



Laboratory of Economics and Management
Sant'Anna School of Advanced Studies

Piazza Martiri della Libertà, 33 - 56127 PISA (Italy)
Tel. +39-050-883-343 Fax +39-050-883-344
Email: lem@sssup.it Web Page: <http://www.sssup.it/~LEM/>

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Social Heterogeneities in Classical New Product Diffusion Models

Piero Manfredi*
Andrea Bonaccorsi*
Angelo Secchi

*University of Pisa

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Piero Manfredi*, Andrea Bonaccorsi* and Angelo Secchi

* University of Pisa

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Abstract

Within the very broad scientific debate on the role of heterogeneities in models of innovation diffusion, there seems to exist a delay of marketing science in absorbing the fecund developments that took place, in the field of epidemiology of infectious diseases in very recent times. This paper, which constitutes the basis for a future research project, aims to fill this gap. This is done along three main directions. First, the paper discusses in general terms the role of the concept of heterogeneity in innovation diffusion. Second it reviews those recent epidemiological results which are of crucial interest for marketing models. Finally it offers a sample of results which are suggestive of the critical role played by heterogeneities within deterministic models for the diffusion of new products.

1 Introduction

This paper aims to start a systematic discussion of the role of behavioural heterogeneities within the classical deterministic mathematical models describing new products diffusion. As it is well known, the backbone of the mathematical theory of marketing of new products, the Bass model, superimposes the effects of two "classical forces" operating during the adoption process: those due to "innovation" and those due to imitative behaviours, well described since longtime ago in the classical papers by Fourt and Woodlock (1960) and Mansfield (1961). These two forces describe the action of the two basic communication mechanisms operating during the adoption process: the mass-media and word-of mouth, i.e. the so called "external" and "internal" diffusion mechanisms (see the recent syntheses by Mahajan et al. (1990,1993)). Well recognised merits of these "classical" models for new product diffusion are, first, that of providing simple empirically testable numerical predictions, and second, that of introducing parameters which give invaluable insight about the nature of the diffusion process itself: the intensity of diffusion by the external source and the imitation coefficient (denoted respectively as α and q in the sequel).

All the previously mentioned models describe *homogeneous* situations: the social environment (i.e.: the population) in which the rumour spreads is homogeneous in that i)all the individuals are homogeneously exposed to the action of the public sources of informations, ii)all the individuals mix homogeneously, i.e. are homogeneously exposed to internal communications.

The aim of this paper is to discuss several issues connected with the possibility that social heterogeneity exist in the population, i.e. that, on the contrary, individuals may be exposed in an inhomogeneous manner to the action of the media and/or individuals do not mix homogeneously among them. Typical questions we will pose are: are the individuals in the population exposed in the same way to the information source? Do the individuals in the population share identical imitation patterns? What happens if the answer is not? More precisely: what happens when we wrongly treat as homogeneous an underlying heterogeneous situation?

The relevance of these questions is indisputable, as may be realised from some limit cases, such as the case of a group excluded from information in pure external diffusion world, or of a group which does not mix with the rest of the population in a pure internal diffusion world. In both cases the individuals of the involved groups would not adopt the product in the long term, but any statistical apparatus of market surveys based on a wrong homogeneous model would hardly discover the problem in real time.

To our knowledge, up to the eighties the only serious tentative to embed heterogeneities within the classical models for new product diffusion and/or models for the diffusion of news is the classical Bartholomew (1967), and its subsequent developments by Dimitri (1987). Both these contributions, largely based on the old epidemiological paper by Rushton and Mautner (1955), took place before the "heterogeneity revolution" and the development of mixing theory which represented a true structural breakdown in theoretical epidemiology. The heterogeneity revolution has started at the beginning of the eighties, more or less closely tied to the need to understand the peculiar features of the spread of AIDS, and is still very active. Several results produced during this very intensive debate, which proved to be of extreme fecundity in theoretical and applied epidemiology, such as the debate on social mixing, are definitive acquisition of social sciences.

Nonetheless the spread toward allied fields, such as in the marketing of new products has been, at least to our knowledge, quite scarce, compared to the development of the concept of heterogeneity in other fields. In fact, up to now, at least to our knowledge, the only contribution based on the emerged concepts is Putsis et al. (1997).

The aim of the present paper, which should constitute the basis of an applied research project to be developed at the University of Pisa in the forthcoming year is threefold. First we aim to provide a review of the existing literature on the subject of heterogeneity within the broad family of models for the diffusion of innovation, which may be roughly subdivided into two large subsets: models for the diffusion of new products, and models for the diffusion of new technology. These subclasses have important interactions, and in any case, a detailed overview of the many facets of the concept of heterogeneity seemed to us a necessary starting point.

Our second goal is that of providing a sufficiently complete review of the many acquisition due to heterogeneity revolution in epidemiology for modelling processes and analysis of models for the diffusion of new products. Rather than supply a meticulous discussion of a huge and hard literature, we have tried to concentrate on the main buiding blocks, showing how new epidemiological concepts can be useful by directly working on the marketing models.

Finally, we have tried to provide a discussion concerning the extension of the basic classical model of innovation theory to heterogeneous situations. Our basic assumption in the present work is that the social environment ("population") within which the diffusion process occurs, is stratified in k distinct subgroups (not necessarily of the same size), which may differ for what concerns the magnitudes of one or the other or both their α, q coefficients. Two main sections which correspond to the two ingredients of the Bass model and to Bass's model itself.

In the first section we consider the impact of heterogeneity within the basic model for external diffusion: individuals acquire the informations from the external source at different speeds. In the second section we consider the role of heterogeneities within the epidemiological model which drives the "pure imitation" process. This part of the discussion heavily relies on the role played by heterogeneities and mixing within epidemiological models, as emerged by the recent debate on the dynamics of sexually transmitted diseases, HIV for example.

Finally we combine the results of the two preceding sections to consider the role of heterogeneities within Bass models (and its basic variants). The qualitative and quantitative roles of heterogeneity is sketched out, by stressing several aspects previously evidenced in other disciplines, namely demography and epidemiology, such as the aggregation biases that would arise from an homogeneous treatment of an underlying heterogeneous situation.

The present paper is organised as follows. The second section provide our more general overview on the concept of heterogeneity within models for the diffusion of innovation. The third section is just of help in the reading through the paper: it recalls the classes of models we will discuss throughout, and makes some terminological clarifications. The fourth section considers heterogeneities within external diffusion processes. Here the treatment is rather complete. The subsequent section, which is the main section of the paper, considers internal diffusion at the light of the heterogeneity/mixing revolution. Several particular results are presented the aim of which is that of suggesting the role of the new concepts, possibly also in field research. The section on heterogeneities in the Bass' model is purely sketched, as its results are incomplete and will be presented in more detail elsewhere. A "pot-pourri" of numerical illustrations on some selected examples of heterogeneous situations follows, with brief conclusions on the need for the opening a serious applied work on the problem.

2 Heterogeneities in models of innovation diffusion: a review

2.1 Heterogeneities in new product diffusion models

Since the notion of heterogeneity in diffusion models is not simple, it requires a careful review. The starting point is the absence of heterogeneity in the three classical models of Mansfield (1961), Fourt and Woodlock (1960) and Bass (1969). Let us discuss the later, more comprehensive model.

The Bass model assumes homogeneity in at least two different respects: with respect to characteristics of members of the population and with respect to the information diffusion mechanism. In the former sense, all members have the same probability to adopt the new product at the time defined by a stochastic allocation to various timing classes; in the latter sense all members are exposed homogeneously to external influence (advertising) and to internal influence (word-of-mouth of previous adopters). Let us call behavioral homogeneity the former, social homogeneity the latter. In both cases there have been attempts to incorporate heterogeneity in the basic model, but, as we shall see, there is still much to be done.

In the former sense, one can see that the typical diffusion path is the result of the fact that the level of innovativeness is not distributed homogeneously among members of the population; rather, innovators and early adopters are more incline to try the new product at the beginning of the process, while late adopters and laggards wait until sufficient word-of-mouth is available regarding the product. Following Rogers (1962) one can assume that individuals are characterized by intrinsic degrees of innovativeness. However, it is only the distribution of

individuals across timing of adoption that makes them different to each other, while no other structural difference is allowed. It has been correctly noted that the mathematical formulation of the Bass model requires the population to be homogeneous, in the sense that any difference related to the timing of adoption is of stochastic type (Tanny and Derzko, 1988; Chatterjee and Eliashberg, 1990). The distribution of individuals across timing of adoption proposed in the early work of Rogers (1962) may in fact be easily obtained from a Bass diffusion model (Mahajan, Muller and Srivastava, 1990) without any specification about intrinsic differences among members of the population. The distribution of adopting time is merely an aggregate property, while individuals are behaviorally homogeneous at the microlevel.

In the original formulation, the distribution was assumed to be normal around the mean adoption time. The bell-shaped normal curve can be considered as the first derivative of a logistic function, hence the cumulative adoption function is a symmetric, S-shaped logistic curve. The mathematical properties of the logistic curve have been the object of much criticism, however, since they imply not only the symmetry of the diffusion curve, but also that the probability to adopt an innovation (or the degree to which the innovation decision is “infectuous”) is constant over time. Basically, the Bass model is not flexible (the point of inflection, which represents the maximum rate of diffusion, is not allowed to be at any time of the process) and is rigidly symmetric (the diffusion pattern after the inflection point is the mirror of the pattern before it). These features have been modified in further developments, still maintaining the basic epidemic framework, with a large variety of asymmetric and skewed diffusion curves with (see a review in Grübler, 1992). In these flexible models the distribution of individuals across timing of adoption is allowed to be non-normal, taking into account the specific shape of diffusion curves in different product settings.

However, simply modifying the mathematical properties of the original model is not enough to characterize heterogeneity. The most challenging efforts to introduce heterogeneity have been developed in response to another criticism to the epidemic models, namely the indeterminacy of the behavioral foundations. Why are some people more inclined to innovate than others? Which is the microfoundation of their adoption behavior? As it has been noted (Gatignon and Robertson, 1986; Grübler, 1992) epidemic models are empty of substantive theory regarding the adoption process of individuals. In response to this weakness of the basic model, there have been several attempts to develop microfounded models; it is in this context that heterogeneity is included into the basic framework. It is important to underline this point: heterogeneity is a major concern of the new generations of diffusion model builders, but only inasmuch as it is defined as behavioral heterogeneity, or heterogeneity of parameters that enter into the adoption decision process. Here it is abandoned the idea that individuals are stochastically assigned to various classes of potential adopters, but rather an explicit formulation of the decision process is figured out. In some sense, therefore, heterogeneity arises as an internal theoretical problem within an intrinsically epidemic approach. It is also considered one of the frontiers of research. Let us give a brief review of these developments, following the reviews of Sultan, Farley and Lehmann (1990), Mahajan, Muller and Bass (1990; 1995), Bass, Krishnan and Jain (1994) and Cestre (1996).

In general, heterogeneity is allowed in the population with regards to tastes and income, the initial perception of the quality of the new product or the expected benefit from the use, the price-performance trade-off, the perceived risk and risk propensity (Cestre, 1996).

Kalish (1985) develops the idea that potential adopters differ with respect to tastes and income. This translates, following an initial suggestion by Schmalensee (1982) into differences in the level of the reservation price. Each consumer has his own valuation of the intrinsic value of the new product, which is expressed in monetary terms, and is willing to adopt it

only if this value is larger than actual market price. Given some uncertainty regarding the true quality of the new product, the reservation price can be considered as a function of the expected value of a distribution of quality outcomes. Therefore only those consumers whose risk-adjusted valuation exceeds the product's price will be potential buyers.

A model which tries to develop formally the idea of heterogeneity has been proposed by Chatterjee and Eliashberg (1985). Here consumers are assumed to maximize a von Neumann-Morgestern utility function and to update their beliefs following Bayes' rule. Each consumer has an initial perception of the quality of the new product, which is compared to a reservation price. In the evaluation of the product, price and quality attributes are traded off. Information about the product is then socially diffused, either through advertising and word of mouth. Consumers update their perception of the quality of product and come to converge to the "true" evaluation. Consumers are allowed to be heterogeneous with respect to a large number of parameters: risk aversion, initial perception of product quality, relative weight of price and quality in the evaluation of product, information sensitivity.

Heterogeneity is also accounted for in models of diffusion that explicitly incorporate stochastic elements. As an example, agents may be associated to different probabilities to be exposed to advertising, to be influenced by advertising and word-of-mouth, to purchase or to repeat the purchase (Tapiero, 1983; Boker, 1987, Wheat and Morrison, 1990; Mahajan and Peterson, 1978). In all these cases, heterogeneity is again obtained by representing agents as drawn from an appropriate distribution.

One of the striking features of this stream of literature is the complete independence of behavioral heterogeneity from social heterogeneity. Individuals that differ by income, tastes, quality perception or risk attitudes are otherwise entirely similar in their social behavior: they do not differ in the speed they acquire information from public sources, nor in the intensity with which they communicate interpersonally regarding the new product. One author is explicit in stating that heterogeneity in consumers' reservation prices has no relation with heterogeneity in communication patterns: "information diffusion is homogeneous (geographically, as well as by population segments)" (Kalish, 1985, 1572). In other words, information acquisition and dissemination is not considered as a typical human activity which interacts with the internal decision making process. Works that claim they have reached a truly microfounded theory of diffusion exhibit a remarkable incompleteness in the characterization of human behavior.

