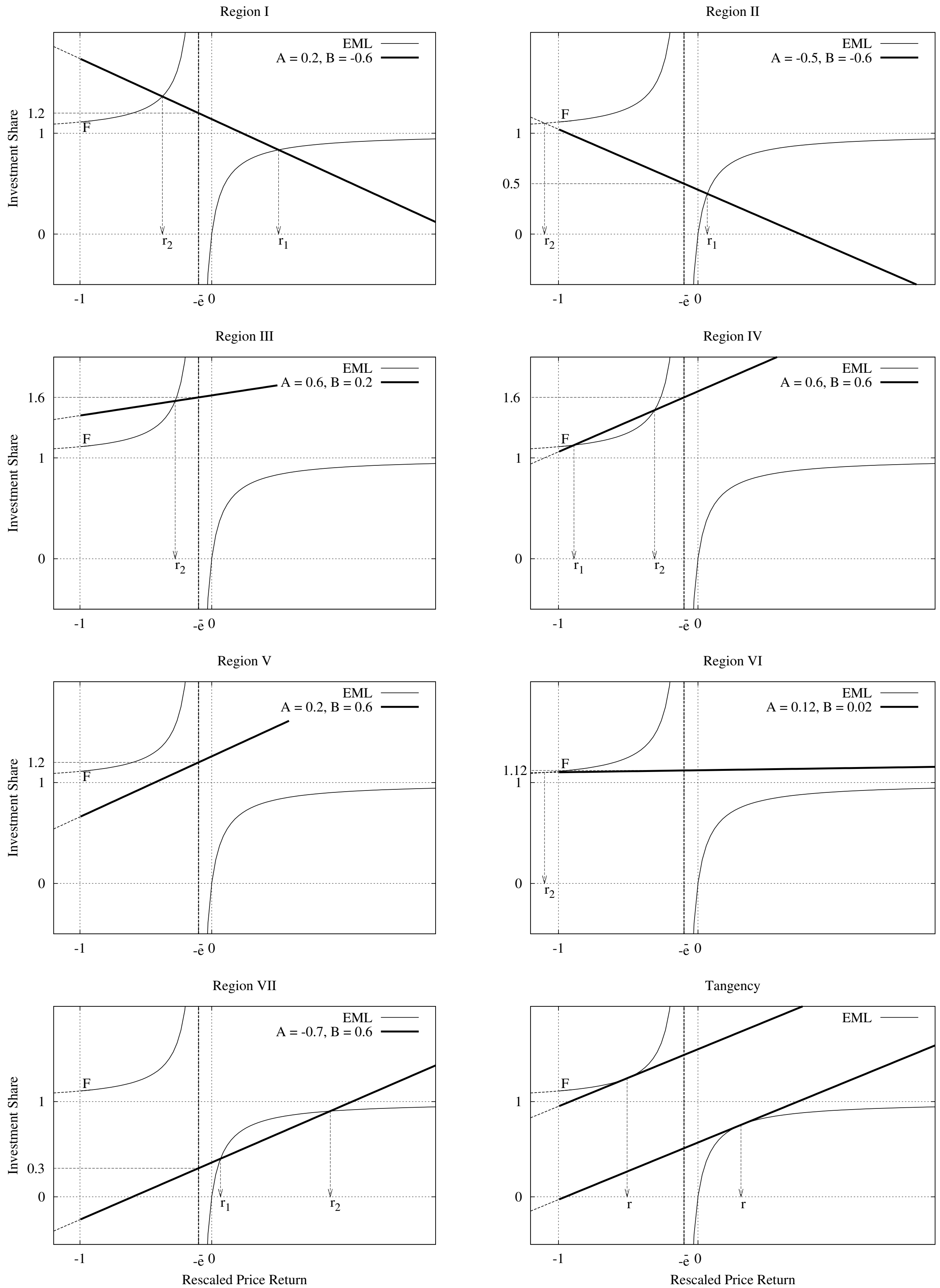

Figure 4: Examples of equilibria with linear investment functions. The titles of the panels correspond to the regions in Fig. 5. See text for explanation.

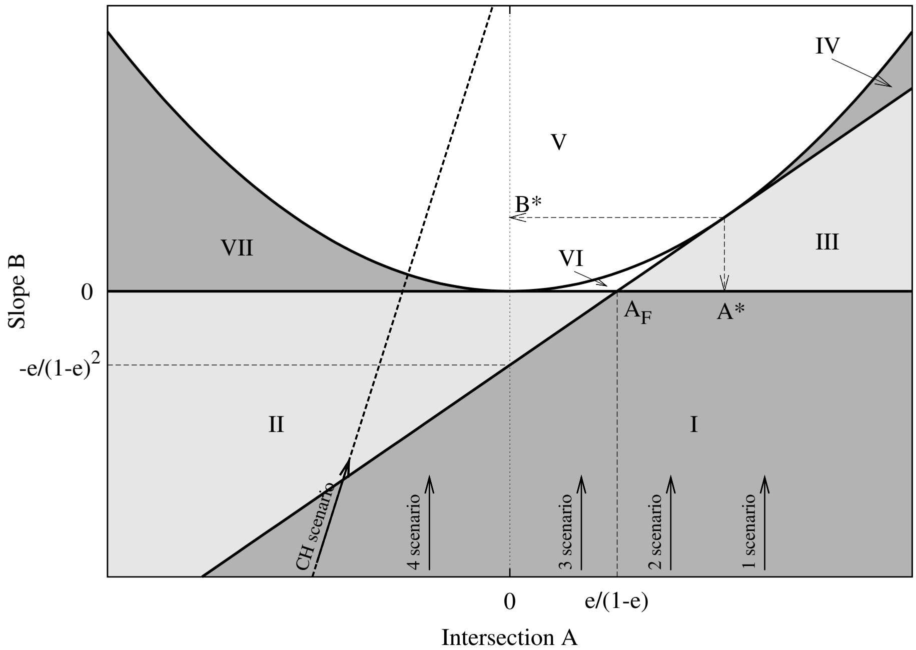

Figure 5: Stratification of the parameter space (A, B) according to the number of feasible equilibria. The dark gray (light gray, white) area represents the parameters for which there are two (one, zero) equilibria. See the text for the explanation of the regions marked by Roman numerals and Fig. 4 for the illustrative example for each of these regions. See the text for the explanation of the "scenarios" denoted by arrows in the down part of the picture and Fig. 6 for the illustrations.

Proof. See appendix B. □

This Proposition provides all possible equilibrium values of the return for different linear investment functions (4.1). When $B = 0$ the agent's investment does not depend on the past information and his function represents the horizontal line as it is shown in the right panel of Fig. 3. If $A > 0$, in addition, the only equilibrium belongs to the upper-left branch of the EML and $r^* < -\bar{e}$. Obviously, this equilibrium is not feasible when $A < A_F$. When $A < 0$ the equilibrium generated by constant investment function belongs to the lower-right branch of the EML, so that $r^* > -\bar{e}$ and it always generates positive prices.

When $B \neq 0$ one can distinguish between two cases. If the investment function is decreasing, so that $B < 0$, it is always the case that $D > 0$ and, therefore, two equilibria exist, as in example in the left panel of Fig. 3. From (4.4) it follows that $r_1^* > -\bar{e} > r_2^*$ in this case. Therefore, the first equilibrium belongs to the upper-left branch of the EML (and may be unfeasible), while the second equilibrium is always feasible and belongs to the lower-right branch of the EML. In the opposite case, when $B > 0$, the investment function increases and can have 0, 1 or 2 equilibria. In the latter situation, which is also illustrated in the left panel of Fig. 3, $r_1^* < r_2^*$ and both equilibria belong to the upper-left (lower-right) branch of the EML when $A > 0$ ($A < 0$).

In Fig. 4 we provide a geometric illustration of all possibilities described in the last Proposition. Two upper panels represent examples of decreasing investment functions. In both cases two equilibria exist, either both feasible (the left panel, first row), or one feasible and one unfeasible (the right panel, first row). The second and third rows of panels provide four examples when both parameter A and slope B are positive. If two equilibria exist, then only the largest of them may be feasible (the left panel, second row). For higher values of B, i.e. steeper investment function, both equilibria become feasible (the right panel, second row). It is also possible that there exist no equilibrium (the left panel, third row), or that there exist no feasible equilibrium (the right panel, third row). Finally, two equilibria in the lower-right branch of the EML coexist when slope is positive but A is negative (the left panel, the last row). With increase of A or B, these two equilibria r_1^* and r_2^* approach each other and, eventually, coincide in the non-generic situations of tangency of the investment function and the EML (the right panel, the last row). With sufficiently high A both equilibria reappear through the tangency again. Such tangency scenario happens when $D = 0$ in Proposition 4.1*(ii)*.

In Fig. 5 we show the stratification of the parameter space (A, B) according to the number of feasible and unfeasible equilibria. For the parameter pairs from the white area there are no feasible equilibria, for those pairs which belong to the light gray area only one feasible equilibrium exist, and, finally, if parameters belong to the dark gray area there exist two different feasible equilibria. Three important loci which are important for more detailed stratification of the parameter space are shown by the thick curves and divide the space on seven different regions marked by the Roman numerals[8]. The first locus is a horizontal straight line corresponding to $B = 0$. In this case the investment function is horizontal and one equilibrium exist. Any change of B leads to the appearance of the second equilibrium which can be unfeasible, though. The curve with parabolic shape contains the points with $A^2 = 4B\bar{e}$, i.e. those parameters for which the equilibrium is unique due to the tangency between the EML and the straight line (4.1). This parabola, therefore, separates the parameters for which there are no equilibria (region V) from those points for which two equilibria exist. We call this locus "tangency curve".

Finally, the third locus is represented by the increasing line $A = \bar{e}/(1 - \bar{e}) - B(\bar{e} - 1)$, corresponding to the parameter pairs for which linear symmetrization (4.1) passes through the point F of the upper-left branch of the EML. With the crossing of this locus, which we call "feasibility curve", one feasible equilibrium is lost. If B is negative, the equilibrium on the upper-left branch of the EML disappears with decrease of A, so that two regions I and II are determined. If $B > 0$ and $A < 0$ then, as we mentioned above, both equilibria (if exist) belong to the lower-right branch of the EML and both are feasible, so that area VII is determined. Finally, if both A and B are positive, let us denote as (A^*, B^*) the parameter pair defining the investment function which passes through F and, at the same time, is tangent to the EML. In region III only the equilibrium with the smallest return r_1^* is feasible. When A decreases there are two possible options: either r_1^* also becomes infeasible or r_2^* becomes feasible. From the EML plot it is easy to see that the first case happens for $B < B^*$, i.e. when in the point F the investment function is flatter than the tangency line. In this case from region III we move to region VI. Respectively, the second case happens when $B > B^*$ and we move from region III to IV.

The EML can be effectively used to study the effects of change of different parameters. For instance, if the value of A is fixed, then increase of B from $-\infty$ to $+\infty$ corresponds to the counter clock wise rotation of the vertical line $r = -\bar{e}$ around the point with the ordinate $A+1$ on the graph in Fig. 3. One can easily sketch the graphs of the roots behavior like we do in Fig. 6. These four graphs can, alternatively, be understood from the stratification diagram in Fig. 5. We fix abscissa A and move up vertically through this figure (see the arrows in the down part of the picture). Four different scenarios simply correspond to the different order of the points of intersection with the "tangency curve", "feasibility curve" and curve $B = 0$.

[8] Each of the first seven panels in Fig. 4 gives the example for the corresponding region in Fig. 5.