The problem of social heterogeneity is perceived as a separate weakness of the Bass model, but again the responses available in the literature are rather weak. As an example, in the model by Mahajan, Muller and Kerin (1984), it is taken into consideration the possibility that word-of-mouth has a negative, instead of positive, sign (i.e. dissatisfied customers). The population is therefore divided into groups according to the stage of the adoption process they are in (non aware, aware, adopter or non-adopter) and to the sign of the information they produce for other members of the population. The flows between these groups determine the aggregate dynamics of diffusion. However, heterogeneity is admitted only for senders of messages, not for receivers. All members of the population have the same probability of receiving any given piece of information, so that the aggregate probability is proportional to the relative size of the message sender groups.

Another notable attempt has been directed towards endogenizing the coefficient of imitation. Note that the mathematical formulation of the Bass model implies that the interaction between adopters and non-adopters, as reflected in the constant coefficient of imitation or parameter b , does not change during the diffusion process. This is unfortunate, since "there is no theoretical rationale for the word-of-mouth influence or the imitation effect to remain uniform over the entire penetration span" (Easingwood, Mahajan and Muller, 1983). Several alterna-

tive formulations of the model have been proposed, which allow for flexible, non-symmetric diffusion curves and imply variations of the imitation coefficient (see Easingwood, Mahajan and Muller, 1983 for a review and a general model). However, social effects of communication are allowed to vary over time, but only as an aggregate effect. The reason why imitation coefficients may vary over time is predicated at the level of the aggregate dynamics, not at the level of the communication activity of individuals.

A more complete treatment of social heterogeneity may be found in a new generation of stochastic models of diffusion, which explicitly introduce the idea that individuals may differ in the structure of their social activity. As an example, LeNagard and Steyer (1995) use the stochastic theory of avalanches to model a situation in which imitation takes place only within a restricted circle of individuals. Adopters may reach only a subset of individuals, who in turn speak to other restricted circles of individuals. The propagation effect is the result of the topological structure of the population.

To our knowledge, the only paper that explicitly uses the epidemiological concept of heterogeneity is Putsis, Balasubramanian, Kaplan and Sen (1997). They study mixing behavior in several European countries and analyze the rate of contacts between individuals both intra- and cross-country and find that mixing is indeed an important consideration in new product diffusion.

Information diffusion is not only the object of the specialized literature on new products. A related stream of literature in microeconomic models of consumer demand is devoted to the study of several informational structures which are of interest to our discussion. The point of departure of this literature is the effort to model a situation in which the utility function of individuals is not only defined on their own preferences, but also on preferences of other individuals. This effort can be seen as the conceptualization and mathematical treatment of phenomena long time ago discussed by many, mainly institutionalist, economists. Examples of these phenomena include the consumption behavior of leisure class illustrated by Veblen, the bandwagon and snob effects of Leibenstein (1950; 1976), the demonstration effect of Duesenberry (1949), the conspicuous consumption further analyzed by Mason, the arousal effect discussed by Scitovski (1992), down to the analysis of lifestyles of Earl (1986).

The incorporation of others' preferences can nevertheless take several directions. In the Becker (1996) formulation, it is used to enlarge and confirm the predictive power of standard utility theory. In non-neoclassical settings, on the contrary, the social influence on preferences is accounted for by modeling a form of non-optimizing, partly routine-driven, partly novelty-seeking behavior, in which preferences are socially embedded and emerge from the interaction of individuals. The simulation proposed by Aversì, Dosi et al. (1997) is based on a genetic algorithm model, in which each agent is defined by his income level and income class and is endowed with strings that represent the pattern of consumption of goods. Goods are ordered within strings according to a lexicographic order, meaning to capture the structure of lifestyle of individuals. The social embeddedness of preferences is evident in the imitation mechanism, which is restricted to the same income class, or possibly higher ones.

With many subtle variations, all these effects point out to situations where individual preferences are shaped by what agents observe in their social environment. In some cases it is simple observation that matters, so that the revealed preferences of other individuals enter into others' utility calculation simply by being reflected in the adoption of some "visible" goods. In other cases individuals send signals, or actively communicate their own consumption experience to others. Still in other cases, individuals that must decide which product to buy enter into a search process, whereby they may "sample" previous adopters, look at the type of product they have purchased and ask experiential information on it.

In all cases, the theoretical effect of the endogenization of preferences is that the specific pattern of communication between individuals become of highest importance. If agents make decisions based on others' preferences, it becomes absolutely crucial to specify which others.

In this perspective, some prototypical information diffusion structures have been explored. Let us briefly review some of them.

- *mimetic contagion* (Orlean, 1989). It is the social situation corresponding to the Keynesian beauty contest, in which all individuals imitate each other. Each individual enters into the decision process with his own initial belief, but loses it over time, by observing and assuming the beliefs of others'. Under some circumstances, financial markets seem to exhibit some of the properties of mimetic contagion.
- *information cascades* (Bikhchandani, Hirshleifer and Welch, 1992; Welch, 1992). Here individuals are ordered in sequences, in which one can observe only the action of the immediate preceding individual, and is in turn observed only by another individual. After observing the action of others, individuals must decide whether to adopt or reject.
- *local interaction* (Schelling, 1960). Here the central idea, pioneered by Schelling, is that each member of the population is matched with agents in a neighborhood of a given size. A typical spatial representation involves a torus, i.e. a 2-dimensional lattice, in which members of the population are spatially distributed on the nodes.

So far, the typical field of application of the local interaction paradigm has been game theory, and particularly evolutionary game theory (see for example Ellison, 1993).

Several stochastic mechanisms have also been included in these models. The size and configuration of the neighborhood can also change endogenously. A rather general result of this stream of literature is that the evolution due to local interaction may strongly contrast with the evolution due to global interaction (Umbhauer, 1995). Our results in the following sections of the paper firmly support this finding.

- *percolation* (David and Foray, 1993). Applications of the local interaction idea to diffusion have been also proposed. David and Foray (1993) have used a percolation model to study the diffusion of EDI techniques. In a percolation system there are many elements between which can exist a certain number of interactions, When the number of interactions is randomly increases, there exists a critical level (percolation threshold) beyond which all elements of the system can be reached from any other element. Markov random fields and Gibbs random fields can also be used to model this type of situations (see Cohendet, 1995) in diffusion studies.

As we shall see, this line of inquiry is still at the infant stage in the theory of diffusion of new products, while it is much more robust in the technology diffusion field. Let us turn to the latter for a while, in order to make our review of heterogeneity effects more complete.

2.2 Heterogeneity in technology diffusion models

Interestingly, there has been strong separation between two streams of literature: diffusion of new products in a population of consumers and diffusion of new technologies in a population of firms. Although the basic epidemic mechanisms are assumed to be the same, the literature has explored specific mechanisms of diffusion in a largely unrelated way. Models of new product

diffusion are published in marketing journals, models of new technology in economics journal, with very infrequent cross-citations.

In technology diffusion theory heterogeneity plays a major role. However, as we shall see, the way in which it is dealt with is still unsatisfactory.

In technology diffusion there is a new source of heterogeneity that must be accounted for, which is instead absent in product diffusion among consumers, namely, competition or strategic effects.

Karshenas and Stoneman (1993) refer to three classes of effects in diffusion theory that may be considered sources of heterogeneity among potential adopters: rank, stock and order effects.

Rank effects arise from heterogeneity in intrinsic characteristics of the members of the population that may influence the decision to adopt. In the case of technology diffusion, rank effects mainly refer to the size of the firm (larger firms may reap greater benefits from a given new technology), the level of expenditure in R&D, the corporate status (independent vs. subsidiary of a large company), and the like. Additional sources of heterogeneity may lie in differences across firms in attitudes or expertise towards the new technology (expert vs. novices) or in differences in previous experience with similar innovations (Jensen, 1982). In empirical works, all these differences are accounted for in the estimation of coefficients, since they refer to observable variables. In modeling terms, however, the differences are thought to influence adopters' reservation acquisition costs. The distribution of expected benefits from the new technology across the members of the population is reflected in a distribution of reservation costs, which potential adopters compare to the actual acquisition cost (price). Since price is assumed to fall over time, the very existence of heterogeneity in reservation costs generates a diffusion path. This is the route explored by the classical works of David (1975) and Davies (1979) and by Ireland and Stoneman (1986).

Stock effects arise when the benefit to the marginal adopter from the acquisition of the new technology decreases as the number of previous adopters increases. This effect means that for any given cost of acquisition (price) there will be a maximum number of adopters that make the adoption profitable for each of them. What is relevant is therefore the number of adopters at the date in which each of the members of the population must make his decision. Again, since actual acquisition costs are assumed to fall over time, a diffusion path may result. (To be more precise, things are made more complicated by the possibility of endogenizing output decisions in the evaluation of the new technology).

Finally, order effects imply that the return from adoption depends on the position in the order of adoption, with higher order position associated to greater returns. In this case, what matters is not the total number of adopters after the date of adoption, but only the relative position. As an example, first mover adopters may benefit of better geographic locations of gain access to more skilled labor (Ireland and Stoneman, 1985). In another context, this idea is at the heart of the literature on first mover advantages in the launch of new products.

Models of technology diffusion which incorporate heterogeneous rank effect have been developed by Feder and O'Mara (1982) and Jensen (1982). Here agents (firms) have an initial perception of the value of the new technology, which is updated over time through a Bayesian mechanism. The importance of heterogeneity is such that "differences among firms in their subjective beliefs that the innovation will be profitable may be sufficient to explain the commonly observed pattern of diffusion" (Jensen, 1982, 184).

It is interesting to observe how, even in this case, behavioral heterogeneity is entirely de-coupled from social heterogeneity. Rather, it is crucial for such an inferential mechanism to work that all agents are exposed to information in the same way. In Jensen's terms,

an exogenous source of information sends messages on the value of the technology at discrete intervals. Since the messages may be positive or negative regarding the value of the technology, in Jensen's model the information source is approximated by a Bernoulli process. In the model of Feder and O'Mara, the information content of an agent's own trial of the new technology is the same as that of another agent's. Hence, "each agent acquires the same information about the new technology in each time period, whether or not he is an adopter" (Feder and O'Mara, 1982, 146). Therefore a complete separation between the process of evaluation updating and information diffusion is postulated: heterogeneity in the former do not require heterogeneity in the latter.

Other forms of behavioral heterogeneity can be found in various non-neoclassical models of diffusion, in which firms exhibit bounded rationality in the adoption process. The role of heterogeneity in these models is theoretically committing, since the hypothesis of the "representative firm" (more generally, the very notion of representative agent) is refused (see on this Kirman, 1989; 1991). Firms are instead represented as entities that differ with respect to cost structure, production technique, ability to innovate and master the new technology, ability to learn after the adoption. Acquisition of information does not follow the Bayesian conditioning rule. Economic agents are heterogeneous cognitively, structurally and behaviorally (Antonelli and Gottardi, 1988). The most prominent examples of this tradition are Dosi, Orsenigo and Silverberg (1988) and Silverberg, Dosi and Orsenigo (1988).

One of the most important streams in technology diffusion, on the other hand, involves the analysis of the effects of increasing returns to adoption and positive feedback. The setting of these models is not the diffusion of innovations per se, but rather the pattern of diffusion of competing technologies (Arthur, 1988; 1989; David 1985). The typical result of these models is that one of the competing technologies, not necessarily the more efficient one, comes to dominate the market. Interestingly, this strong convergence result is not due to the fact that agents share the same preferences, which instead are allowed to be distributed homogeneously across product alternatives. It is due, on the contrary, to the existence of positive feedback effects. In Arthur's model, the benefits from adoption increase with the number of previous adopters. This may be considered, using the words of Karshenas and Stoneman, a positive, rather than negative, order effect. This idea is also at the core of the large literature on adoption with network externalities, which we cannot review here.

In a related model proposed by Arthur and Lane (1993) the positive feedback is generated by pure information advantage. Individuals try to obtain information on new products, but they can "sample" only a small subset of all past purchasers, because "in most circumstances there is no means by which to collect information about performance from all past purchasers" (Arthur and Lane, 1993, 82). Because of this constriction of information, the valuation of products for each agent depends on the choice made by the small subset of individuals that randomly enter into the initial sample. Therefore preferences may become self-reinforcing, since products that are initially adopted have a higher probability to produce information, influencing cumulatively the decisions of potential adopters.

Interestingly, this bandwagon effect is obtained without assuming any ex-ante heterogeneity among potential adopters. Instead, all purchasers have the same priors on the value of the new product, the same utility function, and, more importantly, follow the same search strategy (have the sampling and stopping rule). At the same time, the authors remark that the model might be usefully modified allowing for heterogeneity: "when agents obtain private information, they typically do not sample randomly from all previous adopters, but instead interrogate previous purchasers that are 'close' to them, geographically, professionally or in some other appropriate sense. Various models could be constructed, taking this 'spatial' aspect

of sampling into account” (Arthur and Lane, 1993, 98). This seems to be a promising area for further integration with recent contributions on heterogeneity in epidemiology, following the lines of this paper.

2.3 The empirical relevance of the heterogeneity hypothesis

Interestingly, while heterogeneity seems to play a minor role in the mainstream diffusion theory, and there are only limited and very recent attempts to take it into account, its empirical importance is out of discussion in consumer behavior. A rather impressionistic tour into the large, qualitatively heterogeneous literature on personal influence and opinion leadership on consumer behavior shows a remarkably consistent pattern.

2.3.1 Word-of-mouth and opinion leadership in consumer behavior

Since the early ‘60s several researchers underlined how large is the amount of social communication that goes around products (Dichter, 1966). Word-of-mouth is generally considered a reliable and trustworthy type of information, whose influence on consumer behavior is high because it allows two-way communication, is associated to strong social support and is often backed up by social-group pressure. Arndt (1967) carried out a survey among members of a large community (a campus for married students) and found that product-related word-of-mouth closely follows the pattern of social relations, so that those individuals that are more central in the social network are also those that generate more communication about the new product. This is an important finding, since it shows that product-related conversation follows the (relatively) invariant channels of communication that individuals open and keep active in their social life. He also found that word-of-mouth has a strong influence on purchase decisions, with negative word-of-mouth having a larger effect than positive one. Finally, product-related communication is a powerful risk-handling strategy: the individuals that perceive a higher risk in the adoption decision are also those that look for word-of-mouth more intensively.

The amount of product-related conversation may indeed be huge for some categories of product: in a study of attendance to movies Mahajan et al. (1984) found that more than 60% of subjects engaged in movie-related conversation with friends. Interestingly, “friends were the major source of information”, much more than advertising or reviews in newspapers.

According to a survey on external search behavior, consumers perceive personal information sources to be more important than non-personal sources because the former provide two-way communication, and bring at the moment of communication more knowledge, generating more attention and interest (Pinson and Roberto, 1988).

One of the most robust findings is that individuals sharply differ in the amount of information they search for and, moreover, in the intensity of product-related conversation. The concept of opinion leader captures the characteristics of those individuals that engage in more information diffusion. A survey of several studies (e.g, Robertson and Myers, 1969; see the surveys in Mullen and Johnson, 1990; Loudon and Della Bitta, 1993) shows that opinion leaders have the same social and class position as non-leaders, but enjoy better reputation and social status within the group. They have greater interest in the area of influence, are more exposed to mass media and try new products relatively early (although they are not innovators but rather early adopters). They normally develop much more social communication than non-leaders, are more sociable and more loyal to group values and norms. Because of these characteristics opinion leaders’ influence is accepted within the relevant group as trustworthy and relevant for other individuals’ consumer decisions. The proportion of opinion leaders out

of the population is of course an empirical matter, but market surveys seem to converge, approximately, around 10%: the so-called Influential Americans are one of every ten adults in the United States, while the “must-know men” discovered by the Yankelovich research company, to whom many individuals refer for problems of mechanics, electronics and car purchase, are one-quarter of the adult male population (Loudon and Della Bitta, 1993).

An interesting question that comes about is whether opinion leaders have a general influence that cut across several product categories or are rather “specialist” in one product. There are conflicting findings on this question. Specialized opinion leaders seem to prevail: in studies of product areas, it is generally found that few of the respondents are opinion leaders for many products, with overlapping strongly related to similarities in the interest raised by products (King and Summers, 1970). A related important process may be defined as social division of labor in consumption. This relates to the increasing specialization of some classes of goods, particularly those related to leisure, hobbies and recreational activities and many consumer durables. In this areas, products incorporate increasingly sophisticate technical attributes and call for a lot of awareness and expertise of potential buyers. In turn, this require a great deal of product-related conversation among interested individuals, often with a substantial degree of technicality. This trend is reinforced by the development of specialized press, which is in turn an impressive phenomenon in contemporary markets, with deep influences on patterns of media use in advertising. As an example, readers of automotive magazines consider themselves expert and are frequently asked for advice. According to a professional magazine, 39% of the readers of *Road&Track*, a leading US automotive publication, are asked about cars by their friends one or more times a week (Advertising Age, June 22, 1981). One can therefore observe the birth and development of large product-related communities, in which communication regarding products is very intense, while possibly very limited on other topics of social life.

General opinion leaders are also present, however. Feick and Price (1987) have developed the label of market maven (where maven is a Yiddish word for a neighborhood expert) to characterize those individuals that enjoy shopping and collecting information about products, become aware of new products earlier and engage in product-related conversation with many others over a large range of product categories. The interesting finding is that market mavens do not need to be actual users of all products they are asked about by other individuals. In general, market mavens are women.

The emergence of this category is probably related to another important social process in contemporary societies. A substantial amount of sociological research has been devoted, since the ‘60s, to the analysis of the time budget of individuals and families. A good deal of evidence has also been produced in a comparative perspective, so that many countries are covered. One of the more robust piece of evidence is that the long term reduction in weekly work hours has not been used by families to increase “pure” leisure time, but, on the contrary, to manage household related problems. This includes dealing with product-related problems (collection of information, selection, management and maintenance of durables), with social relations and administrative and bureaucratic issues related to the purchase of goods and services. At least part of this additional time is strictly linked to consumption decisions and to the management of consumption. In fact, these studies show that the availability of increasing leisure time has required a more than proportional allocation of resources for consumption. As Offe and Heinze put it, the “organization and management of consumption” requires an increasing share of time budget of families (Offe and Heinze, 1990).

2.3.2 Active consumers

The notion of active consumer has recently been proposed to try to link more explicitly a realistic representation of consumer behavior to the hard core of economic theory. The basic idea is that individuals enjoy in searching for goods, and particularly in experimenting novelty and surprise in consumption. In this perspective, search is not considered a cost (a negative element in the utility function), but rather an important part of the utility itself of agents. This idea, fully developed in Bianchi (1998) is consistent with various institutional and neo-Austrian theories of consumption. It has clearly a linkage with theories of social communication. As the editor explicitly says, “the social dimension of the consumer- which includes the communication side of individual choices- must recover a more refined and articulated place in the motivational structure of consumer decision procedures” (Bianchi, 1998, 8).

These arguments are reinforced by findings in other areas of social sciences, namely behavioral decision theory and cognitive psychology. Much work has been devoted to the discovery and experimental reproduction of anomalies in individual choice behavior in a variety of settings, among which consumer behavior is a privileged area. Let us quote some of the most striking results that may add arguments about heterogeneity in consumer markets, without any attempt to be complete.

2.3.3 Reason-based choice (Shafir, Simonson and Tversky, 1993)

In economic theory, individuals’ preferences must be consistent. Standard rational choice theory does not provide an account of how preferences are formed and modified over time, but rather prefers to assume them as given (although Becker claims that standard utility theory can give an account of how tastes are formed, see Becker, 1997). A recent debate in cognitive psychology suggests that individuals come to prefer one option to another at the end of a mental process which is similar to the balance and counter balance of explicit arguments. A choice is made only when a sufficient amount of reasons are agreed upon for the choice being meaningful. This idea creates an explicit bridge between behavioral decision theory and communication theory, since it is likely that many reasons for choice are constructed through social interaction with other individuals.

2.3.4 Useless information (Bastardi and Shafir, 1998)

According to a recent stream of laboratory evidence in cognitive psychology, people engage frequently in search of useless information. With this word it is denoted that type of information which does not increase the information set of the agent. If information collection and processing is costly, useless information should not be searched for at all. Instead, individuals actively look for such an information. Again, information search may be not an instrumental activity, but an intrinsically pleasant one.

This evidence suggests that looking at information collection as an instrumental activity may be rather reductive. People probably get pleasure in enriching their set of pieces of information regarding the object of their choice, although the additional information does not change at all their preferences. Clearly, part of this information is socially determined: this piece of information may be useless to me, but I can use it for enhancing my position in the web of social communication.

Taken together, these findings support the notion that information diffusion after product innovation is indeed a heterogeneous process, in which several social groups engage with dif-

ferent intensity and with effects mainly localized within the group. Interestingly, there is no mathematically developed theory of such an empirically relevant behavior.

The argument might be pushed further, even at the risk of some exaggeration. In some cases, consumption decisions can be considered intermediate decisions, whose ultimate goal is not only the satisfaction of needs, but the building and maintenance of social relations. This seems to us an interesting perspective from which to explore consumption patterns: instead of viewing communication as instrumental to final consumption choices, we propose to reverse the polarity and look at consumption decisions as (at least partially) instrumental to the building of social identity of agents. Rather than arguments in the utility functions of agents, products may be seen as instrumental variables that agents manipulate, subject to a budget constraint, in order to fulfill social expectations and enjoy social communication.

In other words, consumption decisions may be seen as the preparation of the inventory of arguments that agents can use in social intercourse. There exists a relation between consumption decisions and role expectation of agents. If an agent is frequently referred to by other agents as a source of legitimate information, this agent must continuously update his experience and sources of information in order to keep in line with these expectations. If he does not fulfill the expectations, over time he will lose his position in the network of social relations.

We are not suggesting that consumption decisions are independent on preferences regarding the objects of choice (or their attributes). We are instead suggesting that in many cases consumption decisions are so socially embedded that it is more useful to look at the structure of social interaction in which individual consumers are placed, than to look at their intrinsic characteristics or the shape of their utility function. Therefore we do not advocate a radical over-determination of individual choices by social phenomena. Rather, we suggest to investigate the degree to which structural positions within a population of interacting individuals may influence consumption decisions.

3 Preliminary considerations: the models considered and some remark

As already pointed out, the present work considers the effects of heterogeneities in classical *deterministic* (stochastic effects are not considered) models for the social diffusion of innovations and/or the social diffusion of news and rumours. The models considered in this work and extended with the aim to investigate the effects of heterogeneity phenomena are the following:

- The model for *external innovation* (Fourth and Woodlock, 1960)
- The model for *internal innovation* (Mansfield, 1961)
- The *standard model for innovation* (Bass, 1969), of which the two previously mentioned models are the constitutive elements
- Some more epidemiologically based models (Bartholomew 1967, Dimitri 1987) in which spreaders are allowed to cease to spread.

The first three models will be often referred in the sequel as the "classical" models for the diffusion of news and/or new products. They still are the most frequently used models in the current marketing research (Mahajan et al. (1990a,b,1993), Givon et al. (1995)).

The fourth class of models is much less common. These models are straightforwardly derived from the standard epidemiological literature through the acknowledgement of some extra-epidemiological factors (the removal process, namely) acting during the innovation process. Although these enrichments do not seem to have played a particularly relevant impact on the specialised literature of marketing research, they deserve, in our opinion, a great deal of attention. Despite being interesting by themselves, as they introduce more interesting behaviours, such as equilibria different from market saturation and threshold effects, the heterogeneous variants of these models have been somewhat deeply investigated in the recent epidemiological literature. Therefore they make directly available to marketing scholars a well developed apparatus.

3.0.5 Social spread of information versus actual decision of adoption: a remark

A theory of the diffusion of an innovation should offer a joint explanation of two distinct processes: i) the process of acquisition by the individuals of the population of the relevant information on the involved product: this is mainly a social process of diffusion of information (concerning the involved product); ii) a decision process (the actual decision to adopt the product). The classical models of the marketing of new products, such as the Bass model, fully concentrate on the first aspect, and substantially neglect the second. Strictly speaking, the models for the diffusion of new products are, in effect, nothing else than models for the diffusion of an information in a social environment.¹ Therefore the present work may be essentially regarded as a work on the role of social heterogeneities in models for the social spread of information (news and rumours). We will not be concerned in the present paper with the role of the decision process and involved heterogeneities.

We apologize since now for a possible terminological confusion within the paper. In some cases, being talking about tools useful for the descriptions of a plethora of possible social phenomena, such as the diffusion of news, or of new products, or, finally, of infectious diseases, we somewhat mixed the terminology, borrowing freely from the epidemiological language when this seemed to us more effective, and viceversa shifting to marketing language otherwise. We hope this will not increase exponentially the degree of confusion of the paper.

4 Heterogeneities in innovation due to external information

4.1 The basic model for external diffusion

The basic model for the diffusion of innovation through external information (firstly introduced in the marketing literature by Fourt and Woodlock, 1960) is subsumed by the following ODE:

$$\dot{Y}(t) = \alpha(m - Y(t)) \quad (1)$$

(where $\alpha > 0$) plus the "typical" initial condition $Y(0) = 0$. In (1) Y denotes the cumulative number of adopters ("A" individuals: those who already received the new) at time t , m the market size, ie the size of the population of potential adopters (consequently the difference $X(t) = m - Y(t)$ represents the number of individuals who have not yet been informed, "N")

¹The representation of the decision process actually embedded in models à la Bass is the simplest possible: a constant fraction of those who received the information is assumed to actually adopt the product.

individuals), and α the *innovation* rate, which basically reflects the intensity of the source which spreads the information. As it is well known (1) gives rise to a progressive saturation of the market through a concave dynamics. The best way to introduce model (1), which will also be useful for subsequent developments, is the following: the acquisition of the new transfers, at any moment of time, individuals from the compartment of the "noninformed" to that of the "informed". Let us assume that the compartment of noninformed is depleted at a constant rate α . Then the transfer of material from one compartment to the other may be described by the following couple of differential equations:

$$\dot{X}(t) = -\alpha X(t) ; \dot{Y}(t) = \alpha X(t)$$

where: $\dot{X} + \dot{Y} = 0$, amounting to assume that the total population $m(t) = X(t) + Y(t)$ is constant over time.² Hence $Y(t) = m - X(t)$ which definitively leads to (1).

4.2 Consequences of heterogeneity

Let us now explicitly assume that the population is *heterogenous* from the point of view of information acquisition, i.e. that the individuals of the adopting population are not exposed in the same ways to the information source and, as a consequence, are characterised by different time scales in their process of information reception.

Formally we will suppose that the population of adopters is stratified in n different groups depending on the intensity of their innovation rate α . The adoption process will hence be described by the following system of n independent equations, one for each segment of the population:

$$\dot{Y}_i(t) = \alpha_i(m_i - Y_i(t)) \quad i = 1, 2, \dots, n \quad (2)$$

where m_i is the size of the i -th group. Just to fix the ideas let us assume that $\alpha_1 < \alpha_2 < \dots < \alpha_n$.³

The qualitative behaviour of each equations (2) is identical to that of the basic model: each separate group will saturate to its market size with the same qualitative path of the basic model (2). Quantitatively different modalities will occur however, due to the different time scales existing in the adoption process of the several groups.