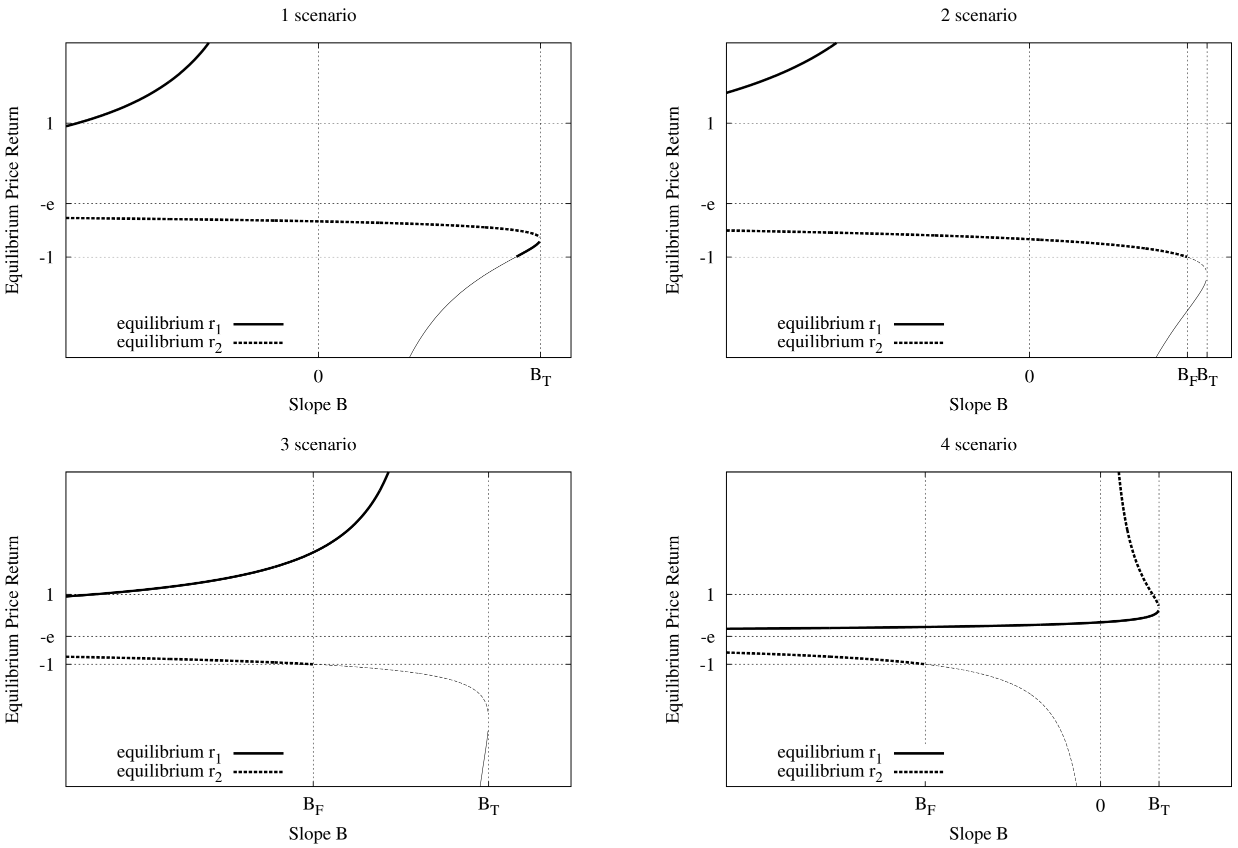

Figure 6: Equilibria r_1^* and r_2^* computed in Proposition 4.1 as functions of the slope B of the linear investment function for four different levels of A. **Upper Panel**: $A > A^*$ (**left**) and $A^* > A > A_F$ (**right**). **Lower Panel**: $A_F > A > 0$ (**left**) and $A < 0$ (**right**). Qualitative behavior can be easily understood from the EML, see text for explanation. These graphs can be also explained from the stratification diagram in Fig. 5 in the following way. For example, in the last scenario with $A < 0$ with increase of B we consequently go through the regions I, II, VII and V on the stratification diagram, first crossing the "feasibility" line in $B = B_F$, then the "fundamental" line in $B = 0$ and finally the "tangency" line in $B = B_T$.

4.2 Stability of Equilibria for $L = 1$ Case

We address in this Section the question of stability of the equilibria for the single agent case. Recall from the general analysis of Section 2.3 that these conditions are also necessary for the stability of equilibria in the multi-agent case with one survivor. Notice that they are also the same in the case with many survivors for the corresponding weighted average of the survivors' investment functions (see condition ***1)*** of Proposition 2.2).

Unfortunately, the general conditions cannot be simplified with the use of the only assumption that the investment function possesses a linear symmetrization. One problem here is that such assumption does not provide any information about L partial derivatives of function f in equilibrium, which appear in the stability conditions through the stability polynomial P_f defined in (2.13). Even if this polynomial is simplified somehow, there is another problem to make explicit the requirement for its roots to be inside the unit circle.

The stability conditions can be obtained in explicit form for the case $L = 1$ as we showed above in page 10. Therefore, we strengthen here the assumption about the linear form (4.1) for the “symmetrization” of the investment function and assume that the investment function *itself* is linear:

$$f(r) = \big(A+1\big) + B\,\big(r+\bar{e}\big) \quad . \tag{4.5}$$

Linear investment choice based on a naïve forecast of the future return represents one possible interpretation of such behavior.

The stratification in Fig. 5 showed the number of different equilibria, in general, and feasible equilibria, in particular. The question about their stability leads to the following conditions:

$$\frac{B - l'(r^*)\,r^*}{r^*} < 0\ , \qquad B - l'(r^*) < 0 \qquad \text{and} \qquad \frac{B(2+r^*) + l'(r^*)r^*}{r^*} > 0\ , \tag{4.6}$$

where r^* stands for the equilibrium return. Corresponding values of the return were computed in Proposition 4.1. In the case when the investment choice is constant and investment function is horizontal, the unique equilibrium has return r^* provided by (4.3). When investment function is not horizontal, two equilibria returns are given by (4.4). Plugging the corresponding values of the returns in (4.6), one can express stability conditions and bifurcation loci through parameters A and B. The resulting expressions are quite cumbersome, so we provide only their geometric illustration.

In Fig. 7 we consider the parametric space (A, B) and produce its stratification in accordance to the validity of the stability conditions for both equilibria found in Proposition 4.1. More precisely, in each point of the space we compute the corresponding equilibrium (if it exists) and check whether each of the three inequalities (4.6) holds. In the gray regions the corresponding equilibrium exists, it is feasible and stable. Otherwise, the parameter couple belongs to the white region. Apart from the “tangency” and “feasibility” curves shown as thick curves, we show in Fig. 7 different *bifurcation loci* as dotted thick lines. They correspond to the points where one of the inequalities (4.6) change its sign. For example, the convex parabola corresponds to those points where the first inequality changes its sign. In these points the system exhibits the Neimark-Sacker bifurcation. Analogously, the concave parabola in the left panel and another concave parabola in the right panel represents points of flip bifurcations, where the third inequality (4.6) changes its sign.