Even very simple cases, as the present one, give us quite remarkable informations. Let us start by deriving the full market dynamics by adding up all the (2): the goal of this operation is to investigate the behaviour of the aggregate innovation rate in presence of heterogeneities. We obtain the aggregate equation:

$$\begin{aligned} \dot{Y} &= \sum_i \alpha_i(m_i - Y_i(t)) = \\ &= \left(\sum_i \alpha_i \frac{(m_i - Y_i(t))}{m - N(t)} \right) (m - N(t)) = \alpha(t)(m - N(t)) \end{aligned} \quad (3)$$

where $\alpha(t)$ is the aggregate innovation rate, thereby defined as:

$$\alpha(t) = \sum_i \frac{\alpha_i(m_i - Y_i(t))}{m - Y(t)} = \sum_i \alpha_i \frac{X_i(t)}{X(t)} = \bar{\alpha}(t) \quad (4)$$

²We have just introduced the way to represent the process at hand: a pool of "virgin" individuals which decays exponentially over time.

³There are no difficulties in treating heterogeneities in a continuous framework, rather than discrete as the present one.

The last equation shows that the aggregate innovation rate observed during the dynamics is not constant. More precisely, the aggregate innovation rate appears to be defined as the weighted average⁴ of the several innovation rates of the different groups weighted with the fractions of noninformed in the various groups. Which is the actual dynamics of $\bar{\alpha}(t)$? The following remarkable facts hold:

•

$$\bar{\alpha}(0) = \sum_i \alpha_i \frac{X_i(0)}{X(0)} = \sum_i \alpha_i \frac{m_i}{m} \quad (5)$$

• The aggregate innovation rate strictly decreases over time⁵:

$$\frac{d}{dt}(\bar{\alpha}(t)) = (-1)Var_t(\alpha) < 0 \quad (6)$$

where:

$$Var_t(\alpha) = \left(\sum_i \alpha_i^2 \frac{X_i(t)}{X(t)} - (\bar{\alpha}(t))^2 \right) \quad (7)$$

• The asymptotic aggregate innovation rate coincides with the innovation rates of the slowest group (the true "laggards" of the process):

$$\bar{\alpha}(\infty) = \sum_i \alpha_i \pi_i(\infty) = \alpha_1 \quad (8)$$

Consequently the time scale of the market saturation process will necessarily coincide with the time scale typical of the slowest group.

The proof of the basic result (6) is straightforward:

$$\frac{d(\bar{\alpha}(t))}{dt} = \frac{d}{dt} \sum_i \alpha_i \frac{X_i(t)}{X(t)} = \sum_i \alpha_i \frac{\dot{X}_i(t)X(t) - X_i(t)\dot{X}(t)}{X^2(t)} \quad (9)$$

Noticing that:

$$\dot{X}(t) = - \sum_i \alpha_i X_i = -X(t) \sum_i \alpha_i \frac{X_i}{X(t)} = -\bar{\alpha}(t)X(t)$$

we obtain:

$$\begin{aligned} \frac{d}{dt}(\bar{\alpha}(t)) &= \sum_i \alpha_i \frac{(-\alpha_i X_i(t))X(t) + X_i(t)\bar{\alpha}(t)X(t)}{X^2(t)} = \\ &= \sum_i \alpha_i \frac{(-\alpha_i X_i(t)) + X_i(t)\bar{\alpha}(t)}{X(t)} = \\ &= (-1) \left(\sum_i \alpha_i^2 \frac{X_i(t)}{X(t)} - \bar{\alpha}(t) \sum_i \alpha_i \frac{X_i(t)}{X(t)} \right) = \\ &= (-1) \left(\sum_i \alpha_i^2 \frac{X_i(t)}{X(t)} - (\bar{\alpha}(t))^2 \right) = (-1)Var(\alpha) \end{aligned}$$

⁴This implies that the dynamics of the aggregate innovation rate will actually be bounded in the set (α_1, α_n) .

⁵More detailed analytical informations on the dynamics of the aggregate innovation rate may be obtained by studying the dynamical behaviour of the weights π_i . Some calculations are reported in the appendix.

The proof of (8) is elementary and is postponed in the appendix where short considerations on the formal dynamics of independent exponential populations are added.

What are consequences of these facts? As we have seen heterogeneity does not introduce any qualitative difference with respect to the basic homogeneous model. Rather it is essentially responsible of quantitative differences.

A good way to illustrate them is through the notion of *aggregation bias*. Let us suppose we are unaware of the heterogeneous structure of the population and would try to estimate only a general innovation rate (wrongly applying a homogeneous model (1)) from sales dynamics. Independently on the length of the data set we would unavoidably overestimate the true innovation rate, thereby systematically underestimating the true time scale of the adoption process, and definitively the true market saturation time (a quite undesired effect). The behavioural explanation is simply that in presence of heterogeneity the initial dynamics of the innovation process will exceedingly reflect the role of the groups with faster adoption time scale.

This aspect is illustrated with more detail in the last section.

5 Mixing mechanisms and heterogeneities within the internal adoption process

5.1 The classical epidemiological model for internal diffusion

The basic model for diffusion of innovations through internal information, i.e. "word of mouth" contagion, goes back to the classical paper by Mansfield (1961) and it is subsumed by the following logistic ODE:

$$\dot{Y}(t) = \frac{q}{m}Y(t)(m - Y(t)) = rY(t)(m - Y(t)) \quad r = \frac{q}{m} \quad (10)$$

plus the initial condition $Y(0) = 1$ (needed to start the transmission process). The q coefficient ($q > 0$) has been defined by Bass as the *imitation coefficient*.

There exists several justifications of (10). The one we give here is typically epidemiological in nature and will be useful for subsequent developments. Let us assume that at every time t further spread of information totally depends on the rate of encounters between "A" (let us call them now "infected") and "N" ("susceptible" to the acquisition of the information) individuals. Let's make the following assumptions:

- each individual in the population encounters exactly C individuals p.u.t (independently on the fact he already adopted or not). C is the *meeting rate*, or the *rate of social activity* (defining the total rate of "social partners" encountered per unit time). The quantity C^{-1} defines the mean between two different episodes of encounters.
- the pattern of encounters be perfectly at random
- there exists a constant probability β that a meeting between an "A" individual and an "N" individual gives rise to a new infection, i.e. to a new "A".

The previous three assumptions describe a population of (homogenous) individuals mixing homogeneously. The absolute rate of change in the number of infectious individuals (whose who received the information), is obviously given by the number of new infections per unit time.

This latter can be easily evaluated by tracing the impact of the single infected individual⁶, and then by cumulating the action of all the infected individuals. The single infected individual meets C individuals p.u.t. As individuals mix homogeneously among them, a fraction $S(t) = X(t)/(X(t) + Y(t))$ of these C encounters will take place with susceptibles ($0 \leq S(t) \leq 1$ is the *susceptible fraction*), and hence will be a possible source of new infections. A fraction β of these encounters will result in new infections. The quantity $\beta CS(t)$ hence defines the load of new infections caused by a single infected individual. By finally adding up the actions of all the infected individuals we find the total load of new infections p.u.t:

$$B(t) = \sum_{i=1}^{Y(t)} \beta CS(t) = Y(t)\beta CS(t) = \beta C \frac{X(t)Y(t)}{X(t) + Y(t)} \quad (11)$$

By defining $\beta C = q$, and using the relations: $X(t) + Y(t) = m$ (total population size), $Y(t) = m - X(t)$, the basic equation (10) is quickly obtained. In the epidemiological jargon model (10) is called an SI ("*susceptible-infective*") model. Notice that the epidemiologically based foundation followed here leads to define the *imitation coefficient* as the product between two constitutive parameters, the meeting rate C prevailing in the population times the probability of infection per single encounter β . Let's remark that the imitation coefficient is the rate of growth of the initial exponential phase of the dynamics of (10).

5.2 A preliminary case: heterogeneities only in the process of acquisition of information but non in the social interaction process

Let us consider first a simplified situation in which individuals are still assumed to mix homogeneously, as in the basic model (10), but differ in what concerns the speed at which their acquire the information⁷ from encounters (epidemiologically we would say that individuals differ among them in terms of their susceptibility). This leads to the following extension of (10)⁸:

$$\dot{Y}_i(t) = r_i(m_i - Y_i(t))Y(t) \quad i = 1, 2, \dots, n \quad (12)$$

where: $m_i - Y_i(t) = X_i(t)$, $r_i = q_i/m$, and obviously:

$$\dot{X}_i(t) = -r_i(m_i - Y_i(t))Y(t) = -r_i X_i(t)Y(t) \quad i = 1, 2, \dots, n$$

where again $Y(t) = \sum_i Y_i(t)$.

The equations (12) are, compared to the previous case of external diffusion, more difficult to treat. Nonetheless the main qualitative feature of the basic model (10), asymptotic saturation of the whole population, still holds.

For model (12) considerations quite similar to those developed for the model for external diffusion hold. Let us assume for simplicity $r_1 < \dots < r_n$. The aggregate dynamics is described by:

$$\begin{aligned} \dot{Y}(t) &= Y(t) \sum_i r_i(m_i - Y_i(t)) = X(t)Y(t) \sum_i r_i \frac{X_i(t)}{X(t)} = \\ &= r(t)X(t)Y(t) \end{aligned} \quad (13)$$

⁶We may obtain the same result by working on susceptibles.

⁷Hence in a way completely similar to that considered in (2).

⁸In this case the overall susceptible fraction at any time is given by: $(X_1 + X_2)/m$. Hence the action of the single susceptible individual leads to $\beta C(X_1 + X_2)/m$ new infections per unit time. Of these $\beta C X_1/m$ will take place in group one and the remaining in group two.

where $r(t)$ is given, as expected, by:

$$r(t) = \sum_i r_i \frac{X_i(t)}{X(t)} = \sum_i r_i \pi_i = \bar{r}(t) \quad (14)$$

Hence, quite similarly to the external diffusion case, the overall r coefficient results to be defined as the mean of the r_i weighted with the (time varying) susceptibility weights. The following facts, which straightforwardly extend results from the case of external diffusion, hold:

•

$$r(0) = \sum_i r_i \frac{X_i(0)}{X(0)} \cong \sum_i r_i \frac{m_i}{m} \quad (15)$$

•

$$\left(\frac{d}{dt} \bar{r}(t) \right) = (-1) Y(t) \text{Var}_i(r) \quad (16)$$

The proof of (16) is straightforward and it is postponed to the appendix. Its substantive meaning is that the aggregate imitation coefficient is strictly decreasing over time, for reasons identical to those discussed in the section on external diffusion.

- The model exhibits an initial phase of exponential growth at the rate:

$$r(0)X(0) = m \sum_i r_i \frac{m_i}{m} = \sum_i r_i m_i = \sum_i \frac{q_i}{m} m_i = \sum_i q_i$$

which is exactly the arithmetic mean of the imitation rates specific to the several groups weighted with the initial susceptible weights.

- For what concerns the time scales needed to the saturation of the market results quite similar to the case of external diffusion hold.

Definitively, the same type of aggregation biases discussed in the basic model for external diffusion are observed in the model (12) as well.

5.3 Heterogeneities in social activity: the mixing problem

Do individuals participating to the process of social diffusion of information exhibit different imitation patterns? And if yes, which are the concrete patterns by which individuals interact in the social process of imitation? And, finally, which consequences arise from a wrong homogeneous treatment of an underlying heterogeneous situation? Classical new product diffusion theories being homogeneous in nature do not have answers to such questions.

The introduction of heterogeneities in the rates of social activity makes the internal mechanism much more complex and interesting of the external one: a new problem, the so called mixing problem, largely investigated in the recent epidemiological literature, necessarily arise. Heterogeneity and the mixing problem ("who mixes with whom?") have represented a core problem in the field of mathematical epidemiology in the last 10-15 years, roughly corresponding to the world onset of HIV. A key motivation to the birth of a general theory of heterogeneity and mixing (HMT since now on) in epidemiology has been in fact represented by the need to provide flexible tools for the mathematical analysis of HIV and other sexually

transmitted diseases (STD), where a central aspect of the dynamics is ”...*the marked heterogeneity in degrees of sexual activity within the overall population*” (Anderson and May 1991, 228). A huge literature has been developed since then, of which we may quote here only a very small, although highly representative, sample. Even if an overall HMT in theoretical epidemiology is still far from being complete, nonetheless several quite general problems have been nowadays quite well understood.

Although most of the efforts produced up to now have been connected with the analysis of STD, the HT approach is completely general and it may be applied to fairly different situations involving general types of social interactions. In very recent times several serious tentatives (Castillo-Chavez et al. (1995), Edmunds et al. (1997)) have been made aimed to extend the basic framework for STD to other areas, ranging from nonsexually transmitted diseases to general contact patterns in biology.

It is not out intention in what follows to provide an extensive review of the existing state of the HMT in epidemiology. This operation would be too long and complex for our objectives. Rather we will try to show how the recent epidemiological developments could be transferred into the area of marketing models. We will develop the following points (corresponding to the subsections of the present section):

- We will first introduce the tool-box of mixing theory, by showing how the HM approach provides a very convenient framework for the investigation of the effects of heterogeneities within internal diffusion models, in that it clarifies which are the crucial parameters in the representation of social interaction process: *activity levels* plus *mixing functions*. During this phase we will systematically exploit background materials derived from the recent epidemiological literature (for instance Anderson and May (1991), or Jacquez et al. (1995)). In particular we will see how the typical epidemiological representation, as used for (10), suggests a parameterisation of the diffusion process which could be quite fruitful in applied marketing work.