Fig. 7 allows to understand the effects of different parameters on the stability of equilibria in a straight-forward way. One can, for example, repeat the same procedure as we applied to Fig. 5 when we studied four different scenarios, emerging when intersection A is fixed and slope B is changing. For instance, it is immediate to see that in the first three scenarios, where $A > 0$ and which were represented by the first three panels of Fig. 6, equilibrium r_1^* is unstable for any value of B, while equilibrium r_2^* is stable when the absolute value of B is small enough. Furthermore, in the latter case, if B increases, the equilibrium exhibits a flip bifurcation, while when B decreases there is a Neimark-Sacker bifurcation. In the fourth scenario with negative A, equilibrium r_2^* is unstable, while r_1^* is stable for B close to zero.

Fig. 7 suggests that even if two feasible equilibria can coexist for linear investment functions, at least one of them will be unstable. For the case of increasing linear investment functions it is, indeed, obvious from the EML plot. If such function intersects the EML twice as in Fig. 4 for regions IV and VII, then in one of these intersections the slope B is greater than the slope of the EML, and, therefore, the second inequality in (4.6) is violated. Generally we have the following

Proposition 4.2. *There is at most one feasible stable equilibrium in the market with single linear investment function* (4.5).

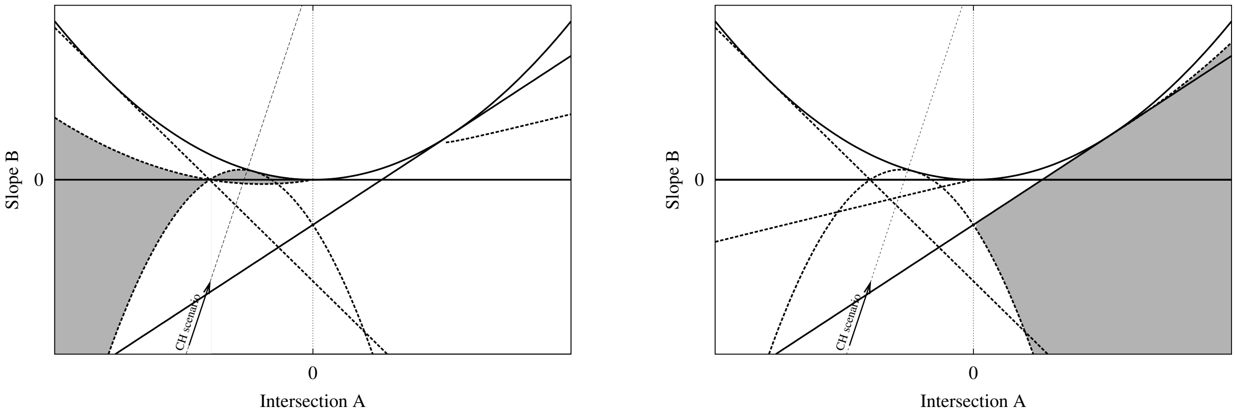

Figure 7: Stratification of the parameter space (A, B) according to the stability of equilibria. **Left panel**: stability of the first root r_1^*. **Right panels**: stability of the second root r_2^*. Corresponding root is stable if parameters belong to the gray area.

Proof. See appendix C. □

This Proposition is the main result of this Section. It shows that the restriction of the analysis on the market populated by the agents with linear investment functions (in particular, those who derive their demand through the MV optimization) leads to the impossibility to have the phenomenon of multiple *stable* equilibria in the single agent case. If non-linear investment functions were allowed, many stable equilibria could co-exist as we show geometrically in the left panel of Fig 8.

As a consequence of this limitation, the range of possible market dynamics can be over-simplified if only "linear" behaviors are considered. This is clear in the single agent case. Furthermore, it can also be the case in the market with many agents, since the equilibria and their stability in such market are characterized through the equilibria and stability of the single agent equilibria. For instance, one can be easily convinced[9] that the market with many agents having linear investment functions cannot possess more than one stable equilibrium with $r^* > -\bar{e}$. Another example will be discussed in the next Section.

It is important to stress that Proposition 4.2 does not, in general, hold in the market with many linear investment functions. It can be seen from the situation depicted in the right panel of Fig 8, where the market possesses one stable equilibrium S with one surviving agent and also another "no-risk-premium" equilibrium where two agents survive (cf. Proposition 2.1*(i)* and *(iii)*). Both equilibria are asymptotically stable, since investment functions are horizontal.

5 Mean-Variance Investment Functions

Geometric interpretation of the equilibria and investigation of particular case with linear investment functions developed in the previous Section can now be straight-forwardly applied to the analysis of the market with "rational" behavior derived from the MV optimization and introduced in Section 3. Such analysis has been performed in Chiarella and He (2001),

[9]Consider the lower-right branch of the EML and recall that the equilibrium will be unstable in any of two following cases. First, if there exist more aggressive investor in this equilibrium. Second, if the increasing investment function intersects the EML from below.

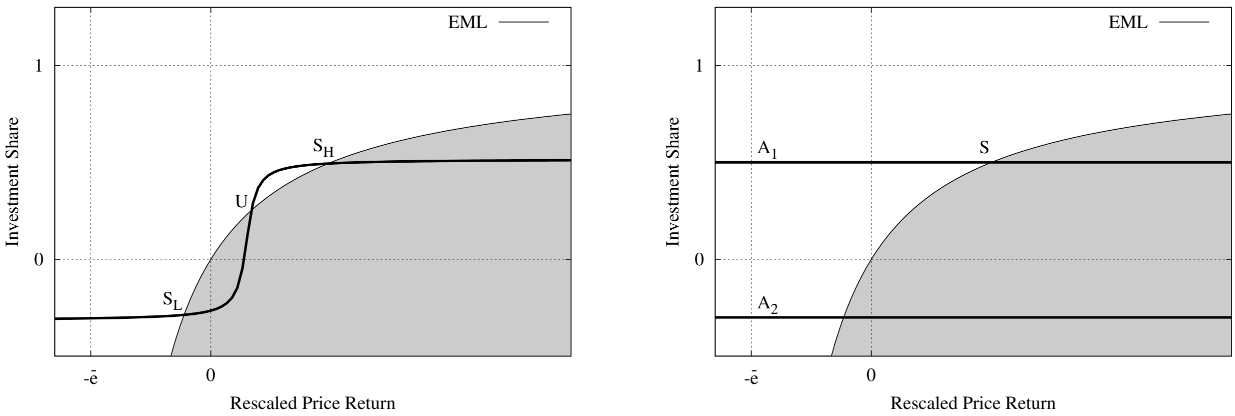

Figure 8: **Left panel:** In the market with single agent multiple stable equilibria can be generated by non-linear investment function. S_H and S_L are stable while U is unstable. **Right panel:** In the market with two agents multiple stable equilibria can be generated with linear investment functions. S is stable with one survivor. (A_1, A_2) generates unique stable equilibrium where both agents survive.

henceforth CH. We, first, review the results of this model, and then explain them using the geometric tool of the EML.