A brief excursus will then be made on the ”mathematical mixing theory”, a highly theoretical subject developed from the concept of mixing functions and the connected notion of *consistency requirement*.

- Some highly stylized prototype models of mixing patterns of the most frequent use in the recent epidemiological literature, useful to organize the boundaries of the discussion, will then be presented.
- By exploiting to the typical mixing of parameters we will show how the basic simple model (10) for internal diffusion modifies when social heterogeneity is explicitly assumed.
- We will then report some interesting examples of the effects of heterogeneities taken from the recent epidemiological literature.

The fact that a wrong homogenous treatment of an underlying heterogeneous situation may give rise to dramatically wrong predictions, i.e. that ”heterogeneity matters”, raises perhaps the main question of the present paper (a question we intend to develop in subsequent theoretical and field work): how do individuals mix in the actual processes of social diffusion of innovations?

5.4 Mixing frameworks; mixing parameters

To formulate our model of internal diffusion with heterogeneity we will not merely assume that the population is subdivided in groups depending on their different imitation coefficients. This does not seem to constitute a transparent nor a fruitful approach. Rather, as suggested by the typical epidemiological approaches (and implicit in our "epidemiologically based" formulation of (10)), we will assume, as a starting point, that the individuals of the involved population are stratified on the basis of their different rates of *social activity*. More precisely we assume that social activity is characterised by a prescribed distribution of *activity levels*:

$$C = \begin{matrix} C_1 & C_2 & \dots & C_k \\ m_1 & m_2 & \dots & m_k \end{matrix} \quad (17)$$

where C_i is the number of encounters⁹ p.u.t. of individuals belonging to group i and $\sum m_i = m$ is the total population. Sometimes it may also be convenient to think of C as a continuous variable, with classes $[C_{i-1}, C_i)$ and so on.

The heterogeneity in the levels of social activity of the involved individuals appears, in absence of a strong empirical evidence, a quite natural heterogeneity in the diffusion processes relevant for the marketing of new products, and in general for processes of diffusion of informations. Once this heterogeneity has been recognised, the definition of the rate of change over time in the number of adopters requires the specification, at the very least, of the *functions of social interactions* (the *mixing functions* studied in the recent epidemiological literature) p_{ij} , denoting the fraction of his C_i encounters p.u.t that the generic individual of group i has with individuals belonging to group j . Provided all the groups are socially active, the mixing functions satisfy the following properties (sometimes called the mixing axioms in the more theoretically involved developments):

- $p_{ij} \geq 0$
- $\sum_{j=1}^n p_{ij} = 1$
- $C_i m_i p_{ij} = C_j m_j p_{ji}$

The first two properties quickly follow from the probability nature of the mixing functions, connected with the process of allocation of the social activity of the various groups:

$$\begin{aligned} C_1 p_{11} + C_1 p_{12} + \dots + C_1 p_{1k} &= C_1 \\ C_2 p_{21} + C_2 p_{22} + \dots + C_2 p_{2k} &= C_2 \\ \vdots & \\ C_k p_{k1} + C_k p_{k2} + \dots + C_k p_{kk} &= C_k \end{aligned}$$

The last property has a more substantive nature and it represents a kind of *law of conservation of social activity*.¹⁰ The physical meaning of the relations $C_i m_i p_{ij} = C_j m_j p_{ji}$ is simply that the social activity (i.e. the number of relations) of individual type "i" with individuals type "j"

⁹We will sometimes call the number of encounters as the *number of relations*. Let us simply assume that these encounters are of a type which is adequate for the transmission of information. In the mathematical theory of STD this quantity typically represents the mean number of (new or total) sexual partners p.u.t. of an individual belonging to group i .

¹⁰The case we are considering is actually the simplest mixing framework: discrete groups all having social activity. Several other cases have been treated in detail.

cannot be different from the total number of relations of "j" individuals with "i" individuals. As these relations involve quantities which are not necessarily constant, they represent *concrete restraint operating at every time during the social process*. Let us assume that for some reason the number m_j of individuals in group j should diminish, coeteris paribus the numbers in the other groups: this necessarily implies that all the restraints involving group j are not anymore satisfied and that, therefore, something has to change to preserve the restraints satisfied. To stress the dynamical nature of the mixing axioms it is useful to write therefore: $C_i(t)m_i(t)p_{ij}(t) = C_j(t)m_j(t)p_{ji}(t)$.

Remark 1 *In compact terms the overall process of social interaction may be represented by the so called mixing matrix: $P = [p_{ij}]$ (which is obviously a markov matrix).*

Heterogeneities in social activity do not exhaust the range of possible heterogeneities in our framework. A possible further source of heterogeneity may be represented by the probability of infection per single encounter. For a maximum of generality we will denote by β_{ij} the probability that a single encounter between an infected individual of group "j" and a susceptible of group "i" gives rise to a new infection. This type of heterogeneity is, a priori, indisputable. Nonetheless, in this initial part of the discussion, we will devote scarce attention to it, by concentrating our attention on the heterogeneities in social activity. In the subsequent developments β_{ij} will always be taken as constant across groups. When $\beta_{ij} = \beta$ the overall characterization of the mixing problems rely on the following triple of parameters (C_i, m_i, p_{ij}) .

Finally a related important notion is that of *contact matrix*, which is nothing that the matrix having for elements the actual number of contacts taking place between pair of groups in a given community characterised by a prescribed triple (C_i, m_i, p_{ij}) :

$$CM = \begin{pmatrix} C_1 m_1 p_{11} & C_1 m_1 p_{12} & \dots & C_1 m_1 p_{1n} \\ C_2 m_2 p_{21} & C_2 m_2 p_{22} & \dots & C_2 m_2 p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_n m_n p_{n1} & C_n m_n p_{n2} & \dots & C_n m_n p_{nn} \end{pmatrix}$$

Remark 2 *The approach to social mixing presented here is taken as simplest as possible. The unique source of heterogeneity considered is the pure level of social activity, measured by an index of speed in social circulation. But nothing prevents from much more involved treatment. For instance several covariates could be necessary for the description of the mixing patterns. Typical examples could be age (a), or income (y), or geographic location (s) and so on. In these cases we would only have to stratify the triple (C_i, m_i, p_{ij}) with respect to these variables, by considering the functions $C(a, y, \dots), m(a, y, \dots), p(a, y, a', y')$ but without further problems in the description.*

Remark 3 *In applied epidemiology, the previous tools have been, up to the present times, essentially applied in the study of STD. As our presentations clearly shows, in effect the approach is completely general in that it may be applied to every type of social interactions and not only to STD's.*

5.5 Mathematical mixing theory

When we explicitly assume that the quantities m_i, C_i are known, i.e. are taken as the data of the problem, then the mixing axioms define a system of well-posed functional equations (the third axiom) having for unknowns the mixing functions p_{ij} :

$$C_i m_i p_{ij} = C_j m_j p_{ji}$$

under the restraints given by the first two axioms:

$$p_{ij} \geq 0, \sum_{j=1}^n p_{ij} = 1$$

The mathematical analysis of such equations, and several generalisations, has been attacked by Busenberg, Castillo-Chavez and coworkers in a series of recent important papers (Busenberg and Castillo-Chavez (1989, 1991a,b)), Blythe et al. (1991, 1995), Castillo-Chavez et al. (1995)).

In particular it has been proven (Busenberg and Castillo-Chavez (1989, 1991a)) that the mixing problem has one and only one separable solution (i.e. a solution p_{ij} which can be factored in the form: $p_{ij} = A_i B_j$), which is given by the proportionate mixing function discussed in the next section. The role of the proportionate mixing solution is central, in that it can be proven, moreover, that all the solutions of the problem can be represented as special multiplicative perturbations of the basic PM solutions. These two basic results have been definitively the starting point for the development of a unified theory of mixing. In fact, although their nature is highly theoretical, several consequences arise which have a lot of potentially fertile applications. For instance an important recurrent question in social theory has been for instance how to measure the degree of social affinity between different social groups. Clearly, in a framework as the present one, the p_{ij} 's do not answer the question: they are simply an observed footprint of the entanglement of a plethora of forces. Viceversa, thanks to the aforementioned representation formula, it is possible to put in evidence a whole set of stable *latent* parameters φ_{ij} , termed affinities (Blythe et al. 1991), which in a reasonably large set of situations represent measures of the degree of social interactions between groups.

5.6 Special mixing structures

Even if the mixing theory has provided, among other results, a general representation formula of mixing patterns, able, at least in principle, to represent in a unitary way whatever actual mixing pattern, its practical application still poses formidable problems. Therefore, the analysis of special cases is of great help in defining the boundaries of the problem. Noteworthy examples of mixing structures employed in the recent epidemiological literature (Jacquez et al.(1989), Uche and Anderson (1996)), are the following:

1. *Restricted mixing (perfect assortativeness or "like with like")*: 100% of the social activity of the individuals of a given group is confined within their own group. The corresponding form of the mixing matrix is the identity matrix:

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \quad (18)$$

2. *Proportionate mixing (P.M.)*: the concept of PM, first introduced in theoretical epidemiology by Barbour (1978) and Nold (1980), extends to heterogeneous situations the notion of homogeneous mixing with random selection of encounters. To understand this notion let us first consider the limit case in which all individuals in all groups have the same rate of social activity, i.e. the same number of relations per unit time (per day

for instance). In this case, provided individuals choose their relations at random, the mixing functions p_{ij} would simply have the form:

$$p_{ij} = \frac{N_j}{N} = p_j \quad (19)$$

as the probability for a "i" individual to encounter a "j" individual would only depend on the relative frequency of "j" individuals.

Let us now consider the more general case in which individuals are stratified, as it actually is the case, on the basis of their rates of social activity. In this case, in the computation of the mixing functions, we need to take into account of the different rates of social activity. Let us consider the case of a community compounded by two groups of identical size; individuals of the first group have a low rate of social activity, while individuals of the second group have a large rate of social activity. This means that the group of "very social" individuals contribute more than the other group to the the overall social activity of the community. This implies in turn that if an individual choose a relation at random, he will have a larger probability to meet an individual from the "very social" rather than from the "unsocial" group. This leads to the following definition:

Definition 1 (*proportionate mixing*) Given the distribution of activity levels $C = \{C_i; m_i\}$, the corresponding proportionate mixing function is given by:

$$p_{ij}^* = \frac{C_j m_j}{\sum_{j=1}^n C_j m_j} \quad i = 1, 2, \dots, n \quad (20)$$

The meaning of the last expression is clear: the quantity $C_i m_i$ is a measure of the number of social relations by "i" individuals, while $\sum C_i m_i$ is a measure of the total social activity taking place in the overall community.

1. *Preferred mixing*: in this type of mixing (Jacquez et al. 1988) an arbitrary fraction h_i of each group's contacts are reserved for within-group contacts; the remaining contacts of each group (given by: $(1 - h_i)C_i N_i$) are subject to the proportionate mixing rule. Hence:

$$p_{ij} = h_i \delta_{ij} + (1 - h_i) \frac{(1 - h_j) C_j N_j}{\sum_{j=1}^K (1 - h_j) C_j N_j} \quad (21)$$

where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

2. *Perfectly disassortative mixing*: we say that a mixing pattern is perfectly disassortative when all the social activity of a group is allocated within a unique different group. The mixing matrix has therefore the structure of a non-identical zero-one elements matrix. The matrix:

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

constitute the simplest example of a fully disassortative pattern.

5.7 The model for internal diffusion embedding heterogeneity and mixing

By using the three sets of parameters C_i, p_{ij}, β_{ij} defined in the previous section, the rate of change over time in the number of adopters may be computed following the same reasoning used for the simpler model (10). The final model is given by:

$$\dot{Y}_i = C_i(m_i - Y_i) \sum_j \beta_{ij} p_{ij} \frac{Y_j}{m_j} \quad (22)$$

5.7.1 Justification of the previous formulation

The justification of (22) follows the same arguments previously used for the homogeneous case (10). Let us therefore follow the load of new "infections" caused p.u.t by the single infected individual belonging to group j . Such individual encounters C_j (new) individuals p.u.t.; of these C_j encounters a fraction p_{ji} is allocated within group i , i.e. $C_j p_{ji}$ relations are with type "i" individuals. Let us assume, just to begin, that the probability of infection per single encounter be constant (β) across groups; therefore our infected individual would cause βC_j new infections p.u.t, of which $\beta C_j p_{ji}$ within group "i", provided all the encounters are with "susceptible" individuals (i.e. individuals who have not yet received the information). Clearly not all encounters will be with susceptible individuals; at time t in group i there will be still X_i susceptible individuals on a total number of $m_i = X_i + Y_i$. Therefore, by assuming that the mixing process between "i" and "j" individual is perfectly at random, our single "j" infected individual will meet p.u.t $C_j p_{ji} (X_i/m_i)$ susceptible individuals in group i , causing $\beta C_j p_{ji} (X_i/m_i)$ new infections. To finally find the total load of new infections caused p.u.t by all the infected individuals in group j within group i , we have to multiply the previous quantity by Y_j , leading to:

$$B_{i,j}(t) = Y_j \beta C_j p_{ji} \frac{X_i}{m_i} = \beta C_j p_{ji} Y_j \frac{X_i}{m_i} \quad (23)$$

Last, to find the total number of new infections p.u.t in group "i" we have to add up all the quantities $B_{i,j}(t)$ with respect to all groups of possible infected "partners":

$$B_i(t) = \sum_j B_{i,j}(t) = \sum_j \beta C_j p_{ji} Y_j \frac{X_i}{m_i} \quad (24)$$

The case of a β variable across groups does not change the point. We simply have:

$$B_i(t) = \sum_j B_{i,j}(t) = \sum_j \beta_{ij} C_j p_{ji} Y_j \frac{X_i}{T_i} \quad (25)$$

The last quantity may also be expressed as follows. Notice that:

$$B_i(t) = X_i \sum_j \beta_{ij} C_j p_{ji} \frac{Y_j}{T_i} \quad (26)$$

As, from the third mixing property, we have: $m_i C_i p_{ij} = m_j C_j p_{ji}$, it holds:

$$C_j p_{ji} = \frac{m_i C_i p_{ij}}{m_j}$$

Therefore we may write (25) as follows:

$$\begin{aligned}
B_i(t) &= X_i \sum_j \beta_{ij} C_j p_{ji} \frac{Y_j}{m_i} = X_i \sum_j \beta_{ij} \frac{m_i C_i p_{ij}}{m_j} \frac{Y_j}{m_i} = \\
&= X_i \sum_j \beta_{ij} C_i p_{ij} \frac{Y_j}{m_j} = C_i X_i \sum_j \beta_{ij} p_{ij} \frac{Y_j}{m_j}
\end{aligned} \tag{27}$$

The quantities (25) or (27) define the total rate of change per unit time in the number of infected individuals (who have received the information) in group i . Remembering that: $X_i = m_i - Y_i$ the equation (22) is definitively obtained.