5.1 Model of Chiarella and He: Review of the Results

CH consider agents with investment function f^{CH} given in (3.10). All these agents have the same risk aversion coefficient $\gamma = 1$, i.e. the same demand functions. Two different cases are analyzed. The first case is the model with homogeneous expectations. In this model the realized demand functions of all agents are identical. They are characterized by the (rescaled) risk premium $\tilde{\delta} \in (0, 1)$. Accordingly with the sign of the extrapolation parameter $\tilde{d}$ the situations of fundamental, trend-following or contrarian behavior as described in Section 3.2 are possible. CH provide complete equilibrium analysis in each of these cases (Proposition 3.1). Stability analysis is performed for the case $\tilde{d} = 0$, when the unique equilibrium is asymptotically stable and for the case when $\tilde{d} \neq 0$ and $L = 1$ when sufficient conditions for stability are derived (Corollary 3.3). For larger "memory span" L the numerical approach is exploited which shows that the stability can be brought to the system through increase of the memory span. The qualitative aspects of the equilibrium and stability analysis of the single-agent case are summarized in Figure 1 of that paper.

After the analysis of the homogeneous expectations case, CH proceed to the market with two investors and consider four different scenarios. In the first scenario there are two fundamentalists with different risk premium. The equilibrium analysis shows that there are two equilibria in such market (Proposition 4.2), however only one of them is stable (Corollary 4.3). It leads to "optimal selection principle" for this scenario, which states that the investor with the higher risk premium will survive.

The second scenario corresponds to the market with one fundamentalist and one contrarian. There exist three steady-states for such market, but (unscaled) price return is positive in only two of them (Proposition 4.4). The fundamentalist dominates the market in one of these two steady-states and contrarian dominates the market in another one. The stability analysis can be performed analytically only for the former steady-state (Corollary 4.5). As a result of numerical analysis of the stability of the latter steady-state, CH conclude that the long-run return dynamics depends on the relative levels of the returns in these two steady-states and follows a similar optimal selection principle. Namely, the steady-state is stable if it generates the highest return.

In the third example of heterogeneous market fundamentalist meets trend-follower. Such market has one equilibrium where fundamentalist survives. It also can have zero, one or two equilibria with surviving trend-follower (Proposition 4.6). Similar to the previous example, the stability conditions for the latter equilibria are obtained through the numerical investigation. It is found that for small extrapolation rates (i.e. for relatively small value of $\tilde{d}$ of the trend-follower) there exist two equilibria where trend-follower survives. The highest return is generated in one of these equilibria which is, however, unstable. Between the two remaining equilibria "the stability switching follows a (quasi-)optimal selection principle", depending where the return is higher.

Finally, in their last example CH consider the market with two chartists. In this case there exist multiple steady states. If traders extrapolate strongly (i.e. in particular they both are trend-followers) none of the steady-states is stable. For weak extrapolators, "the stability of the system follows the (quasi-)optimal selection principle – the steady-state having relatively higher return tends to dominate the market in the long run".

To summarize, Chiarella and He have found *quasi-optimal selection principle* which allows to predict a long-run market dynamics in the case, when there are multiple equilibria. Comparing this principle with optimal selection principle which we formulated in Section 2.3, one see an important difference.

The principle in CH has a *global* character. When the ecology of the traders is fixed, it can be applied to the market, so that unique possible outcome is predicted. Our optimal selection principle has a *local* character, instead. For a given traders' ecology there can be different possibilities of the market long-run behavior, i.e. multiple equilibria. The final outcome depends on the initial conditions and, in the stochastic case, on the yield dynamics, and cannot be predicted *a priori*. However, independently of the realized equilibria, the survivors will be chosen in "optimal" way: to allow the highest possible growth rate of the economy in this point. In some sense, our principle selects among investment functions, while principle in CH chooses among equilibria.

5.2 Model of Chiarella and He: Geometric Approach

Let us show that cumbersome analytic results in Chiarella and He (2001) may become much more clear if one uses the geometric tools. We have already computed the coefficients (4.2) which, in terms of parameterization (4.1), define the investment functions from CH. Now we use these relations to study the impact of different parameters on equilibria and their stability.

We start with the single agent situation. On the stratification diagram of Fig. 5 all CH investment functions with fixed risk premium $\tilde{\delta}$ and risk-free interest rate r_f can be represented as the straight line with positive slope which we label as "CH scenario". The bottom-up movement along this line corresponds to increase in the extrapolation parameter $\tilde{d}$. This parameter reaches the zero value in the point $B = 0$. The results of equilibrium analysis of Proposition 3.1 in CH can now be reproduced straight-forwardly.

Indeed, the dotted line "CH scenario" subsequently intersects regions I, II, VII, V and IV in Fig. 5. When the rate of extrapolation of the contrarian is high (in absolute value), parameters belong to the region I and, therefore, two feasible equilibria coexist. With increase of extrapolation parameter, the "feasibility line" is intersected for some $\tilde{d} = d_F$. At this point

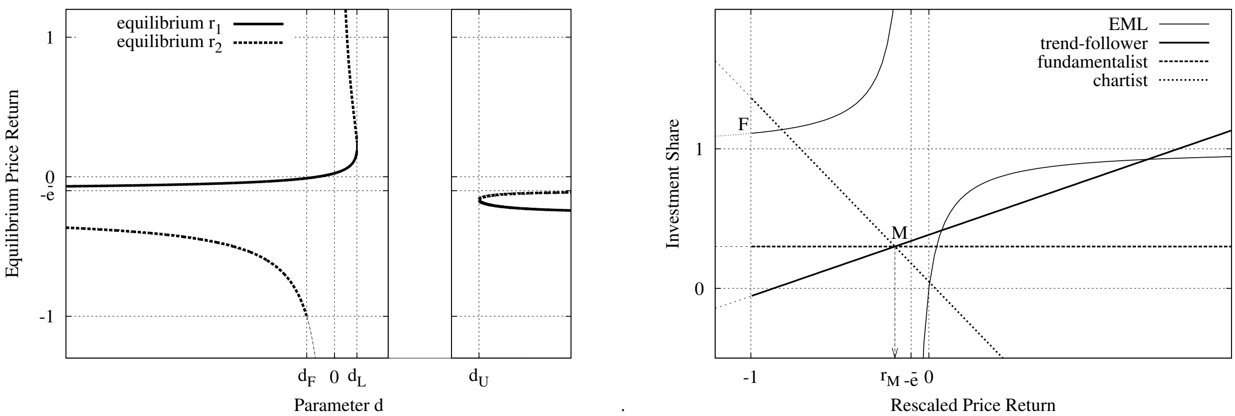

Figure 9: The case of homogeneous agents in the model of Chiarella and He (2001). **Left panel:** Equilibria as function of extrapolation parameter d. (Cf. Figure 1 from the original paper.) **Right panel:** Investment functions for the contrarian, fundamentalist and trend-follower.

one of two equilibria becomes unfeasible. The remaining feasible equilibrium is unique for all $\tilde{d} \in (d_F, 0]$. When $\tilde{d} > 0$, the agent is trend-follower and parameters belong to region VII. Here again there are two coexisting equilibria. With further increase of the rate of extrapolation, the "tangency line" is intersected for some $\tilde{d} = d_L$ and both equilibria disappear. In region V there exist no equilibria, but when extrapolated parameter is very high, i.e. agent extrapolates strongly, the "tangency line" is intersected again in some point $\tilde{d} = d_U$. After this intersection two equilibria coexist. As a result of such consideration, we reproduce (and improve) Figure 1 from Chiarella and He (2001) in the left panel of Fig. 9.

Alternative, and more explicit way to understand the last graph is to exploit the EML. Notice that symmetrization (4.1) of function f^{CH} always passes through the point

$$M = \left(r_M,\ \tilde{\delta}\,\right) \quad , \quad \text{where} \qquad r_M = -\bar{e} - \frac{r_f}{1+r_f} \quad , \tag{5.1}$$

which does not depend on $\tilde{d}$. The slope of the symmetrization is equal to $\tilde{d}\,(1+r_f)$. Three typical behavior are presented in the right panel of Fig. 9. The horizontal investment function corresponds to $\tilde{d} = 0$, i.e. to the fundamentalist type of behavior. Analogously, any trend-follower possesses an increasing investment function, while the chartist's function is decreasing. Notice also that return r_M in (5.1) corresponds to the zero level of gross unscaled return.

Counter-clockwise rotation of the straight vertical line passing through point M immediately explains the left panel of Fig. 9. In particular, notice that d_F represents the value of the extrapolation parameter, when the corresponding investment function of contrarian passes the point F of the upper-left branch of the EML. Values d_L and d_U correspond to the trend-followers whose investment functions are tangent to the EML.

Stability analysis which CH performed for the case $L = 1$ can be easily illustrated in Fig. 7. In particular, any horizontal (fundamental) investment function is stable, and such equilibrium remains to be stable for $\tilde{d}$ close to 0. Moreover, equilibrium r_1^* is stable for very small negative $\tilde{d}$, while equilibrium r_2^* is stable for very large positive $\tilde{d}$.

Further advantages of the geometrical application of the EML can be seen in the case of market with N agents. In Section 5.1 we described four different scenario considered in CH for two-agents case: two fundamentalists with different risk premium, fundamentalists vs. contrarian, fundamentalists vs. trend-follower, and two chartists with different extrapolation coefficients. We illustrate all these possibilities in Fig. 10 and discuss below how all the results of CH can be re-obtained geometrically.

Consider, first, the case of two fundamentalists with different risk premium $\tilde{\delta}_1 > \tilde{\delta}_2$ (the left panel, first row). These traders have horizontal investment functions passing through points M_1 and M_2 defined in (5.1). From the assumption on the risk premium it follows that M_1 is above M_2. There are two equilibria in such market: S and U. Each of these equilibria would be stable if the corresponding agent would operate alone. When two agents operate together, then equilibrium S with the highest risk premium is stable, while U is unstable. Notice that this result can be immediately generalize for the arbitrary number of fundamentalists.

Let us now suppose that fundamentalist with risk premium $\tilde{\delta}_1$ encounters in the market contrarian with risk premium $\tilde{\delta}_2$, so that horizontal and decreasing investment functions are competing. CH distinguish between two cases depending on which of these risk premium is higher. Geometrically, it corresponds to the location of points M_1 and M_2. We start with the case in which $\tilde{\delta_1} \geq \tilde{\delta}_2$, i.e. when point M_1 is above M_2 (the right panel, first row). With respect to the previous case we have made a rotation of the lower investment function around point M_2. It is obvious that equilibrium S_f is always stable in this case, while equilibrium S_c cannot be stable. Thus, the left plot in Figure 3 of CH illustrating the qualitative features of this situation is obtained[10]. In the second case, when $\tilde{\delta}_1 < \tilde{\delta}_2$, there are different possibilities. If contrarian extrapolates not very strongly, so that an absolute value of $\tilde{\delta}_2$ is small enough (left panel, second row), then S_f is, certainly, unstable equilibrium. Therefore S_c remains to be the only candidate for the stable equilibrium on two-agents market. It will be stable only when it is stable in the single agent case, which happens for relatively small $\tilde{d}_2$ (see the left panel in Fig. 7). Otherwise, there is no stable equilibria in the market. If, on the other hand, contrarian extrapolates strongly (the right panel, second row), then S_f is the only stable equilibrium. Comparing this analysis with the second graph in Figure 3 in CH, we can see that the situation of possible absence of any stable equilibrium in the market has been overlooked.