Remark 4 (*the imitation rates in presence of heterogeneities*) *The contribution to the rate of change of the number of infecteds of group i in (22) due to encounters with individuals belonging to group j :*

$$\beta_{ij} C_i p_{ij} \frac{Y_j}{m_j} (m_i - Y_i)$$

shows that the rate of imitation of "j" individuals by "i" individuals is given by the quantity:

$$q_{ij} = \beta_{ij} C_i p_{ij} \tag{28}$$

5.7.2 Some facts concerning the basic model for internal diffusion with mixing

Mathematically the model (22) constitutes an instance of a heterogeneous SI model. Its qualitative properties are quite easy to understand. Two main qualitative situations are of concern: i) the case in which all the groups are connected (i.e. the case of an imprimitive mixing matrix),¹¹ ii) the case in which isolated groups exist (a primitive mixing matrix). In the former case, independently on the initial conditions, the whole community will be reached by the new in the long term, while in the latter one isolated groups will never be informed unless an informed individual reaches the group at some stage (formally: if the initial condition $Y_i(0)$ of an isolated group is zero, than the epidemics will never start). Hence in a community characterised by restricted mixing, or in any case, by the presence of isolated groups, special care should be reserved by planners to reach isolated groups.

Viceversa it is quite hard to provide a synthetic view of the overall quantitative behaviour of the system (22) for arbitrary mixing patterns. By resorting to special assumptions on the mixing patterns, valuable information is nonetheless obtained. We report here an important results for the case of PM, useful for the understanding of a fundamental result that will be presented later.

The case of proportionate mixing An interesting information concerns the aggregate rate of growth which characterizes the initial exponential phase in the overall number of infectives/adopters (which is a counterpart of the imitation rate in the homogeneous model). By explicitly assuming that mixing patterns are of the PM type, and that $\beta_{ij} = \beta$ the equation (22) reduces to:

$$\dot{Y}_i = \beta^* C_i (m_i - Y_i) \sum_j C_j Y_j \tag{29}$$

or else:

$$\dot{Y}_i = \beta^* C_i m_i \sum_j C_j Y_j - \beta^* C_i Y_i \sum_j C_j Y_j \tag{30}$$

¹¹Of course the connection may be direct, by direct interaction, or indirect, due to the intermediate interaction with other groups.

where:

$$\beta^* = \frac{\beta}{\sum_j C_j m_j} \quad (31)$$

The initial exponential phase may be characterised by studying the eigenvalues of the Jacobian matrix $J(0)$ which characterizes the stability properties of the $(0, \dots, 0)$ equilibrium of model (22). Since:

$$J(0) = \beta^* \begin{pmatrix} m_1 C_1^2 & m_1 C_1 C_2 & \dots & m_1 C_1 C_n \\ m_2 C_2 C_1 & m_2 C_2^2 & \dots & m_2 C_2 C_n \\ \vdots & \vdots & \ddots & \vdots \\ m_n C_n C_1 & m_n C_n C_2 & \dots & m_n C_n^2 \end{pmatrix}$$

we are quickly leaded¹² to the characteristic equation:

$$\lambda^n - \left(\beta^* \sum_j C_j^2 m_j \right) \lambda^{n-1} = 0$$

This implies that the (dominant) eigenvalue which lead the initial exponential case is given by:

$$\lambda_0 = \beta^* \sum_j C_j^2 m_j = \beta \frac{\sum_j C_j^2 m_j}{\sum_j C_j m_j} = \beta \left(E(C) + \frac{Var(C)}{E(C)} \right) \quad (32)$$

The result (32) is of special interest (we will find a generalisation in the forthcoming model with removal) as it indicates the special role played in heterogeneous phenomena by highly socially active groups.

5.7.3 Internal diffusion: the case of spreaders who cease to spread

The possibility of (permanent) removal from the state of spreader in models of diffusion of news was considered by Bartholomew (1970, 223), borrowing from theoretical epidemiology the so called model of general epidemics (which dates back to the seminal contribution by Kermack and McKendrick (1927)): "*People may cease to be spreader for a variety of reasons: they may forget, loose interest, or gain the impression that every body knows*". The explicit introduction of permanent removal modifies the SI internal diffusion process into a classical SIR epidemiological process. The internal diffusion model with removal takes then the form:

$$\begin{aligned} \dot{X} &= -\frac{q}{m}XY \\ \dot{Y} &= \frac{q}{m}XY - vY \\ \dot{Z} &= vY \end{aligned} \quad (33)$$

where we have distinguished the three distinct classes of susceptibles (X =those who have not yet received the information), infectives (Y =those who have been informed), and permanently removed (Z =those who cease to spread). As usual $q = \beta C$; v is the removal rate. Typical initial conditions are $X(0) = m - 1$; $Y(0) = 1$; $Z(0) = 0$.

In presence of heterogeneities, some nice results are available for the previous model from the recent epidemiological developments (Anderson and May (1988, 1991)), in particular for the remarkable case of proportionate mixing. To fully understand their effects it is useful to ground them against the properties of the basic homogeneous model (33), which we quickly recall in the next subsection.

¹²By developing the coefficients of the characteristic polynomial we see that the determinants of all the principal minors of orders 2, 3, ..., n vanish.

Basic facts concerning the homogeneous model with removal In the "classical" models of innovation diffusion, the diffusion dynamics is qualitatively very simple, unavoidably ending with the asymptotic saturation of the whole population. This is a necessary consequence of the fact that the spreading source never stops to spread and the whole population is homogeneously exposed in the model for external diffusion, and of the fact the spreaders never cease to spread (i.e.: they circulate for an infinitely long time compared to the diffusion time scales) and the population is seemingly homogeneously exposed in the model for internal diffusion. Both these characteristics are of course present within Bass model.¹³ In these models, therefore, the only questions of interest in applied works concerns purely quantitative aspects, such as the magnitudes of the relevant parameters.

Viceversa, when permanent removal is explicitly introduced two main news appear, which make predictions about market dynamics more complex:

- The epidemics/new will not necessarily spread. This will in fact actually happen only provided the removal rate is not too large compared to the imitation rate. Intuitively this is obvious: if the removal rate is larger, compared to the imitation rate, spreaders will get exhausted before being able to infect a relevant part of the susceptible population. Mathematically this fact is expressed via a remarkable *threshold theorem*¹⁴, leading among other things to a central epidemiological parameter, namely the so called *basic reproduction ratio (BRR) R_0* . The BRR expresses the number of secondary infections caused by a single infected individual during his whole infective period (i.e.:before being removed) in a wholly susceptible population. Hence R_0 naturally acts as a threshold parameter: when $R_0 < 1$ we expect the disease dies out, while when $R_0 > 1$ the disease will invade the population. In the case of model (33):

$$R_0 = \frac{\beta C}{v} \quad (34)$$

All relevant facts concerning model (33) are driven by R_0 . In particular the rate of growth r_0 of the initial exponential phase of the epidemics is linked to R_0 by the relation:

$$r_0 = v(R_0 - 1) = \beta C - v \quad (35)$$

- Even if $R_0 > 1$ (so that the disease invades the host population), the disease will not be able, in any case, to reach the whole population. In the long term the disease dies out without having infected the whole population. As expected, the fraction not infected depends inversely on R_0 .

5.7.4 Some results on the heterogeneous model with removal: the case of proportionate mixing

Let us now consider the heterogeneous version of (33):

$$\begin{aligned} \dot{X}_i &= -B_i(t) \\ \dot{Y}_i &= B_i(t) - vY_i \\ \dot{Z}_i &= fvY_i \end{aligned} \quad (36)$$

¹³The only qualitative difference lies in the fact that the model for internal diffusion exhibits an inflection point needed to reabsorb the initial phase of exponential growth, while the "external" model does not. This feature will be present or not within Bass model depending on whether internal forces prevail or not on the external ones.

¹⁴These theorem dates back to the historical papers by Ross (Ross and Hudson, 1917), and Kermack and McKendrick.

where:

$$B_i(t) = C_i X_i \sum_j \beta_{ij} p_{ij} \frac{Y_j}{m_j} \quad (37)$$

and the transmission parameter is assumed to be constant for simplicity: $\beta_{ij} = \beta$.

Even the apparently most innocent heterogeneous assumption, that of proportionate mixing, leads to striking quantitative results, compared to a reference homogeneous situation. In particular the following results by Anderson and May (1990, 1991) nowadays constitute a classical theorem of mathematical epidemiology

For the heterogeneous model (36)-(37) it holds:

$$R_0 = \frac{\beta C^*}{v} \quad (38)$$

where:

$$\begin{aligned} C^* &= \frac{\sum_{j=1} C_j^2 N_j}{\sum_{j=1} C_j N_j} = \frac{E(C^2)}{E(C)} = \frac{E(C)^2 + Var(C)}{E(C)} = \\ &= E(C) + \frac{Var(C)}{E(C)} \end{aligned} \quad (39)$$

The last result is partly explained by our previous findings. The actual BRR (and hence the initial rate of growth) of the heterogeneous model does not simply depend on the mean level of the distribution of levels of social activity $E(C)$: a variance term is also present which has the effect to strictly increase the BRR compared to the corresponding homogeneous situation. *"This result simply reflects the disproportionate role played by individuals in the more (socially) active groups, who are both more likely to acquire the infection and more likely to spread it"* (Anderson and May (1990,287). This variance effect has proven to be very valuable in the study of the transmission dynamics of STD, where the patterns in the distribution of levels of sexual activity usually generate very large values of the variance (ibidem).

Mutatis mutandis we may reformulare the last sentence in terms of diffusion of news, just by observing that heterogeneities exaggerate the role of the more socially active groups (the "superspreaders") who are both more likely to acquire the new and to retransmit it.

- The asymptotic fraction of the population which experiences the disease in the long term, is a strictly decreasing function of the degree in heterogeneity in social activity. As Anderson and May (1990) were noticing: *"This is essentially because, other things being equal, the epidemics tend to burn itself out among those in the highly active classes, thus driving the effective value of the BRR below unity before a large fraction of those in the low activity classes have been infected"*.

This last result in particular appears of crucial relevance for the marketing of new products. Its substantive meaning in terms of diffusion of news is the following: when within the population there are groups which are very socially active, the information tends to spread very quickly among these groups. But if permanent removal exists, these individuals will also, soon or later, cease to spread the new, leaving then the charge to continue to spread the information to the less active groups, those characterised by the lowest degrees of social activity. This fact will sharply reduce the intensity of the disease/diffusion of the new in the long term, thereby reducing the *actual reproduction ratio*, and finally leading to a smaller final prevalence. But a smaller prevalence

5.7.5 Other results

The results of the previous subsections do merely constitute an instance of the actual consequences that heterogeneities may generate. They are in fact derived under one special assumption on mixing patterns and hence do not at all exhaust the problem of the overall effects of heterogeneities. Still presently the understanding of the mathematical properties of complex SIR models as (36) in presence of general mixing patterns is far from being complete (Jacquez et al. 1988, Jacquez et al. 1995; viceversa the understanding of simpler models, such as SI and SIS is much more complete, see Jacquez and Simon (1992), Jacquez et al. (1995)). The detailed classification of the whole range of possible dynamics of (36) in the domain of possible mixing patterns is still, therefore, a very far result.

Nonetheless several important problems have been quite fairly well understood. In particular the theoretical foundations of the central concept of basic reproduction ratio have been strongly deeped and a general solution to the problem of the computation of R_0 in presence of a general heterogeneity pattern has been given (Diekmann et al. 1990, Heesterbeek and Dietz (1996) and references therein).

6 Heterogeneities within some Bass-type models

In this section we merely list some general Bass-type models with heterogeneities the study of which will be presented elsewhere (Manfredi et al. 1998). No one of the models here described have represented, at least to our knowledge, a specific research interest in the marketing research literature.

6.1 A Bass-type model with heterogeneities in susceptibility

This case merge the two basic cases (2) and (12), leading to the model:

$$\begin{aligned} \dot{Y}_i(t) &= \alpha_i(m_i - Y_i(t)) + r_i(m_i - Y_i(t))Y(t) = \\ &= (\alpha_i + r_iY(t))(m_i - Y_i(t)) \quad i = 1, 2, \dots, n \end{aligned} \quad (40)$$

As expected this case preserves some features of its components. We have in fact:

$$\dot{Y} = (\alpha(t) + r(t)Y(t))(m - Y(t)) \quad (41)$$

where:

$$\alpha(t) = \sum_i \alpha_i \frac{m_i - Y_i(t)}{m - Y(t)} = \sum_i \alpha_i \frac{X_i(t)}{Y(t)} ; \quad r(t) = \sum_i r_i \frac{m_i - Y_i(t)}{m - Y(t)} = \sum_i r_i \frac{X_i(t)}{Y(t)}$$

and:

$$\begin{aligned} \dot{X}_i(t) &= -(\alpha_i + r_iY(t))(m_i - Y_i(t)) = -(\alpha_i + r_iY(t))X_i(t) \\ \dot{X}(t) &= -(\alpha(t) + r(t)Y(t))(m - Y(t)) = -(\alpha(t) + r(t)Y(t))X(t) \end{aligned}$$

In particular it quickly follows:

$$\frac{d}{dt}(\alpha(t)) = \sum_i \alpha_i \frac{(-(\alpha_i + r_iY(t))X_i(t))X(t) - (-1)X_i(t)(\alpha(t) + r(t)Y(t))X(t)}{X^2(t)} =$$

$$\begin{aligned}
&= (-1) \left[\sum_i \alpha_i \frac{(\alpha_i + r_i Y(t)) X_i(t)}{X(t)} - (\alpha(t) + r(t) Y(t)) \sum_i \alpha_i \frac{X_i(t)}{X(t)} \right] = \\
&= (-1) \left[\left(\sum_i \alpha_i^2 \frac{X_i(t)}{X(t)} - \alpha^2(t) \right) + Y(t) \sum_i \alpha_i (r_i - r(t)) \frac{X_i(t)}{X(t)} \right] = \\
&= (-1) [Var_i(\alpha) + Y(t) Cov_i(\alpha, r)]
\end{aligned}$$

6.2 A Bass-type model with mixing

By combining the basic models (2) and (22) we obtain the more general formulation:

$$\dot{Y}_i = \alpha_i(m_i - Y_i) + C_i(m_i - Y_i) \sum_j \beta_{ij} p_{ij} \frac{Y_j}{m_j} \quad (42)$$

where the first term reflects external diffusion, and the second term internal diffusion.

6.3 The Bass model with removal

By introducing external diffusion as well within the model (36) we obtain the general SIR type model:

$$\begin{aligned}
\dot{X}_i &= -B_i(t) \\
\dot{Y}_i &= B_i(t) - vY_i \\
\dot{Z}_i &= fvY_i
\end{aligned} \quad (43)$$

where transfer from susceptible to infectious contains also an external diffusion term:

$$B_i(t) = \alpha_i(m_i - Y_i) + C_i X_i \sum_j \beta_{ij} p_{ij} \frac{Y_j}{m_j} \quad (44)$$

7 Numerical illustrations

In this section we present several examples drawn from the previous discussion. Their aim is to provide practical illustrations of the effects of heterogeneities.

7.1 The model for external information

As already pointed out, in this case no qualitative differences arise with respect to the corresponding homogeneous model. The effects of heterogeneities are purely quantitative and lie essentially in the aggregation biases which arise every time we wrongly treat as homogeneous the true underlying heterogeneous situation.

We already pointed out in the previous pages how this can result in serious overestimation of the true nonconstant adoption rate of the overall market which in turn may cause severe underestimation of the time scales needed to the saturation of the market. A good indicator of the extent of these aggregation biases can be found, therefore, in the length of the times interval needed to the coverage of prescribed percentages of the overall population in the two cases. More precisely, the time needed to reach the $(1 - q)\%$ of the population in the homogeneous model (1) is given as the solution for the t parameter in the equation:

$$e^{-\alpha t} = q \quad (45)$$

	10%	50%	75%	90%	95%	99%
$\alpha(0)=0.027877$	3.77	24.8	49	83	107	165
$\alpha_1 = 0.08, \alpha_2 = 0.02735$ $w_1 = 0.01, w_2 = 0.99$ $\sigma^2 = 0.000027$	3.78	25.0	50	84	109	168
$\alpha_1 = 0.08, \alpha_2 = 0.02513$ $w_1 = 0.05, w_2 = 0.95$ $\sigma^2 = 0.000142$	3.81	26.0	53	90	117	181
$\alpha_1 = 0.08, \alpha_2 = 0.02210$ $w_1 = 0.1, w_2 = 0.9$ $\sigma^2 = 0.000176$	3.85	27.6	58	99	131	204
$\alpha_1 = 0.08, \alpha_2 = 0.01867$ $w_1 = 0.15, w_2 = 0.85$ $\sigma^2 = 0.0004792$	3.90	29.9	66	115	152	238
$\alpha_1 = 0.08, \alpha_2 = 0.01484$ $w_1 = 0.2, w_2 = 0.8$ $\sigma^2 = 0.0006791$	3.96	33.5	78	140	187	295
$\alpha_1 = 0.08, \alpha_2 = 0.00553$ $w_1 = 0.3, w_2 = 0.7$ $\sigma^2 = 0.0184440$	4.11	61.6	186	352	477	768

Table 1: Saturation times of different heterogeneous situations in a two groups model $\alpha(0) = 0.027877$.

In the heterogeneous model, where an analytical solution is still available, the same information is given as the unique solution of the transcendental equation:

$$H(t) = q \quad (46)$$

where:

$$H(t) = \sum_{i=1}^n e^{-\alpha_i t} \frac{m_i}{m} \quad (47)$$

In this first experiment we consider a grid of possible heterogeneous situations in highly stylized market situations, with $n = 2$ (a fast group and a slow group) or $n = 3$ subgroups. These situations are characterized by distributions at time $t = 0$ of the innovation rates characterized by a common mean value¹⁵ and by increasing levels of heterogeneity, measured through the variance σ^2 of the distribution of the innovation α rates.

All these heterogeneous cases are therefore characterized by the same aggregate innovation rate $\alpha(0)$ at time $t = 0$. The homogeneous model characterized by (constant) innovation rate $\alpha(0)$ acts therefore as a reasonable comparison benchmark against which to evaluate the effects of different underlying heterogeneous models. For $\alpha(0)$ chosen the value $\alpha(0) = 0.027877$ (drawn from Bass,1969). In table 1 are reported the values of the times needed to reach different market shares (obtained as solutions of equations (45) and (47)) in our external diffusion models with two groups. In Tab.2, quite similar to Tab.1, we have considered the impact on saturation times of different heterogeneous situations in which the relative sizes of the two groups $w_1 = 0.1, w_2 = 0.9$ are kept fixed (of course preserving $\alpha(0)$).

¹⁵The mean value of the initial distribution is obviously the arithmetic mean of the innovation rates α_i weighted with the population weights m_i , i.e. the aggregate innovation rate at $t = 0$.

	10%	50%	75%	90%	95%	99%
$\alpha(0)=0.027877$	3.77	24	49	83	107	165
$\alpha_1=0.04 \quad \alpha_2=0.0265$ $\sigma^2 = 0.00001$	3.78	26	51	84	110	170
$\alpha_1=0.06 \quad \alpha_2=0.0243$ $\sigma^2 = 0.00011$	3.80	27	53	91	119	185
$\alpha_1=0.08 \quad \alpha_2=0.0221$ $\sigma^2 = 0.00017$	3.85	28	58	99	131	204
$\alpha_1=0.12 \quad \alpha_2=0.0176$ $\sigma^2 = 0.00094$	4.03	34	73	125	164	256

Table 2: Saturation times of different heterogeneous situations characterized by constant relative sizes of the two groups $w_1 = 0.1$, $w_2 = 0.9$.

	10%	50%	75%	90%	95%	99%
$\alpha(0)=0.027877$	3.7	24	50	83	107	165
$\alpha_1 = 0.09 \quad \alpha_2 = 0.04 \quad \alpha_3 = 0.02047$ $w_1 = 0.07, w_2 = 0.13, w_3 = 0.8$ $\sigma^2 = 0.00033$	3.8	28	59	103	136	214
$\alpha_1 = 0.12 \quad \alpha_2 = 0.02058 \quad \alpha_3 = 0.01$ $w_1 = 0.1, w_2 = 0.65, w_3 = 0.25$ $\sigma^2 = 0.01391$	4.0	34	77	139	191	330

Table 3: Saturation times of different heterogeneous situations in a three groups model (as usual) $\alpha(0) = 0.027877$.

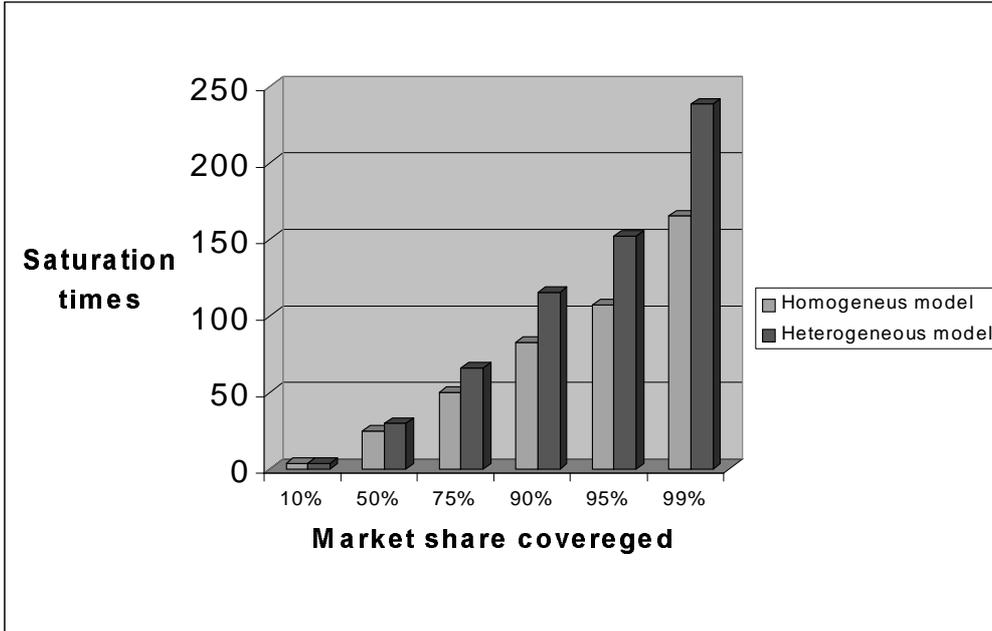


Figure 1: Times needed to saturate several market shares: differences between the heterogeneous model with: $\alpha_1 = 0.08$, $\alpha_2 = 0.01867$, $w_1 = 0.15$, $w_2 = 0.85$, and the corresponding homogeneous one with $\alpha^* = 0.027877$.

Finally, Tab.3 reports results relative to two different situations in which the overall population is divided in three groups; also in these cases we can easily see the bias deriving from a wrong specification of the model.

7.2 The model for internal diffusion with mixing

The approach followed in the previous section to evaluate the impact of heterogeneities is applied, *mutatis mutandis*, in all the subsequent illustrations.

7.2.1 The case of Proportionate mixing

See Fig.2, Fig.3 and Fig.4.

7.2.2 An example of restricted mixing

See Fig.5, Fig.6, Fig.7 and Fig.8.