In the third example we consider the case when fundamentalist with the risk premium $\tilde{\delta}_1$ competes with the trend-follower with the risk premium $\tilde{\delta}_2$. In this example, we again distinguish between two cases depending on which of the risk premium is greater. Let us, first, assume that $\tilde{\delta}_1 \geq \tilde{\delta}_2$. There are two possibilities. If the trend follower extrapolates not too strong, equilibrium S_t is not stable (the left panel, third row). Equilibrium S_f is stable in this case. If the trend follower extrapolates stronger, his investment function rotates and equilibrium S_f looses its stability. S_t remains to be the only candidate for the stable equilibrium. If it exist and stable in the market with trend-follower alone, it is also stable in the two-agents situations (the right panel, third row). Otherwise, there are no stable equilibria in the market with two agents. It is the case for $d_U > \tilde{d}_2 > d_L$, since in this situation there is no equilibrium in the market with surviving trend-follower. But it also happens for some $\tilde{d}_2$ lower than d_L. Finally, for very strong extrapolation, when $\tilde{d}_2 > d_U$ the market may have a stable equilibrium, if it exists for trend-follower. In the case when $\tilde{\delta}_1 < \tilde{\delta}_2$ (the left panel, fourth row) it is obvious that equilibrium S_f cannot be stable, therefore market will have a stable equilibrium S_t whenever it is stable for trend-follower, that is for small enough $\tilde{d}_2$. On

[10]All plots in Chiarella and He (2001) which we mention here and below are just sketches obtained from the mixture of the analytic and numerical analysis. The advantage of our approach is that these qualitative sketches can be obtained from the EML plot. Thus, on the one hand, they all become justified on the analytic basis. On the other hand, they also become more clear and, thus, can be easily generalized for the situations of three and more agents, and also corrected. For example, notice that in this case the return in equilibrium S_c *does not* approach the return in equilibrium S_f when $\tilde{d}_2 \to 0$.

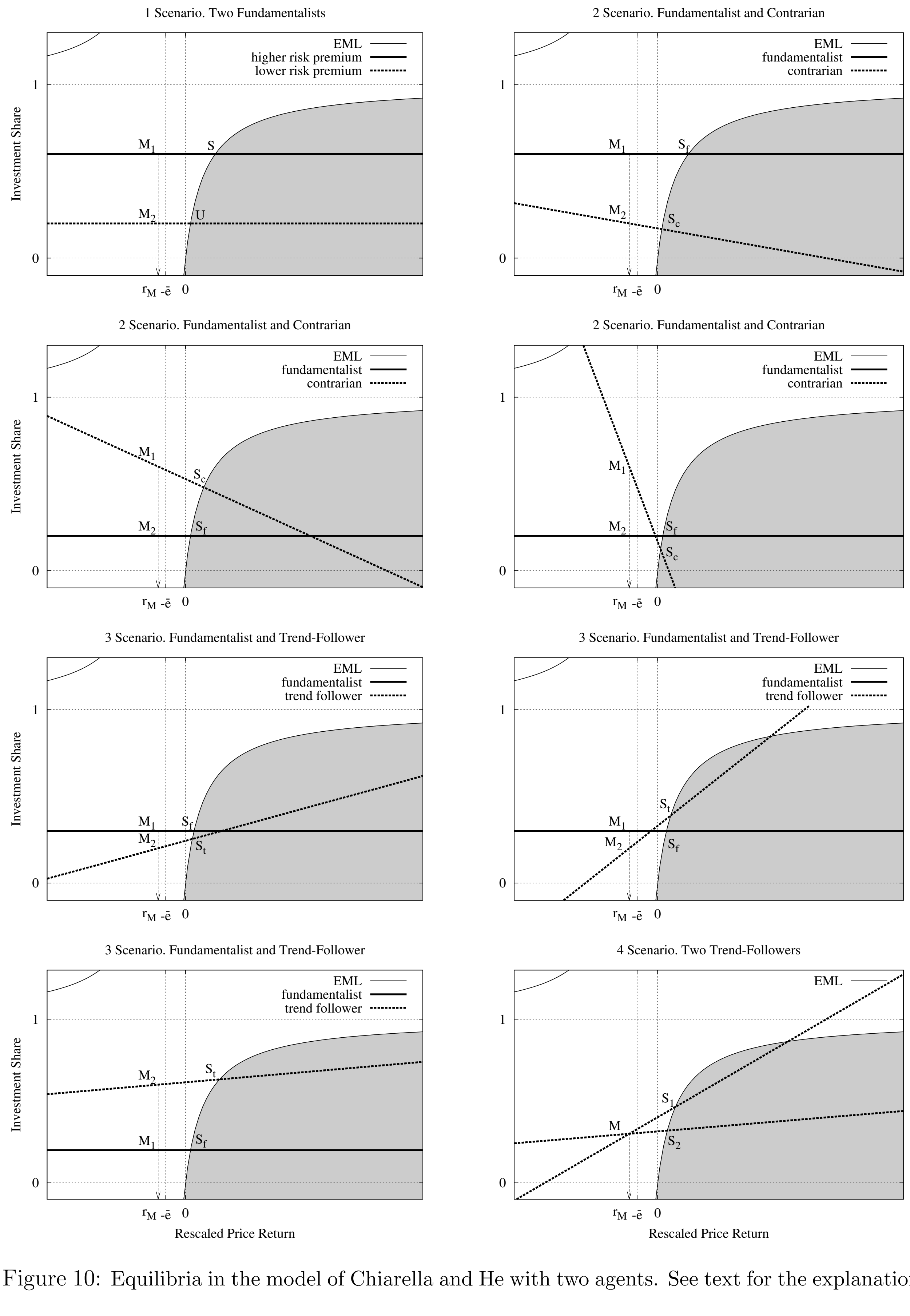

Figure 10: Equilibria in the model of Chiarella and He with two agents. See text for the explanation.

the base of this discussion we can immediately see that Figure 4 in CH is not always correct.

Finally, in the right panel of the fourth row of Fig. 10 we consider an example when two technical traders coexist in the market. We draw the situation when both of them are trend-followers and have the same risk premium, so that their investment functions pass through the same point M. It is clear, that the agent with the lowest extrapolation rate will generate equilibrium S_2 which will always be unstable. Instead, equilibrium S_1 generated by the second agent will be stable if and only if it is stable in the single agent market. Comparing it with the panel (b) in Figure 5 in CH, we notice that with further increase d_2 the stable equilibrium (with growing return) becomes unstable and, eventually, disappears. So that for higher extrapolation rates market does not have any equilibrium.

6 Conclusion

In this paper we have applied the general model of Anufriev and Bottazzi (2005) to the special class of agents' behavior. For the application we have chosen a class which is the most common in economics, namely the class of optimal behavior and demonstrated that the model have implications for a very large subset of this class.

The generality of the Anufriev and Bottazzi framework together with the geometric representation of their results allowed us to overcome well-known technical difficulties in the expected utility maximization setting. We have shown, for instance, that investment functions derived in this setting, which are only implicitly defined, shift downward with the risk aversion. This immediately implies, given the geometric nature of the locus of all possible equilibria (the Equilibrium Market Line), that the price return will decrease when the risk aversion coefficient of the agents increases. This result is not new in the economic literature: if the agents are willing to take a small amount of risk, they will also get a smaller return. What is new, however, is that we have rigorously obtained this result in the framework with *endogenous* price setting.

We have analyzed also the setting where the agents have mean-variance demand. In this case we have demonstrated that the qualitative results about market dynamics can be easily obtained using the EML plot. As an application, we have shown that the analytic model with heterogeneous agents presented in Chiarella and He (2001) can be easily understood and generalized in many directions. Namely, the analysis can be extended for arbitrarily large number of agents with arbitrary risk aversion and expectation rules. Probably, the easiest way to illustrate the advantages of the general approach is to have a look on the stratification diagrams at Fig. 5 and 7, drawn for a special, "linear" case of the agent's behavior. Even in this particular case, the scope of the model of Chiarella and He is represented by the one-dimensional straight line. Moreover, only small interval of this line is analyzed in that model, since risk premium is assumed to be bounded inside an interval $(0, 1)$.

In our view, the most interesting implication of this paper is that some features of the long-run market dynamics, like multiple equilibria, cannot occur in a market with these specific population ecology. The global, quasi-optimal selection principle of Chiarella and He may hold when all demand functions are derived from the mean-variance optimization, but it does not hold in general. In this respect, it seems promising, for the further research, to apply the general framework from Anufriev and Bottazzi (2005) to another, non-rational, types of behavior, e.g. to those advocated by the prospect theory or to the behaviors based on the threshold levels.

APPENDIX: Proofs of Propositions

A Proof of Proposition 3.1

Let us introduce the following function

$$h(x_t, \gamma) = \int y \, (1 + x_t \, y)^{-\gamma} \, g(y) \, dy \quad , \tag{A.1}$$

where $g(y)$ is the perceived distribution of the next period return y. This distribution, in general, depends on the return history. The value of the investment function f^{EP}, or in other words, the investment share x^* of the agent who solves the EU maximization problem (3.2) with power utility function (3.1) is the solution of the first-order condition (f.o.c.) $h(x_t, \gamma) = 0$.

Let us, first, assume that $x^* > 0$. Then, for both positive and negative y we have $y > y \, (1 + x^* \, y)^{-\gamma}$. Multiplying both parts of this inequality on the function g, integrating with respect to y, and applying the f.o.c., we get $\bar{y} > 0$. Analogously, if $x^* < 0$, then $y < y \, (1 + x^* \, y)^{-\gamma}$ for any $y \neq 0$, and, therefore, $\bar{y} < 0$. Finally, when $x^* = 0$ f.o.c. implies that $\bar{y} = 0$. This proves the first part of the statement.

The f.o.c. actually defines x^* as an implicit function of the risk-aversion coefficient γ. Applying the implicit function theorem we get that

$$f^{EP}_\gamma = -\frac{1}{\gamma} \, \frac{\int y \, \log(1 + x^* \, y) \, (1 + x^* \, y)^{-\gamma} \, g(y) \, dy}{\int y^2 \, (1 + x^* \, y)^{-\gamma - 1} \, g(y) \, dy} \quad . \tag{A.2}$$

Denominator of the last expression is always positive, while numerator is positive when $x^* > 0$ and negative, otherwise. This proves the second part of the statement.

B Proof of Proposition 4.1

In the case $B = 0$ condition (2.9) implies that $A + 1 = l(r)$, which is a linear equation with respect to r. We get (4.3) as soon as $A \neq 0$. If, instead, $B \neq 0$, then from definition of the EML we get the following quadratic equation with respect to $\bar{e} + r$

$$B \, (\bar{e} + r)^2 + A \, (\bar{e} + r)^2 + \bar{e} = 0 \quad . \tag{B.1}$$

The discriminant of this equation $D = A^2 - 4B\bar{e}$. Solving (B.1) in the case when $D > 0$ one gets (4.4).

C Proof of Proposition 4.2

The constant investment function has one or zero equilibria. For the increasing function consider the second inequality in (4.6). Substitution of the EML's slope in equilibrium leads to

$$B \, (\bar{e} + r^*)^2 - \bar{e} < 0 \qquad \Leftrightarrow \qquad -A \, (\bar{e} + r^*) - 2\bar{e} < 0 \quad , \tag{C.1}$$

where we used the relation (B.1). Plugging corresponding equilibrium values from (4.4) and simplifying the resulting inequality, one gets

$$\sqrt{A^2 - 4B\bar{e}} + A < 0 \quad \text{in } r^*_1 \qquad \text{and} \qquad \sqrt{A^2 - 4B\bar{e}} - A < 0 \quad \text{in } r^*_2 \quad .$$

When $A > 0$, the left inequality is violated and therefore r^*_1 is unstable as in the plot for region IV in Fig. 4. If $A < 0$ the right inequality is violated and r^*_2 is unstable as in Fig. 4 for region VII.

Consider now the case of decreasing investment function $B < 0$. Then, as we showed in Section 4.1, the equilibria are such that $r^*_2 < -\bar{e} < r^*_1$ (see also illustrations for regions I and II in Fig. 4). If the equilibrium return is negative, the first inequality in (4.6) leads to

$$B \, (\bar{e} + r^*)^2 - r^*\bar{e} > 0 \qquad \Leftrightarrow \qquad -A \, (\bar{e} + r^*) - \bar{e}(1 + r^*) > 0 \quad , \tag{C.2}$$

When $A \leq 0$, it, obviously, always holds with the opposite sign in feasible r^*_2, i.e. r^*_2 is always unstable in this case. Analogously, when r^*_1 is negative it will be unstable when $A > 0$.

Finally, let us consider the case when $A > 0$ and r^*_1 is positive. The third inequality in (4.6) leads to

$$B \, (\bar{e} + r^*)^2 \, (2 + r^*) + r^*\bar{e} > 0 \qquad \Leftrightarrow \qquad -A \, (\bar{e} + r^*)(2 + r^*) - 2\bar{e} > 0 \quad , \tag{C.3}$$

which is always violated. Thus, in all cases when two feasible equilibria exist one of them is unstable.

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