8 Beyond models: conclusions and data issues

Despite the possible theoretical interest of the results discussed in the paper the main aim of this paper is to raise a main question, namely "*which are the social interactions patterns relevant for the diffusion of new products*"? This central question, which is just a part of the more general question of how do individuals actually socially mix, has been largely neglected

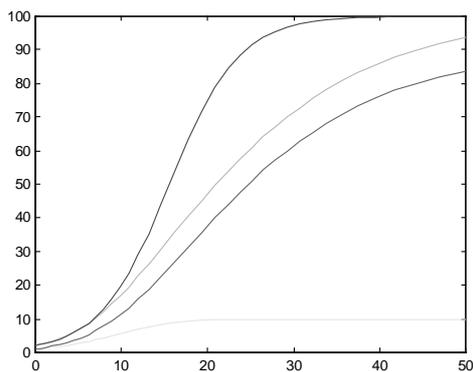


Figure 2: A two groups model for internal diffusion with proportionate mixing $C_1 = 20, C_2 = 3.333, m_1 = 10, m_2 = 90, \beta = 0.025, C_{omo} = 10, q(0) = .25$

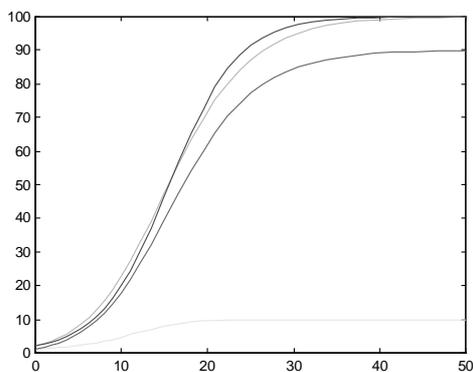


Figure 3: A two groups model for internal diffusion with proportionate mixing $C_1 = 19, C_2 = 7.44, M_1 = 10, M_2 = 90, \beta = 0.025, C_{omo} = 10 (q(0) = 0.25)$

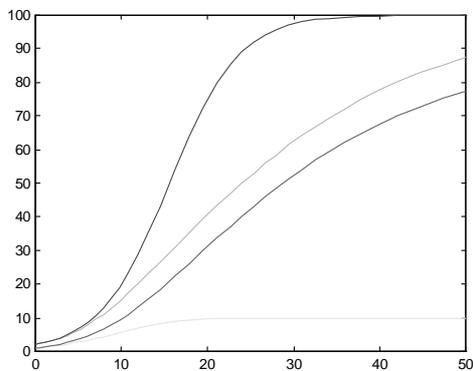


Figure 4: A two groups model for internal diffusion with proportionate mixing $C_1 = 19, C_2 = 2.5505, M_1 = 10, M_2 = 90, \beta = 0.025, C_{omo} = 10 (q(0) = 0.25)$

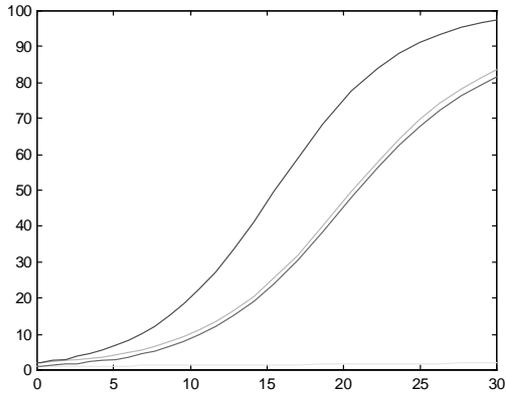


Figure 5: A two groups model for internal diffusion with restricted mixing $C_1 = 1, C_2 = 9, M_1 = 10, M_2 = 90, \beta = 0.025, C_{omo} = 10$ ($q(0) = 0.25$)

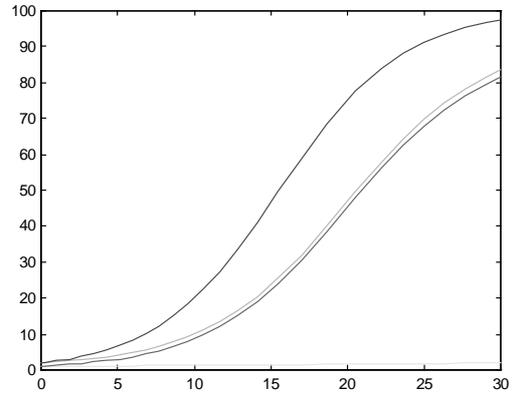


Figure 6: A two groups model for internal diffusion with restricted mixing $C_1 = 1, C_2 = 9, M_1 = 10, M_2 = 90, \beta = 0.025, C_{omo} = 10$ ($q(0) = 0.25$)

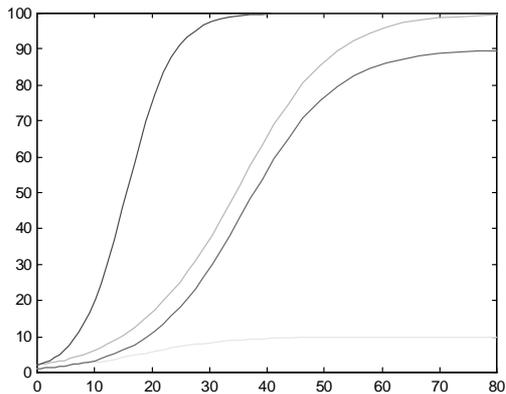


Figure 7: A two groups model for internal diffusion with restricted mixing $C_1 = 5, C_2 = 5, M_1 = 10, M_2 = 90, \beta = 0.025, C_{omo} = 10$ ($q(0)=0.25$)

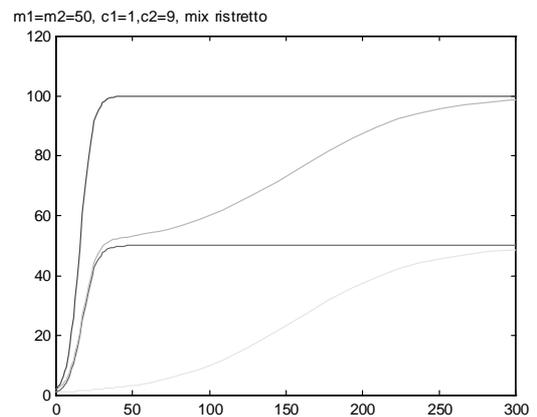


Figure 8: A two groups model for internal diffusion with restricted mixing $C_1 = \quad, C_2 = \quad, M_1 = \quad, M_2 = \quad, \beta = 0.025, C_{omo} = 10$ ($q(0)=0.25$)

in classical new product diffusion models, essentially based on a traditional homogeneous apparatus. The heterogeneity and mixing approach provides a powerful and flexible theoretical framework which has some drawbacks, such as a huge hunger of data (and some not irrelevant statistical issues). Since the beginning of the heterogeneity revolutions theoretical and applied epidemiologists have tried to start to answer the analogous questions, but remaining mainly confined within the domain of sexual interactions. Hence on the empirical side there are no ready answers for fields different from the original one. We believe it is now time to seriously fill this gap.

A Dynamics of the sum of independent exponentially growing populations

Let us consider a population compounded by n non interacting subgroups, each of which is experiencing exponential growth at different constant rate:

$$\dot{x}_i(t) = a_i x_i(t) \quad i = 1, \dots, n \quad (48)$$

The dynamics of the total population $X(t)$ is given by the ODE:

$$\dot{X}(t) = \left[\sum_{i=1}^n a_i \frac{x_i(t)}{X(t)} \right] X(t) \quad (49)$$

The behaviour of (49) depends on the dynamics of the population weights $w_i = x_i(t)/X(t)$. We have:

$$\dot{w}_i(t) = w_i(t) \left[\frac{\dot{x}_i(t)}{x_i(t)} - \frac{\dot{X}(t)}{X(t)} \right] = w_i(t) \left[\alpha_i - \sum_{j=1}^n a_j w_j \right] \quad (50)$$

in which we easily recognise a Lotka-Volterra n -species competitive system on the unit simplex. The dynamics of (50) is so easily ascertained: the system will converge to the long term globally stable equilibrium $(0, 0, \dots, 1, 0, \dots, 0)$ where the only nonzero element pertains to the fastest population (ie the one characterised by the higher growth rate). It is to be noted that this “competitive exclusion” result depends totally on the existence of a population which is fastest compared to others. In practical terms: the fastest population will be, in percentage terms, more and more important as times goes on.

The $n = 2$ case is particularly enlightening. The dynamics of the total population is given by the ODE:

$$\dot{X}(t) = \dot{x}_1 + \dot{x}_2 = a(t)X(t) \quad (51)$$

where the actual growth rate $a(t)$ is defined as:

$$a(t) = \alpha_1 w(t) + \alpha_2 (1 - w) \quad (52)$$

where $w = w_1$ is the weight of the first population. The differential equation for w is:

$$\dot{w}(t) = w(t) \left[\frac{\dot{x}_1(t)}{x_1(t)} - \frac{\dot{X}(t)}{X(t)} \right] = w(t) [\alpha_1 - a(t)] \quad (53)$$

i.e.:

$$\dot{w}(t) = w(t) [\alpha_1 - (\alpha_1 w + \alpha_2(1 - w))] = w(t) [(\alpha_1 - \alpha_2) - (\alpha_1 - \alpha_2) w]$$

and definitively:

$$\dot{w}(t) = (\alpha_1 - \alpha_2)w(t) [1 - w(t)] \quad (54)$$

From the last relation we realise that, assumed without any loss of generality that $\alpha_1 > \alpha_2$ (ie that population one is faster), the weight of the faster population, which is initially determined by the initial size of the two populations, will, as expected, saturate to one with a logistic dynamics. In particular the dynamics of the actual growth rate is totally determined by the dynamics of the weights as stated by:

$$a(t) = \alpha_1 w(t) + \alpha_2(1 - w(t))$$

ie it is the time average of the growth rates of the two subpopulations weighted with their “logistic” weights. This means that, since the asymptotic weight of the faster population will be one, the asymptotic dynamics of the total population will be exponential with a rate which is simply the rate of growth of the faster population. For what regards the transient dynamics of the rate of growth, since:

$$a(t) = \alpha_2 + (\alpha_1 - \alpha_2)w(t)$$

we easily see that this will be logistic as well, starting from the initial value:

$$a(0) = \alpha_2 + (\alpha_1 - \alpha_2)w(0)$$

which is simply α_2 when the initial weight of the faster population is zero.

A The internal diffusion model with heterogeneities in susceptibility

The basic equations are:

$$\dot{X}_i(t) = -r_i X_i(t) Y(t) \quad ; \quad \dot{Y}(t) = r_i X_i(t) Y(t) \quad (55)$$

where we assume $r_1 < \dots < r_n$. The aggregate equations are:

$$\dot{X}(t) = -r(t) X(t) Y(t) \quad ; \quad \dot{Y}(t) = r(t) X(t) Y(t)$$

where $r(t)$:

$$r(t) = \sum_i r_i \frac{X_i(t)}{X(t)} = \sum_i r_i \pi_i = \bar{r}(t)$$

The result (16) of the main text holds:

$$\left(\frac{d}{dt} \bar{r}(t) \right) = (-1) Y(t) Var_t(r)$$

The proof is immediate:

$$\begin{aligned}
\left(\frac{d}{dt} \bar{r}(t)\right) &= \sum_i r_i \frac{d}{dt} \frac{X_i(t)}{X(t)} = \sum_i r_i \frac{\dot{X}_i(t)X(t) - X_i(t)\dot{X}(t)}{X^2(t)} = \\
&= \sum_i r_i \frac{-r_i X_i(t)Y(t)X(t) + X_i(t)\bar{r}(t)X(t)Y(t)}{X^2(t)} = \\
&= (-1)Y(t) \sum_i r_i \frac{r_i X_i(t) - X_i(t)\bar{r}(t)}{X(t)} = \\
&= (-1)Y(t) \left(\sum_i r_i^2 \frac{X_i(t)}{X(t)} - \bar{r}(t) \sum_i r_i \frac{X_i(t)}{X(t)} \right) = \\
&= (-1)Y(t) \text{Var}_t(r)
\end{aligned}$$

The dynamics of the weights $\pi_i(t)$ It holds:

$$\begin{aligned}
\dot{\pi}_i(t) &= \pi_i(t) \left(\frac{\dot{X}_i(t)}{X_i(t)} - \frac{\dot{X}(t)}{X(t)} \right) = \pi_i(t) (-r_i Y(t) + r(t) Y(t)) = \\
&= \pi_i(t) Y(t) (r(t) - r_i) = Y(t) \left(\sum_{j=1}^n r_j \pi_j - r_i \right) \pi_i = \\
&= (-1)Y(t) \left(r_i(1 - \pi_i) - \sum_{j \neq i} r_j \pi_j \right) \pi_i
\end{aligned}$$

The case $n=2$ In the case $n = 2$ the restraint $\pi_1 + \pi_2 = 1$ gives the equation:

$$\begin{aligned}
\dot{\pi}_1(t) &= (-1)Y(t) (r_1(1 - \pi_1) - r_2\pi_2) \pi_1 = \\
&= (-1)Y(t) (r_1(1 - \pi_1) - r_2(1 - \pi_1)) \pi_1 = \\
&= (-1)(r_1 - r_2)Y(t)(1 - \pi_1)\pi_1 = \\
&= (r_2 - r_1)Y(t)(1 - \pi_1)\pi_1
\end{aligned}$$

which can be completed to define a closed system of two equations in the two variables (π_1, Y) :

$$\begin{aligned}
\dot{Y}(t) &= r(t)X(t)Y(t) = (r_1\pi_1 + r_2\pi_2) X(t)Y(t) = \\
&= (r_1\pi_1 + r_2(1 - \pi_1)) X(t)Y(t) = \\
&= (r_2 + (r_1 - r_2)\pi_1) Y(t)(m - Y(t))
\end{aligned}$$

Similarly, full information on the overall dynamics can be derived from the two dimensional system in the variables $(r(t), Y(t))$. The r equation is easily derived by observing that:

$$r = r_1\pi_1 + r_2\pi_2 = r_1\pi_1 + r_2(1 - \pi_1) = (r_1 - r_2)\pi_1 + r_2$$

Hence:

$$\pi_1 = \frac{r - r_2}{r_1 - r_2} ; \quad 1 - \pi_1 = \frac{r_1 - r}{r_1 - r_2}$$

leading to:

$$\begin{aligned}
\dot{r} &= r_1 \dot{\pi}_1 + r_2 \dot{\pi}_2 = (r_1 - r_2) \dot{\pi}_1 = \\
&= (r_1 - r_2)(r_2 - r_1)Y(t)(1 - \pi_1)\pi_1 = \\
&= (r_1 - r_2)(r_2 - r_1)Y(t) \frac{r_1 - r}{r_1 - r_2} \cdot \frac{r - r_2}{r_1 - r_2} = \\
&= Y(t)(r(t) - r_1)(r(t) - r_2)
\end{aligned}$$

Definitively:

$$\begin{aligned}\dot{r}(t) &= Y(t)(r(t) - r_1)(r(t) - r_2) \\ \dot{Y} &= r(t)(m - Y(t))Y(t)\end{aligned}$$

A straightforward qualitative analysis shows that the equilibrium $E = (r_1, m)$, characterised by a long term r_1 logistic rate, is globally asymptotically stable.

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